Solving four-dimensional superconformal Yang-Mills theories using random matrices

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Outline

Ultimate goal is to solve 4d superconformal $\mathcal{N}=2$ and $\mathcal{N}=4$ planar Yang–Mills theories for arbitrary 't Hooft coupling $\lambda = g^2_{\rm YM} N_c$

Existing techniques (localization, integrability, holography) allows us to realize this program for:

- ✔ V.e.v. of half-BPS circular Wilson loop in $\mathcal{N}=4$ SYM
- ✔ Correlation function of infinitely heavy half-BPS operators (= octagon)
- ✔ Flux tube correlators (cusp anom. dim., scattering amplitudes)
- ✔ Free energy and correlation functions in $\mathcal{N}=2$ SYM

A remarkable feature of these observables is that they can be expressed as determinants of certain semi-infinite matrices

$$e^{\mathcal{F}(g)} = \det_{1\le n,m<\infty}\Big(\delta_{nm} - K_{nm}(g)\Big)\,, \qquad\qquad g = \frac{\sqrt{\lambda}}{4\pi}$$

We shall compute $\mathcal{F}(g)$ for arbitrary g

Weak and strong coupling expansion

Simple example: circular Wilson loop in planar $\mathcal{N} = 4$ SYM

$$W = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

Weak coupling expansion

$$W \overset{\lambda \ll 1}{=} 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \frac{\lambda^3}{9216} + \frac{\lambda^4}{737280} + \dots$$

Strong coupling expansion

$$W \overset{\lambda \gg 1}{=} e^{\sqrt{\lambda} - \frac{3}{2}\log\sqrt{\lambda} - \frac{1}{2}\log\left(\frac{\pi}{2}\right) - \frac{3}{8\sqrt{\lambda}} - \frac{3}{16\lambda} + \dots} + O(e^{-\sqrt{\lambda}})$$

Semiclassical asymptotics of observables in AdS/CFT

$$\mathcal{F} = -\sqrt{\lambda} A_0 - A_1 \log(\sqrt{\lambda}) - B - \sum_{n \geq 1} \frac{A_{n+1}}{\lambda^{n/2}} + O(e^{-c\sqrt{\lambda}})$$

- ✔ Expansion coefficients grow factorially $A_n \sim n!$
- ✔ Needs to account for nonperturbative (exponentially small) corrections

Tracy-Widom distribution

Describes statistics of the spacing of the eigenvalues of $N \times N$ hermitian matrices for $N \to \infty$

Gaussian Unitary Ensemble

$$Z_{\rm GUE} = \int d^{N\times N} a \, e^{-\frac{1}{2}\,{\rm tr}\, a^2} = \int_{-\infty}^{\infty} d\lambda_1 \ldots d\lambda_N \, \prod_{i\neq j} (\lambda_i - \lambda_j)^2 e^{-\frac{1}{2}\sum_i \lambda_i^2}$$

Laguerre ensemble (Wishart matrix theory)

$$Z_{\rm Laguerre} = \int_0^{\infty} d\lambda_1 \ldots d\lambda_N \, \prod_{i\neq j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^{N} \lambda_i^{\ell} \, e^{-\lambda_i}$$

where $\ell > -1$ and eigenvalues are located on semi-axis $[0, \infty)$.

The probability density for eigenvalues

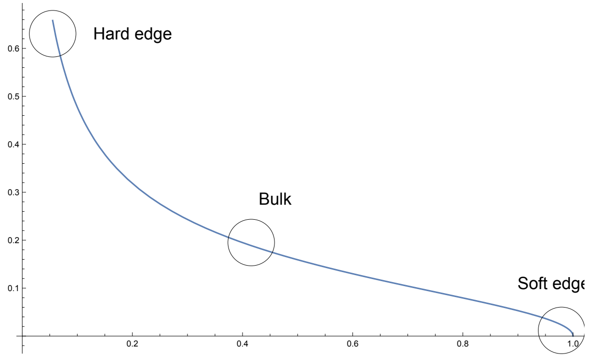
$$R_n(x_1, \ldots, x_n) = \left\langle \prod_{i=1}^{n} \delta(\lambda_i - x_i) \right\rangle = \det K_N(x_i, x_j)\Big|_{i,j=1,\ldots,n}$$

$$K_N(x,y) = \sum_{k=0}^{N-1} \phi_k(x)\phi_k(y)$$

where $\phi_k(x)$ are orthonormal functions $x^k e^{-x^2/2} + \ldots$ (GUE) and $x^k x^{\ell/2} e^{-x/2} + \ldots$ (Laguerre)

