QUANTUM GROUPS AS GLOBAL SYMMETRIES IN THE CONTINUUM

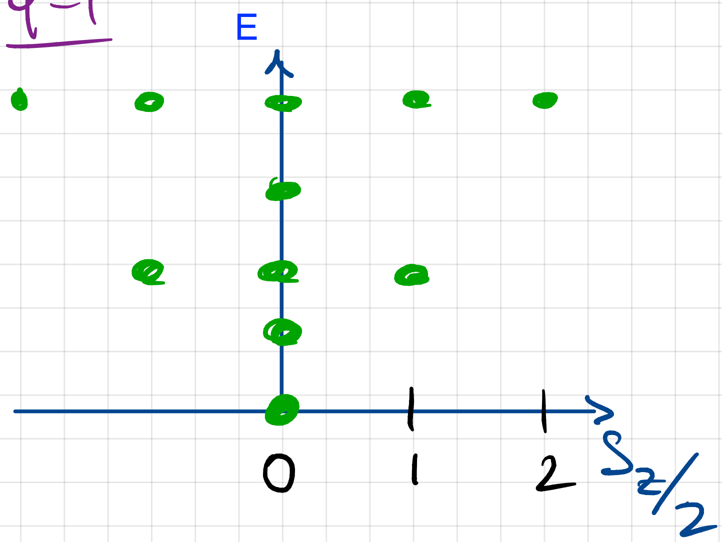
Bernardo Zan

W.I.P. w/ Barak Gabai, Victor Gorbenko,
Jiaxin Qiao, Aleksandr Zhabin

Introduction

$$\mathcal{H} = + \sum_{i=1}^{N-1} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q+q^{-1}}{2} \sigma_i^z \sigma_{i+1}^z \right)$$

$q=1$



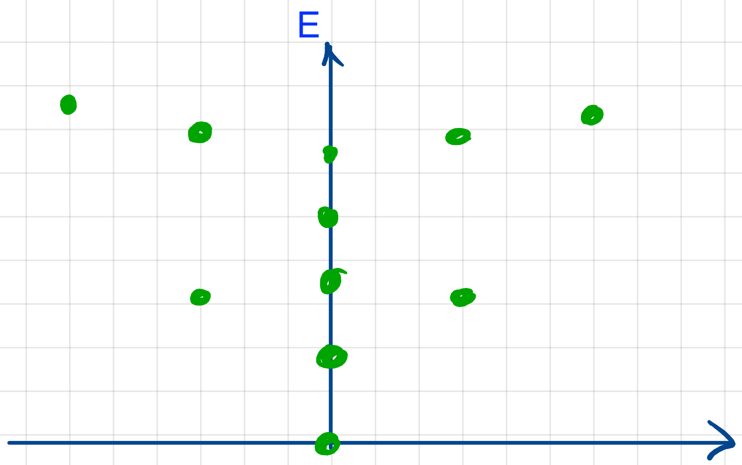
Degeneracy due to $su(2)$

$$S^z = \sum_i \sigma_i^z$$

$$S^\pm = \sum_i \sigma_i^\pm$$

all commute w/ $\mathcal{H}$

$q = e^{i\theta}, \ \theta \in \mathbb{R}$



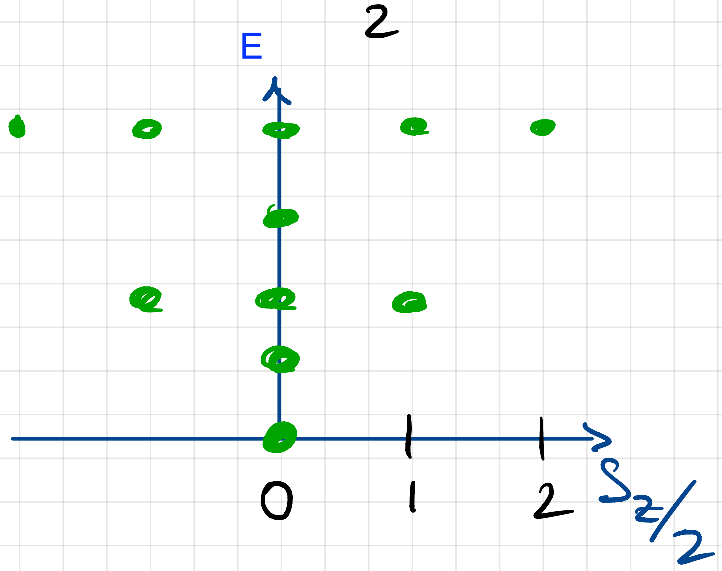
No degeneracy

Only $[S^z, \mathcal{H}] = 0$

Now

$$\mathcal{H}_{PS} = \mathcal{H} + \frac{q-q^{-1}}{2}\left(\sigma_1^z - \sigma_N^z\right)$$

