Backbone exponent and annulus crossing probability for 2D percolation via Liouville quantum gravity

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joint work with Pierre Nolin (CityUHK), Wei Qian (CityUHK), Zijie Zhuang (UPenn)
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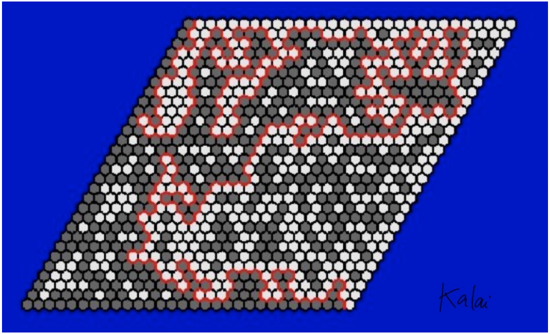
Schramm Loewner Evolution (SLE)

Schramm (1999)

Random interfaces in many 2D statistical physics models should converge to SLE_κ with $\kappa > 0$.

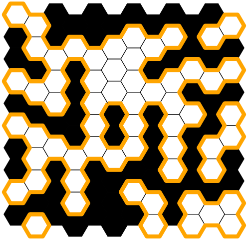
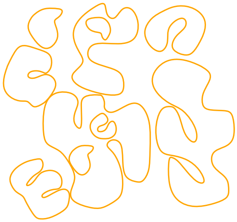


A few scaling limit results, many more conjectures.

Percolation $\to \mathrm{SLE}_6$, Ising model $\to \mathrm{SLE}_3$ (Smirnov et. al.)

Critical Bernoulli percolation

- Bernoulli site percolation on the **triangular lattice**; Color each hexagon either black or white with equal probability independently.

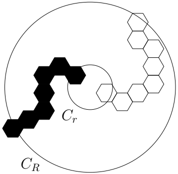



- Full understanding of the scaling limit:
  - Conformal invariance of the quadrangle crossing probability and scaling limit of one interface to $\mathbf{SLE}_6$ (Smirnov '01).
  - Full scaling limit of interfaces as $\mathbf{CLE}_6$ (Camia-Newman '06).

Polychromatic arm crossing events and exponents

- C_R: circle of radius R centered at 0.
- Polychromatic j-arm event:

$$\mathcal{A}_j(r,R) := \{\exists j \text{ disjoint black/white paths connecting } C_r \text{ to } C_R, \text{ and } \textbf{not all of the same color} \text{ for } j>1\}.$$



Polychromatic 2-arm event $\mathcal{A}_2(r,R)$.

Smirnov-Werner '01, Lawler-Schramm-Werner '02

$\forall j \geq 1$,

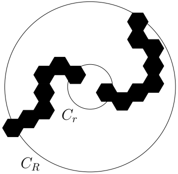
$$\mathbb{P}[\mathcal{A}_j(r,R)] = \left(\frac{r}{R}\right)^{-\alpha_j + o(1)} \quad \text{as } r/R \to 0$$

with $\alpha_j = \frac{j^2-1}{12}$ for all $j \geq 2$; $\alpha_1 = \frac{5}{48}$.

Monochromatic arm events and exponents

- Monochromatic j-arm event ($j \geq 2$):

$$\mathcal{B}_j(r, R) := \{\exists j \text{ disjoint } \textbf{black} \text{ paths connecting } C_r \text{ to } C_R\}.$$



Monochromatic 2-arm event $\mathcal{B}_2(r, R)$.

Beffara-Nolin '11

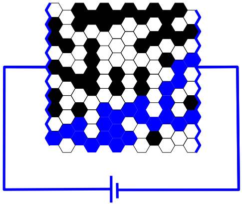
There exists $\beta_j > 0$ such that

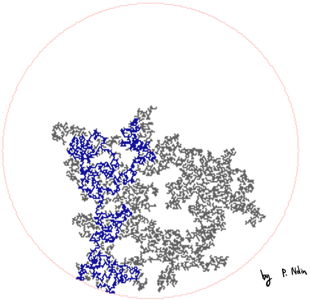
$$\mathbb{P}[\mathcal{B}_j(r, R)] = (\frac{r}{R})^{-\beta_j + o(1)} \quad \text{as } r/R \to 0.$$

β_2: today's topic; β_j ($j \geq 3$): still unsolved.