Tracy-Widom distribution II

The distribution of the eigenvalues in the Laguerre ensemble in the limit $N \to \infty$



$$R_1(4Nx) \sim \frac{1}{2\pi}\sqrt{\frac{1-x}{x}}$$

Scaling behaviour of $K_N(x,y)$ around $x=0$ (hard edge), $x=1$ (soft edge) and $0<x<1$ (bulk)

bulk : $$\frac{\sin\pi(x-y)}{\pi(x-y)}$$

soft edge : $$\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y)-\mathrm{Ai}(x)\mathrm{Ai}'(y)}{x-y}$$

hard edge : $$\frac{J_\ell(\sqrt{x})\sqrt{y}J'_\ell(\sqrt{y})-\sqrt{x}J'_\ell(\sqrt{x})J_\ell(\sqrt{y})}{2(x-y)}$$

The probability that there are no eigenvalues on the interval $[0,s]$

$$E(0;s)=\det(1-K)_{[0,s]}=1+\sum_{n\geq 1}\frac{(-1)^n}{n!}\int_0^s dx_1\ldots dx_n \det\|K(x_i,x_j)\|_{1\leq i,j\leq n}$$

Fredholm determinant of the integral operator: *Sinc* (bulk), *Airy* (soft edge) and *Bessel* (hard edge)

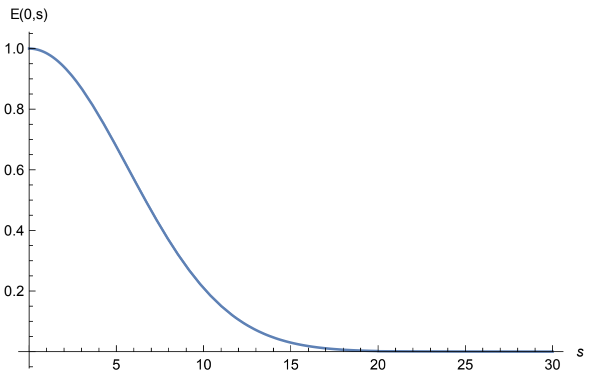
Bessel kernel

Tracy-Widom distribution close to the hard edge

$$E(0,s) = \det(1 - K_{\text{Bessel}})_{[0,s]} = \exp\left(-\frac{1}{4}\int_0^s dx \log(s/x)\, Q^2(x)\right)$$

$Q(s)$ satisfies Painlevé V differential equation

Dependence of the probability $E(0,s)$ on the interval length s



Asymptotics of $E(0,s)$ at small and large s

$$E(0,s) \stackrel{s\lll 1}{=} 1 - \frac{(s/4)^{\ell+1}}{\Gamma^2(\ell+2)} + \dots$$

$$E(0,s) \stackrel{s\ggg 1}{=} \exp\left(-\frac{s}{4} - \frac{\ell^2}{4}\log s + \frac{\ell}{8}s^{-1/2} + \dots\right)$$

Remarkably similar to weak/strong coupling expansion in gauge theory for $s \sim \sqrt{\lambda}$

Bessel kernel at finite temperature

$$K_\ell(x,y) = \sum_{n\geq 1} \phi_n(x)\phi_n(y)\chi\left(\frac{y}{2g}\right)$$

$$\phi_n(x) = (-1)^n\sqrt{2n+\ell-1}\,\frac{J_{2n+\ell-1}(\sqrt{x})}{\sqrt{x}}$$

- For $\chi(x) = \theta(1-x)$ defines the Tracy-Widom distribution $E(0,s)$ for $s=(2g)^2$
- Finite-temperature generalization: $\chi(x) = 1/(1+e^{\frac{x-\mu}{T}})$

Generalized Tracy-Widom distribution

$$e^{\mathcal{F}_\chi} = \det(1-K_\ell) = \det\left(\delta_{nm} - K_{nm}(g)\right)\Big|_{n,m\geq 1}$$

Determinant of a semi-infinite matrix

$$\int_0^\infty dy\, K_\ell(x,y)\,\phi_n(y) = K_{nm}\phi_m(x)$$

$$K_{nm} = \int_0^\infty dx\,\phi_n(x)\phi_m(x)\chi\left(\frac{x}{2g}\right)$$

$\chi(x)$ is the *symbol* of the Bessel operator

Free energy in $\mathcal{N}=2$ super Yang-Mills theory

- $\mathcal{N}=2$ supersymmetric Yang-Mills theory with gauge group $SU(N)$ coupled to matter multiplets

  in rank-2 symmetric $(N_S=1)$ and anti-symmetric $(N_A=1)$ representations

  The beta function vanishes $\beta_0 = 2N - N_S(N+2) - N_A(N-2) = 0$,

- The partition function on sphere S^4 is given by a matrix integral [Pestun]

$$Z_{S^4} = e^{-F} = \int da\, e^{-\frac{8\pi^2 N}{\lambda}\,\mathrm{tr}\, a^2} |Z_{1-\mathrm{loop}}(a) Z_{\mathrm{inst}}(a)|^2$$

  Non-perturbative instanton contribution $Z_{\mathrm{inst}}(a)$ is exponentially small at large N