[Pasquier, Saleur '90]



Recovered $su(2)$? No

Find E, F, H commuting w/ $\mathcal{H}_{PS}$

$$E = \sum_i q^{\sigma_z/2} \otimes \dots \otimes q^{\sigma_z/2} \otimes \sigma^+ \otimes q^{-\sigma_z/2} \otimes \dots \otimes q^{-\sigma_z/2}$$

$\hookrightarrow$ position i

$F =$ same with $\sigma^+ \to \sigma^-$

$$H = \sum_j \sigma_j^z = \sum \mathbb{1} \otimes \dots \otimes \mathbb{1} \otimes \sigma^z \otimes \mathbb{1} \dots \otimes \mathbb{1}$$

Commutation rels not those of $su(2)$

e.g. $$[E, F] = \frac{q^H - q^{-H}}{q - q^{-1}}$$

$\hookrightarrow$ symmetry is actually $\mathcal{U}_q(sl_2)$

Questions

- Continuum limit → CFT

  Studied many CFTs with a lot of symmetries.

  What do a CFT w/ $U_q(sl_2)$ global symmetry look like?

- $U_q(sl_2)$ appear in 2d CFT, but in an indirect manner

  e.g. crossing kernel in minimal models contain 6j symbol of $U_q(sl_2)$

  We look for a genuine internal symmetry

- Any relation to generalized symmetry?

  E.g. non-invertible?

  O(n) model and non-crossing of lines

  [ ]

$U_q(sl_2)$ bases

Commutation relations

Quantum group is an algebra, not a group.

sl_2 $\quad [H,E]=2E \quad [H,F]=-2F \quad [E,F]=H$

↳ raising ↳ lowering

$U_q(sl_2)$ $\quad q^H E = q^2 E q^H \quad q^H F = q^{-2} F q^H$

$$[E,F] = [H]_q \equiv \frac{q^H - q^{-H}}{q - q^{-1}}$$

Coproduct

How do we act on two spins?

sl_2 $\quad \Delta(X) = X \otimes \mathbb{1} + \mathbb{1} \otimes X$

$U_q(sl_2)$: deformed comm. rels. $\rightarrow$ deformed coproduct

Require e.g. $\Delta([E,F]) = [\Delta(E), \Delta(F)]$

$$\Delta(H) = H \otimes \mathbb{1} + \mathbb{1} \otimes H \rightsquigarrow \Delta(q^H) = q^H \otimes q^H$$

$$\Delta(E) = E \otimes \mathbb{1} + q^H \otimes E$$

$$\Delta(F) = F \otimes q^H + \mathbb{1} \otimes F$$

Why not a group?

ordinary algebra $\Delta(X) = \mathbb{1}\otimes X + X\otimes\mathbb{1}$

group element $g = e^{i\alpha X}$ $\Delta(g) = g\otimes g$

No way (so far) of building "g" out of $\mathcal{U}_q(sl_2)$ so that it behaves nicely under $\Delta(g)$.

Representations

Generic q (q not root of unity)

Finite dim. representation same as $su(2)$

$|\ell, m\rangle$ $2\ell \in \mathbb{Z}_{\geq 0}$

$m = -\ell, \dots, \ell$

$$H|\ell, m\rangle = 2m|\ell, m\rangle$$

$$E|\ell, \ell\rangle = 0$$

$$F|\ell, m\rangle = \# |\ell, m-1\rangle$$

$$F|\ell, -\ell\rangle = 0.$$

q root of unity $\to$ much more complicated.