Backbone exponent β_2

- Monochromatic two-arm exponent β_2 is also called the **backbone exponent**.
- Introduced in physics more than 40 years ago to study electric flow within a critical percolation cluster.
- With high probability, there are $R^{2-\beta_2+o(1)}$ points on the support of electric current.





Support of electric current.

Explicit value of the backbone exponent β_2

Nolin-Qian-S.-Zhuang (2023)

β_2 is the unique solution in the interval $(\frac{1}{4}, \frac{2}{3})$ to the equation

$$\frac{\sqrt{36x+3}}{4} + \sin\left(\frac{2\pi\sqrt{12x+1}}{3}\right) = 0.$$

β_2 is a transcendental number. $\beta_2 = 0.35666683671288\cdots$

- $\beta_2 = 0.35661 \pm 0.00005$;
  Match most recent numerical result, Fang-Ke-Zhong-Deng '22.
- $\beta_2 \neq \alpha_1 + \alpha_2 = \frac{5}{48} + \frac{1}{4} = \frac{17}{48} = 0.354...$;
  Disprove conjecture by Beffera-Nolin based on earlier numerical work.

Annulus crossing formulae

Cardy has a famous formula for the left-right crossing probability for percolation on a rectangle.
Smirnov proved it for site percolation on the triangular lattice.

What about the exact probabilities for the annulus crossing?

S.-Xu-Zhuang (2024+)

Let $\tau = \frac{1}{2\pi}\log(\frac{R}{r})$ and $\eta(z) = e^{\frac{i\pi z}{12}}\prod_{n=1}^{\infty}(1-e^{2ni\pi z})$.

$$\mathbb{P}[\mathcal{A}_1(r,R)] = \sqrt{\frac{3}{2}}\cdot\frac{\eta(6i\tau)\eta\left(\frac{3}{2}i\tau\right)}{\eta(2i\tau)\eta(3i\tau)};$$

$$\mathbb{P}[\mathcal{A}_2(r,R)] = \sqrt{3}\cdot\frac{\eta(i\tau)\eta(6i\tau)^2}{\eta(3i\tau)\eta(2i\tau)^2}.$$

Predicted by Cardy ('02, '06) using (non-rigorous) Coulomb gas approach.
Hard to derive via Ito's calculus, in contrast to the rectangle crossing.

CFT style expressions for annulus crossing probabilities

Closed channel expansion: let $\tau = \frac{1}{2\pi}\log(\frac{R}{r})$ and $q = e^{-2\pi\tau}$.

$$\mathbb{P}[\mathcal{A}_1(r,R)] = \sqrt{\frac{3}{2}} \cdot \frac{\sum_{k\in\mathbb{Z}}(q^{2h_{4k-\frac{1}{2},0}} - q^{2h_{4k+\frac{3}{2},0}})}{\prod_{n=1}^{\infty}(1-q^{2n})}.$$

$$\mathbb{P}[\mathcal{A}_2(r,R)] = \sqrt{3} \cdot \frac{\sum_{k\in\mathbb{Z}}(q^{2h_{0,6k+1}} - q^{2h_{0,6k+2}})}{\prod_{n=1}^{\infty}(1-q^{2n})}.$$

$h_{r,s} = \frac{(3r-2s)^2-1}{24}$; Kac table for central charge $c=0$.

Open channel expansion: let $\tilde{q} = e^{-\pi/\tau}$.

$$\mathbb{P}[\mathcal{A}_1(r,R)] = \frac{\sum_{k\in\mathbb{Z}}(\tilde{q}^{h_{1,4k+1}} - \tilde{q}^{h_{1,4k+3}})}{\prod_{n=1}^{\infty}(1-\tilde{q}^{n})}.$$

$$\mathbb{P}[\mathcal{A}_2(r,R)] = \frac{\sum_{k\in\mathbb{Z}}(\tilde{q}^{h_{1,6k+2}} + \tilde{q}^{h_{1,6k+4}} - 2\tilde{q}^{h_{1,6k+3}})}{\prod_{n=1}^{\infty}(1-\tilde{q}^{n})}.$$

A CFT derivation is missing.

Monochromatic two-arm crossing formulae

S.-Zhuang (2024+)

Let $\tau = \frac{1}{2\pi}\log(\frac{R}{r})$ and $q = e^{-2\pi\tau}$.

$$\mathbb{P}[\mathcal{B}_2(r,R)] = \frac{q^{-\frac{1}{12}}}{\prod_{n=1}^{\infty}(1-q^{2n})} \sum_{s\in\mathcal{S}} \frac{-\sqrt{3}\sin(\frac{2\pi}{3}\sqrt{3s})\sin(\pi\sqrt{3s})}{\cos(\frac{4\pi}{3}\sqrt{3s})+\frac{3\sqrt{3}}{8\pi}} q^s,$$

$\mathcal{S} = \{0.440, 2.194 \pm 0.601i, 5.522 \pm 1.269i, 10.361 \pm 2.020i, \dots\}$.
all (complex) solutions to $\sin(4\pi\sqrt{\frac{x}{3}}) + \frac{3}{2}\sqrt{x} = 0$, except 0 and $\frac{1}{3}$.

Relation to a CFT with complex and transcendental spectrum?

Open channel expansion: let $\tilde{q} = e^{-\pi/\tau}$.

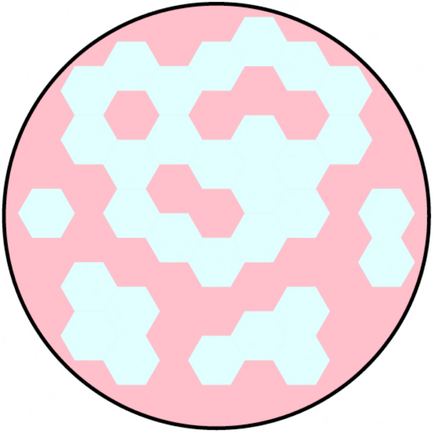
$\mathbb{P}[\mathcal{B}_2(r,R)] = 1 - (1 + \frac{3\sqrt{3}}{4}\tau^{-1})\tilde{q}^{\frac{1}{3}} +$ remaining terms of similar form.

Logarithmic structure. Relation to a log CFT?

Quantum gravity derivation of one-arm exponent

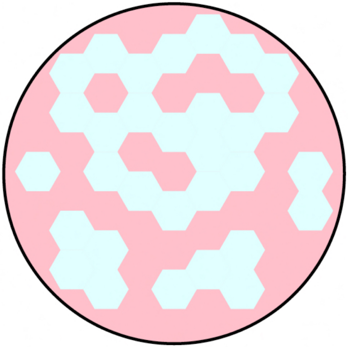
$\mathbb{P}[\text{one arm event}] \sim n^{-\frac{5}{48}}$ is equivalent to:

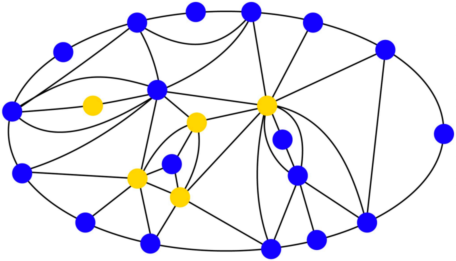
If we have n^2 vertices on the lattice region, the size of the boundary connecting cluster is $\sim n^{91/48}$. $(n^2 \times n^{-\frac{5}{48}} = n^{\frac{91}{48}}.)$



A vertex v is in the boundary connecting cluster if the one arm event occurs at v.