Some things carry over

e.g quantum Clebsch-Gordon coeffs

$$|\ell_1, m_1\rangle |\ell_2, m_2\rangle = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \sum_m \begin{bmatrix} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & m \end{bmatrix}_q \cdot |\ell, m\rangle$$

R-matrix

"Swaps two copies"

Δ treats L, R differently

$$\Delta(E) = E \otimes 1\!\!1 + q^{-H} \otimes E$$

Define other way around

$$\tilde{\Delta}(E) = 1\!\!1 \otimes E + E \otimes q^{H}$$

$$R \Delta R^{-1} = \tilde{\Delta}$$

CFTs w/ $U_q(sl_2)$

- Internal symmetry:

  E, F, H commute w/ $L_n, \bar{L}_n$

- Operators transform under $U_q(sl_2)$

$$\mathcal{O}_\ell^{m}(x) \qquad E\,\mathcal{O}_\ell^{m} = \#\, \mathcal{O}_\ell^{m+1}$$

m: $u(1)$ eigenvalue, $|m| \le \ell$

$\ell \in \frac{\mathbb{Z}_{\geq 0}}{2}$

Correlation fcts satisfy Ward identity

e.g. $\langle X(\mathcal{O}_1 \dots \mathcal{O}_n)\rangle = 0$

X: E, F, H w/ coproduct applied many times

First peculiarity: ops should commute in Euclidean space

spin $1/2$

$$\mathcal{O}_\pm = \mathcal{O}^{\pm\frac{1}{2}}_{\frac{1}{2}} \qquad \Delta(F) = F \otimes q^H + 1 \otimes F$$

$$\langle F(\mathcal{O}_+(x)\,\mathcal{O}_+(y))\rangle = 0$$

$$q\,\underbrace{\langle \mathcal{O}_-(x)\,\mathcal{O}_+(y)\rangle}_{(-1)^{2S}\langle \mathcal{O}_-(y)\,\mathcal{O}_+(x)\rangle} + \langle \mathcal{O}_+(x)\,\mathcal{O}_-(y)\rangle = 0$$

S = spacetime spin

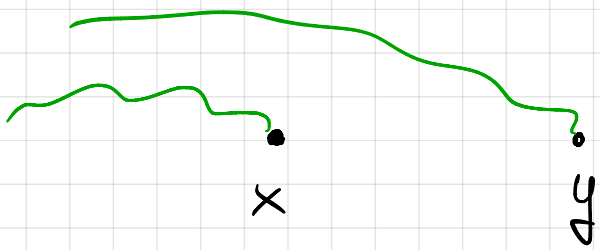
therefore

$$\langle \mathcal{O}_+(x)\, \mathcal{O}_-(y) \rangle = -q\,(-1)^{2s} \langle \mathcal{O}_-(y)\, \mathcal{O}_+(x) \rangle$$

Locality : local operators commute in Euclidean space

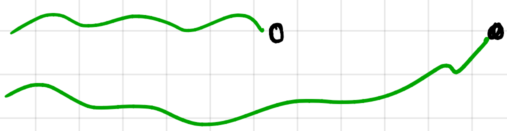
In a $\mathcal{U}_q(sl_2)$ symmetric theory, operators cannot be mutually local.

Simplest modification of locality:
operators attached to lines.
Lines are topological but cannot cross other ops / lines.



$$\langle \mathcal{O}_1(x)\, \mathcal{O}_2(y) \ldots \rangle$$

vs.



$$\langle \mathcal{O}_2(y)\, \mathcal{O}_1(x) \ldots \rangle$$

Lines are swapped by R-matrix

$$R_{ji}\, \mathcal{O}_i\, \mathcal{O}_j = \mathcal{O}_j\, \mathcal{O}_i$$

$$\sum_{m_i',m_j'} [R_{\ell_j,\ell_i}]^{m_j' m_i'}_{m_j m_i}\, \mathcal{O}_{i,\ell_i,m_i'}(x)\mathcal{O}_{j,\ell_j,m_j'}(y) = \mathcal{O}_{j,\ell_j,m_j}(y)\mathcal{O}_{i,\ell_i,m_i}(x),$$

* OPE (ignoring coordinates)

$$\mathcal{O}_{i,\ell_i,m_i} \circ \mathcal{O}_{j,\ell_j,m_j} = \sum_k \lambda_{ijk} \begin{bmatrix} \ell_i & \ell_j & \ell_k \\ m_i & m_j & m_k \end{bmatrix} \mathcal{O}_{k,\ell_k,m_k}$$

e.g 2pf

$$\langle \mathcal{O}_{i,\ell_i,m_i}(0)\, \mathcal{O}_{j,\ell_j,m_j}(x) \rangle$$