Knizhnik-Polyakov-Zamolodchikov (KPZ) Relation





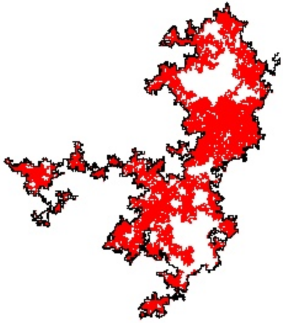
A KPZ derivation of the one-arm exponent

1. On random triangulation, the answer is $n^{\text{quantum exponent}}$.
2. $91/48 = \mathrm{KPZ}(\text{quantum exponent})$

Enumeration on RHS is simpler; $\mathrm{KPZ}(\cdot)$ is explicit quadratic.

History of KPZ

- Derived from the CFT description of LQG. (KPZ '88).
- "Verified" by enumeration of planar maps. (around '90)
  David, Douglas, Gross, Kazakov, Kostov, Migdal, Shenker ...
- Provide a powerful framework to study fractals.



Conjecture Mandelbrot '82

Frontier of planar Brownian motion has fractal dimension $4/3$.

- physics "proof" by KPZ. Duplantier '98.
- rigorous proof via SLE_6.
  Lawler-Schramm-Werner '00.

Duplantier-Sheffield (2011): First rigorous KPZ relation.
Starting point of Liouville quantum gravity in probability

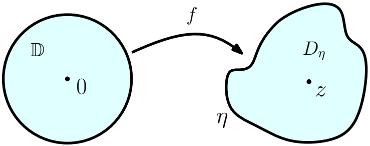
Quantum gravity approach beyond KPZ

The quantum exponent corresponds to β_2 is still a transcendental number. A standard application of KPZ is hard for evaluating β_2.

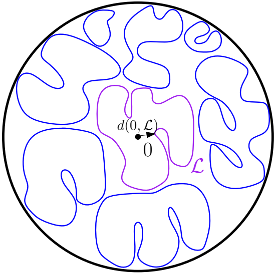
Two crucial tools in our approach:

- Conformal radius encoding of arm-exponent;
- integrability of Liouville CFT that governs LQG surfaces.

Conformal radius: for a loop η and a point z, $\mathrm{CR}(z,\eta) := |f'(0)|$.



Example: One-arm exponent



- $\mathcal{L}$: outermost CLE_6 loop.
- $\mathbb{P}[d(0,\mathcal{L}) \le \epsilon] = \mathbb{P}[\mathcal{A}_1(\epsilon,1)] \approx \epsilon^{\alpha_1}$.
- $\alpha_1 = \inf\{x : \mathbb{E}_{\mathrm{CLE}_6}[d(0,\mathcal{L})^{-x}] = \infty\}$.
- Koebe 1/4 theorem: $\frac{1}{4}\mathrm{CR}(0,\mathcal{L}) \le d(0,\mathcal{L}) \le \mathrm{CR}(0,\mathcal{L})$.

$$\alpha_1 = \inf\{x : \mathbb{E}_{\mathrm{CLE}_6}[\mathrm{CR}(0,\mathcal{L})^{-x}] = \infty\}.$$

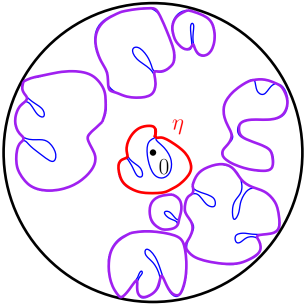
Schramm-Sheffield-Wilson '09

$\mathbb{E}_{\mathrm{CLE}_6}[\mathrm{CR}(0,\mathcal{L})^{-x}] = \frac{1}{2\cos(\frac{\pi}{3}\sqrt{12x+1})}$, hence $\alpha_1 = \frac{5}{48}$.

Conformal Radius Encoding for the Backbone exponent

Consider the outer boundaries of CLE_6 loops: a collection of simple loops.
η: outermost one surrounding 0; locally looks like an $\mathrm{SLE}_{8/3}$ curve.

Exist two monochromatic arms in the annulus bounded by η and $\partial\mathbb{D}$.
$\mathbb{P}[\mathrm{CR}(0,\eta) \le \epsilon] \approx p_{BB}(\epsilon, 1) \approx \epsilon^{\beta_2}$.



$$\beta_2 = \inf\{x : \mathbb{E}[\mathrm{CR}(0,\eta)^{-x}] = \infty\}.$$

Nolin-Qian-S.-Zhuang '23

$$\mathbb{E}[\mathrm{CR}(0,\eta)^{-x}] = \frac{3\sqrt{3}}{4}\sin(\frac{\pi}{2}\sqrt{12x+1})\Big(\frac{\sqrt{36x+3}}{4} + \sin(\frac{2\pi\sqrt{12x+1}}{3})\Big)^{-1}.$$

Derivation of $\mathbb{E}[\mathrm{CR}(0,\eta)^{-x}]$ via Liouville quantum gravity on the disk, applying method from Ang-Holden-S. (2021).

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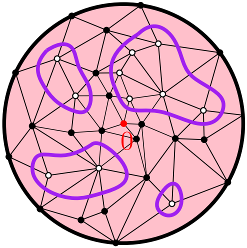
S.-Zhuang (2024+)

Let $\tau = \frac{1}{2\pi}\log(\frac{R}{r})$ and $q = e^{-2\pi\tau}$.

$$\mathbb{P}[\mathcal{B}_2(r,R)] = \frac{q^{-\frac{1}{12}}}{\prod_{n=1}^{\infty}(1-q^{2n})}\sum_{s\in\mathcal{S}}\frac{-\sqrt{3}\sin(\frac{2\pi}{3}\sqrt{3s})\sin(\pi\sqrt{3s})}{\cos(\frac{4\pi}{3}\sqrt{3s})+\frac{3\sqrt{3}}{8\pi}}q^{s}.$$

Derivation of $\mathbb{P}[\mathcal{B}_2(r,R)]$ via Liouville quantum gravity on the annulus, applying method from Ang-Remy-S. (2022).

2D Quantum Gravity coupled with percolation



- Sample random triangulation of the disk decorated with a percolation.
- Conformally embed the random surface on the unit disk $\mathbb{D}$.

Continuum limit: LQG on $\mathbb{D}$ with $\gamma = \sqrt{8/3}$ decorated with CLE_6.

A random geometry on $\mathbb{D}$ with area measure $e^{\gamma\phi}d^2x$ on $\mathbb{D}$;
length measure $e^{\frac{\gamma}{2}\phi}dx$ on $\partial\mathbb{D}$. (Gaussian multiplicative chaos)
ϕ: a random generalized function, locally looks like a Gaussian free field.

(Convergence proved by Holden-S. under a certain conformal embedding.)

Liouville field description of the law of the canonical ϕ

The law of the field ϕ is "$e^{-S_L[\phi]}D\phi$"; $S_L[\phi]$: Liouville action on $\mathbb{D}$.

$$S_L[\phi] = \int_{\mathbb{D}} (\tfrac{1}{4\pi}|\nabla\phi|^2 + \mu e^{\gamma\phi}) d^2x + \int_{\partial\mathbb{D}} (\tfrac{Q\phi}{2\pi} + \nu e^{\frac{\gamma}{2}\phi}) dl.$$

$Q = \frac{2}{\gamma} + \frac{\gamma}{2}$, μ and ν: cosmological constants.

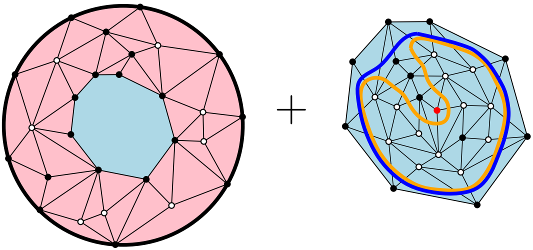
"$e^{-\text{Liouville action}}D\phi$" defines a quantum field theory called Liouville theory, made rigorous by David-Kupiainen-Rhodes-Vargas and follow-up.