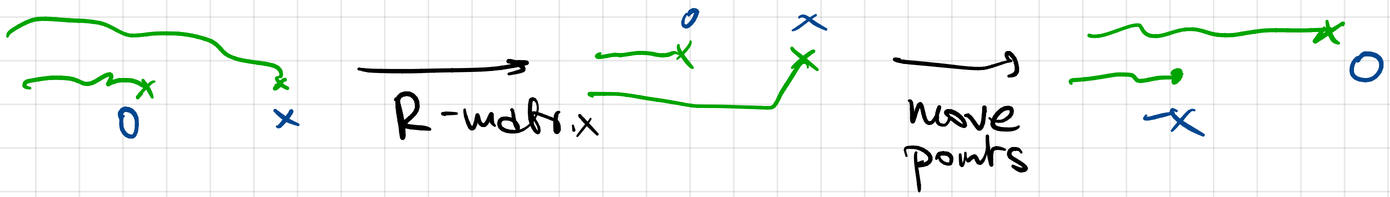
$$\begin{bmatrix} \ell & \ell & 0 \\ m & -m & 0 \end{bmatrix}_q = \frac{(-1)^{m-\ell} q^m}{\sqrt{[2\ell+1]_q}}.$$

$$= \delta_{ij} \begin{bmatrix} \ell_i & \ell_j & 0 \\ m_i & m_j & 0 \end{bmatrix}_q \frac{1}{x^{2h_i}\, \bar{x}^{2\bar{h}_i}}$$

Consequence : constraints on spacetime spin

Non-locality $\Leftrightarrow$ spin not integer

$$\langle \mathcal{O}_{\ell,m_j}(0)\, \mathcal{O}_{\ell,m_i}(x) \rangle$$



$$= \langle \mathcal{O}_{\ell,m_i}(-x)\, \mathcal{O}_{\ell,m_j}(0) \rangle$$

Implies relation b/w $s = h - \bar{h}$ and ℓ.

$$s = \pm \frac{\ell(\ell+1)}{\pi i} \log q + \ell + \mathbb{Z}$$

chosen once and for all for a given theory

XXZ_q example

Spin chain from [Grosse, Pallua, Prester, Raschhofer '94]

Critical, spectrum + central charge known

$$q = e^{i\pi \frac{\mu}{\mu+1}} \rightarrow c = 1 - \frac{6}{\mu(\mu+1)}$$

$$W^{m}_{r,s} \text{ operators} \rightarrow \ell = \frac{s-1}{2}$$

$$\Updownarrow$$

$$(h_{r,s}, h_{r,1})$$

- Check #1: ℓ, $h - \bar{h}$ satisfy our constraint
- Check #2: theory is crossing symmetric

Need to fix OPE coeffs. Two ways

- crossing symmetry
- Coulomb gas: free boson + screening charges

Result $\quad C^{2}_{ijk} = \pm\, C^{MM}_{(r_i s_i)(r_j s_j)(r_k, s_k)} \; C^{MM}_{(r_i,1)(r_j,1)(r_k,s_k)}$

Relation to unitary theory

- Explains 6j symbol in fusion kernels in some case
- Contains minimal models as subsector for $\mu \in \mathbb{N}$

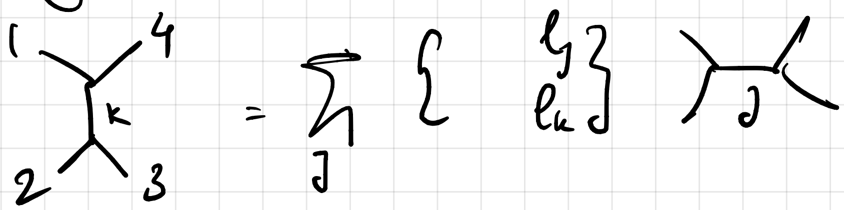
Fusion kernel

Degenerate ops for $c \leq 1$. External ops h_{r_i, s_i}

$$\mathcal{F}^{(t)}_{r_k, s_k} = \sum_J \frac{g_{12}^J \, g_{J3}^4}{g_{23}^k \, g_{k1}^4} \cdot (-1)^{\#} \begin{Bmatrix} \frac{r_1-1}{2} & \frac{r_2-1}{2} & \frac{r_J-1}{2} \\ \frac{r_3-1}{2} & \frac{r_4-1}{2} & \frac{r_k-1}{2} \end{Bmatrix}_{\tilde{q}} \begin{Bmatrix} \frac{s_1-1}{2} & \frac{s_2-1}{2} & \frac{s_J-1}{2} \\ \frac{s_3-1}{2} & \frac{s_4-1}{2} & \frac{s_k-1}{2} \end{Bmatrix}_{q} \mathcal{F}^{(s)}_{r_J, s_J}$$