Liouville theory is a conformal field theory, with rich integrability:

- DOZZ formula: Kupiainen-Rhodes-Vargas.
- Conformal bootstrap on closed surface: Guillarmou-KRV.
- We need the disk analog of DOZZ (with $\mu = 0$) solved by Remy-Zhu.
- We need the annulus conformal bootstrap by B. Wu.

Cut and glue random surfaces: discrete

η: the outermost outer boundary surrounding a marked interior point.
η cuts the triangulation decorated by percolation into:



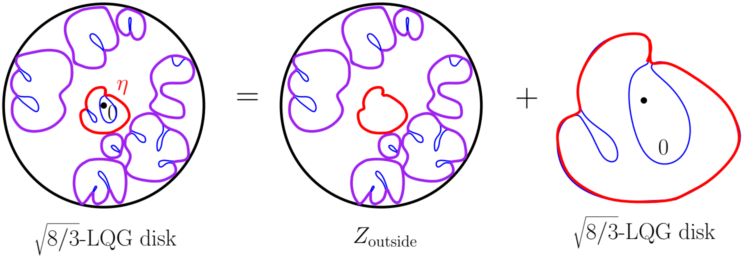
Sample random triangulation according to Boltzmann weight $a^n b^L$;
n: number of triangles L: number of boundary edges.

$$\sum_\ell Z_{\mathrm{outside}}(L,\ell)\times\ell\times Z_{\mathrm{inside}}(\ell)=Z_{\mathrm{whole}}(L).$$

ℓ: the length of η. $Z_{\mathrm{outside}}, Z_{\mathrm{inside}}, Z_{\mathrm{whole}}$: surface partition functions.

Cut and glue random surfaces: continuum

$Z_{\mathrm{whole}}(L) \propto L^{-3/2}$ in the continuum limit for critical weights a and b.
Partition function for the most canonical random disk in $\sqrt{8/3}$-LQG with one interior marked point and boundary length L.



Using SLE/LQG coupling (Sheffield's quantum zipper):
The inside of η is another copy of $\sqrt{8/3}$-LQG disk; $Z_{\mathrm{inside}}(\ell) \propto \ell^{-3/2}$.

$\int_0^\infty Z_{\mathrm{outside}}(L,\ell) \times \ell \times \ell^{-3/2} d\ell = L^{-3/2}$. (proved purely from continuum)

Conformal Radius from Deforming the Bulk Insertion

$$\int_0^\infty Z_{\mathrm{outside}}(L,\ell) \times \ell \times \ell^{-\frac{3}{2}+a} d\ell = L^{-\frac{3}{2}+a} \cdot \mathbb{E}[\mathrm{CR}(0,\eta)^{-\frac{1}{3}a(a-1)}], \quad \text{for all } a.$$

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From $L^{-\frac{3}{2}}$ to $L^{-\frac{3}{2}+a}$ using the field description of the LQG disk.

$$\int_{\phi:\mathbb{D}\to\mathbb{R}} F\left(\int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}\phi} dx\right) e^{\gamma\phi(0)} e^{-S_L[\phi]} D\phi \propto \int_0^\infty F(L) e^{-\nu L} L^{-\frac{3}{2}} dL.$$

ν: boundary cosmological constant in the Liouville action $S_L[\phi]$
$e^{\gamma\phi(0)}$ is inserted since 0 is the marked point on the $\sqrt{8/3}$-LQG disk.

$$\int_{\phi:\mathbb{D}\to\mathbb{R}} F\left(\int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}\phi} dx\right) e^{\alpha\phi(0)} e^{-S_L[\phi]} D\phi \propto \int_0^\infty F(L) e^{-\nu L} L^{-\frac{3}{2}+a(\alpha)} dL.$$

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Same reason for $\ell^{-\frac{3}{2}}$ to $\ell^{-\frac{3}{2}+a}$, except the smaller disk is bounded by η instead of $\partial\mathbb{D}$. This difference is compensated by a power of conformal derivative at 0, which is $\mathrm{CR}(0,\eta)^{-\frac{1}{3}a(a-1)}$.

Solving the conformal radius and the backbone exponent

Equations like $(\star)$ relating boundary length partition function and conformal radius were systematically derived in Ang-Holden-S. (2021).

$$\int_0^\infty Z_{\mathrm{outside}}(L,\ell)\times \ell \times \ell^{-\frac{3}{2}+a}\,d\ell = L^{-\frac{3}{2}+a}\mathbb{E}[\mathrm{CR}(0,\eta)^{-\frac{1}{3}a(a-1)}]. \qquad (\star)$$

No formula for $Z_{\mathrm{outside}}(L,\ell)$ is available for Nolin-Qian-S.-Zhuang (2023) to solve $\mathbb{E}[\mathrm{CR}(0,\eta)^{-\frac{1}{3}a(a-1)}]$ using $(\star)$.

NQSZ found an effective variant of $\mathbb{E}[\mathrm{CR}(0,\eta)^{-\frac{1}{3}a(a-1)}]$ such that:

- still encodes the backbone exponent;
- has an equation like $(\star)$ with a solvable $Z_{\mathrm{outside}}(L,\ell)$.

The NQSZ variant is defined in terms the SLE_6 bubble measure.
The counterpart for $Z_{\mathrm{outside}}(L,\ell)$ is solved
using the boundary analog of DOZZ due to Remy-Zhu.

The ratio between $\mathbb{E}[\mathrm{CR}(0,\eta)^{-\frac{1}{3}a(a-1)}]$ and its NQSZ variant is another conformal radius moment solved using the AHS method, [D. Wu '23].

$$\mathbb{E}[\mathrm{CR}(0,\eta)^{-x}] = \frac{3\sqrt{3}}{4}\sin(\tfrac{\pi}{2}\sqrt{12x+1})\left(\frac{\sqrt{36x+3}}{4} + \sin(\tfrac{2\pi\sqrt{12x+1}}{3})\right)^{-1}.$$

Plugging into $(\star)$ we get

$$\int_0^\infty Z_{\mathrm{outside}}(L,\ell)\ell^{ix}d\ell = \frac{3\sqrt{3}}{4}\frac{\sinh(\pi x)}{\sinh(\frac{4\pi x}{3})+\frac{\sqrt{3}}{2}x}L^{ix-1} \qquad \text{for all } x.$$

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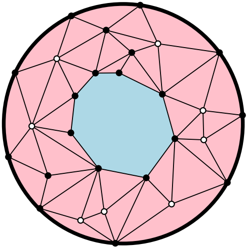
$Z_{\mathrm{outside}}(L,\ell) = Z^{\mathrm{nt}}(L,\ell) + Z^{\mathrm{t}}(L,\ell)$. Z_{outside}: surface between η and $\partial\mathbb{D}$

$Z^{\mathrm{nt}}(L,\ell)$: η does not touch $\partial\mathbb{D}$. $Z^{\mathrm{t}}(L,\ell)$: η touches $\partial\mathbb{D}$.

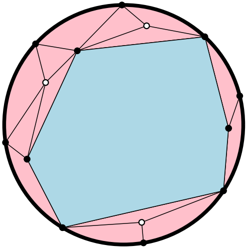
$$\int_0^\infty Z^{\mathrm{t}}(L,\ell)\ell^{ix}d\ell = \frac{3\sqrt{3}}{4}\frac{\sinh(\frac{\pi x}{3})}{\sinh(\frac{2\pi x}{3})}L^{ix-1}; \qquad \text{still by Ang-Holden-S. method,}$$

We now use $Z^{\mathrm{nt}}(L,\ell)$ to solve $\mathbb{P}[\mathcal{B}_2(r,R)]$.

Quantum gravity on the annulus



or



$Z^{\mathrm{nt}}(L, \ell)$: partition function of the random triangulation of the annulus coupled with a percolation with monochromatic two-arm crossing.

Random triangulation of the annulus without any decoration converges to the most canonical random annulus in $\sqrt{8/3}$-LQG (Brownian annulus).

with partition function: $Z^{\mathrm{B}}(L, \ell) = \frac{1}{\sqrt{L\ell}(L+\ell)}$.