$$q = e^{i\pi \frac{\mu}{\mu+1}}, \quad \tilde{q} = e^{i\pi \frac{\mu+1}{\mu}}$$

6j symbol: product of 4 QG coeffs



Why do 6j symbols appear in non $U_q(sl_2)$ symmetric theories?

Explanation for h_{1,s_i}, s_i odd

$$\to \mathcal{F}^{(t)}_{1,s_k} = \sum_j \frac{g^{j}_{12}\, g^{4}_{j3}}{g^{k}_{23}\, g^{4}_{k1}} \begin{Bmatrix} \frac{s_1-1}{2} & \frac{s_2-1}{2} & \frac{s_j-1}{2} \\ \frac{s_3-1}{2} & \frac{s_4-1}{2} & \frac{s_k-1}{2} \end{Bmatrix}_q \mathcal{F}^{(s)}_{1,s_j}$$

Consider $W^{m}_{1,s}$ ops

* $(h_{1,s}, h_{1,1}=0)$ chiral
* one op. per spin

$$W_{1,1+2e_1} \cdot W_{1,1+2e_2} = \sum_{e=|e_1-e_2|}^{e_1+e_2} W_{1,1+2e}$$

Crossing eq

$$\langle W^{m_1}_{1,s_1} \underbrace{W^{m_2}_{1,s_2}}_{W_{1,s_j}} \overbrace{\;W^{m_3}_{1,s_3}}^{W_{1,s_k}} W^{m_4}_{1,s_4} \rangle$$

$$\sum_j{}' \; C_{12j} \begin{bmatrix} e_1 & e_2 & e_j \\ m_1 & m_2 & m_j \end{bmatrix}_q C_{j34} \begin{bmatrix} e_j & e_3 & e_4 \\ m_j & m_3 & -m_4 \end{bmatrix} \cancel{\begin{bmatrix} e_4 & e_4 & 0 \\ -m_4 & m_4 & 0 \end{bmatrix}} \cdot \mathcal{F}^{(s)}_{1,s_j}(z)$$

$$= \sum_k{}' \; C_{23k} \begin{bmatrix} e_2 & e_3 & e_k \\ m_2 & m_3 & m_k \end{bmatrix} C_{1k4} \begin{bmatrix} e_1 & e_k & e_4 \\ m_1 & m_k & -m_4 \end{bmatrix} \cdot \cancel{\begin{bmatrix} e_4 & e_4 & \\ -m_4 & m_4 & 0 \end{bmatrix}} \; \mathcal{F}^{(t)}_{1,s_k}(z)$$

Use orthogonality relation of QG

Get

$$\mathcal{F}^{(t)}_{1,s_k} = \sum_j{}' \frac{C_{12j}\,C_{34j}}{C_{23k}\,C_{14k}} \left(\sum_{m_i}{}' [\quad][\quad][\quad][\quad] \right)$$

$$\mathcal{F}^{(s)}_{1,s_j}$$

$$= \sum_j{}' \frac{C_{12j}\,C_{34j}}{C_{23k}\,C_{14k}} \begin{Bmatrix} \frac{s_1-1}{2} & \frac{s_2-1}{2} & \frac{s_j-1}{2} \\ \frac{s_3-1}{2} & \frac{s_4-1}{2} & \frac{s_k-1}{2} \end{Bmatrix}_q \mathcal{F}^{(s)}_{1,s_j}$$

Blocks are the same in XXZ_q and in generalized minimal models

→ 6j symbol appear b.c. our theory is crossing symmetric.

General $r \neq 1$ → need Coulomb gas construction

Open directions

- Relation to non-invertible symmetries
- $\mathcal{U}_q(sl_{N>2})$?
- A theory w/ 2 QGs ?