When the Brownian annulus is conformally embedded to the flat annulus $\mathcal{C}_\tau$ of modulus τ, we still get Liouville field on $\mathcal{C}_\tau$, but the modulus τ itself is random.

[Ang-Remy-S. '22] Exact Law of τ: Liouville + ghost

$$\iint_0^\infty e^{-\nu_1 L-\nu_2 \ell} Z^{\mathrm{B}}(L,\ell)\,d\ell\,dL = \int_0^\infty Z^{\nu_1,\nu_2}_{\mathrm{Liouville}}(\tau)\cdot Z_{\mathrm{ghost}}(\tau)\,d\tau$$

$Z^{\nu_1,\nu_2}_{\mathrm{Liouville}}(\tau) = \int_{\phi:C_\tau\to\mathbb{R}} e^{-S_L(\phi)} D\phi$; solved by B. Wu (2022).

$S_L(\phi) = \int_{\mathbb{D}} (\frac{1}{4\pi}|\nabla\phi|^2 + \mu e^{\gamma\phi}) d^2x + \int_{\partial_{\mathrm{in}} C_\tau} \nu_1 e^{\frac{\gamma}{2}\phi} dl + \int_{\partial_{\mathrm{out}} C_\tau} \nu_2 e^{\frac{\gamma}{2}\phi} dl$

The formula for $Z^{\mathrm{B}}(L,\ell)$ gives $Z_{\mathrm{ghost}}(\tau) = \eta(2i\tau)^2$,
as predicted in the bosonic string theory, ghost CFT has $c=-26$;
math conjecture formulated by Guillarmou-Rhodes-Vargas; Remy.

From $Z^{\mathrm{B}}(L,\ell)$ to $Z^{\mathrm{nt}}(L,\ell)$

$$\iint_0^\infty e^{-\nu_1 L-\nu_2\ell} Z^{\mathrm{nt}}(L,\ell)\,d\ell\, dL = \int_0^\infty Z_L(\tau)\cdot \mathbb{P}[\mathcal{B}_2(e^{-2\pi\tau},1)]\cdot Z_{\mathrm{ghost}}(\tau)\, d\tau.$$

$$\int_0^\infty \mathbb{P}[\mathcal{B}_2(e^{-2\pi\tau},1)]\eta(2i\tau)e^{-\frac{2\pi x^2\tau}{3}}\,d\tau = \frac{\sqrt{3}}{x}\left(\frac{\sinh(\frac{2}{3}\pi x)\sinh(\pi x)}{\sinh(\frac{4}{3}\pi x)+\frac{\sqrt{3}}{2}x} - \sinh(\tfrac{1}{3}\pi x)\right).$$

S.-Zhuang (2024+)

$$\mathbb{P}[\mathcal{B}_2(e^{-2\pi\tau},1)] = \frac{q^{-\frac{1}{12}}}{\prod_{n=1}^\infty(1-q^{2n})}\sum_{s\in\mathcal{S}} \frac{-\sqrt{3}\sin(\frac{2\pi}{3}\sqrt{3s})\sin(\pi\sqrt{3s})}{\cos(\frac{4\pi}{3}\sqrt{3s})+\frac{3\sqrt{3}}{8\pi}} q^s,$$

$\mathbb{P}[\mathcal{A}_1(r,R)]$ and $\mathbb{P}[\mathcal{A}_2(r,R)]$ are derived using the same strategy.

Outlook

- Determine β_j for $j \geq 3$? (NQSZ variant not found yet.)

- CFT interpretations of the β_2 and $\mathbb{P}[\mathcal{B}_2(r, R)]$ formulas?

  One-arm and dichomatic two-arm exponents naturally appear in a percolation related CFT proposed recently in physics.

  Physics: Jacobsen, Rilbault, Saluer, et.al.: bootstrap formulas for the probability that four points are on the same cluster or the same loop

  Math: Ang, Cai, S., Wu.: rigorously derive the three-point probability formulas involving the Imaginary DOZZ formula.