- Perturbative corrections $Z_{1-\mathrm{loop}}(a) = \exp(-S_{\mathrm{int}}(a))$ only come from one loop

$$S_{\mathrm{int}}(a) = \sum_{i,j} \Big[\log H(\lambda_i + \lambda_j) - \log H(\lambda_i - \lambda_j) \Big] \qquad (\lambda_i \text{ are eigenvalues of } a)$$

$$= 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} \zeta_{2n+1} \sum_{p=0}^{n} \binom{2n+2}{2p+1} \,\mathrm{tr}\, a^{2p+1} \;\mathrm{tr}\, a^{2(n-p)+1}$$

$$H(x) = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right)^n e^{-\frac{x^2}{n}} = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} \zeta_{2n+1}\, x^{2n+2} \right)$$

  Matrix model with double-trace interaction

Large N expansion

$$e^{-F} = \left(\frac{8\pi^2}{\lambda}\right)^{-(N^2-1)/2} \int da\, \exp\left(-N\,\mathrm{tr}\,a^2 + \frac{1}{2}\sum_{k,n} C_{kn} O_{2k+1} O_{2n+1}\right)$$

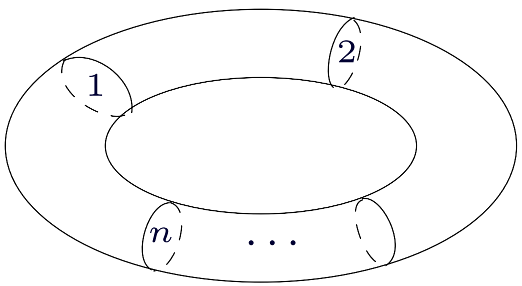
The interaction term is a sum over double traces $O_k = \mathrm{tr}\,a^k$ with the couplings

$$C_{kn} = 4\frac{(-1)^{k+n+1}}{k+n+1}\zeta_{2(k+n)+1}\binom{2(k+n+1)}{2k+1}\left(\frac{\lambda}{8\pi^2}\right)^{k+n+1}$$

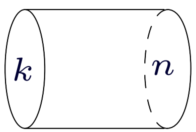
Large N expansion

$$F = N^2 F_0(\lambda) + F_1(\lambda) + F_2(\lambda)/N^2 + \ldots$$

The interaction term does not contribute to F_0 – coincides with the free energy in $\mathcal{N}=4$ SYM

$F_1 = \sum_{n\geq 1}$ 

$= -\sum_{n\geq 1}\frac{1}{2n}\,\mathrm{tr}[(QC)^n] = \frac{1}{2}\log\det(1-QC)$

Cylinders $Q_{kn} =$ 

$= \langle \mathrm{tr}\,a^{2k+1}\,\mathrm{tr}\,a^{2n+1}\rangle_{\mathrm{GUE}}$ are glued together with the weight C_{kn}

Relation to Bessel kernel

Explicit expressions for semi-infinite matrices

$$Q_{kn} = \frac{2\beta_k\beta_n}{k+n+1} + O(1/N^2)\,, \qquad \beta_n = \frac{2^n n\Gamma(n+\frac{3}{2})}{\sqrt{\pi}\Gamma(n+2)}$$

$$C_{kn} = 4\frac{(-1)^{k+n+1}}{k+n+1}\zeta_{2(k+n)+1}\binom{2(k+n+1)}{2k+1}\left(\frac{\lambda}{8\pi^2}\right)^{k+n+1}$$

The matrix (QC) is related to the Bessel kernel by a similarity transformation

[Beccaria,Billò,Galvagno,Hasan,Lerda]

$$\begin{aligned} K_{nm} &= (U^{-1}QCU)_{nm} \\ &= 2\,(-1)^{n+m}\sqrt{2n+1}\sqrt{2m+1}\int_0^\infty \frac{dt}{t}\,J_{2n+1}(t)\,J_{2m+1}(t)\chi\left(\frac{x}{2g}\right) \end{aligned}$$

Special form of the symbol

$$\chi(x) = -\frac{1}{\sinh^2(x/2)}\,, \qquad g = \frac{\sqrt{\lambda}}{4\pi}$$

The free energy coincides with the Tracy-Widom distribution at the hard edge for $\ell = 2$

$$F_1 = \frac{1}{2}\log\det(1-QC) = \frac{1}{2}\,\mathrm{tr}\log(1-\mathbf{K}_\chi)$$

Tracy-Widom distribution in super Yang-Mills theories

Different observables in SYM theories are given by the Tracy-Widom distribution $e^{\mathcal{F}_\chi}$

The symbol function χ depends on the observable:

- ✔ Circular Wilson loop

$$\chi_{\mathsf{W}}(x) = -\frac{(2\pi)^2}{x^2}$$

- ✔ Free energy of $\mathcal{N}=2$ SYM

$$\chi_{\mathsf{free}}(x) = -\frac{1}{\sinh^2(x/2)}$$

- ✔ Four-point correlator

$$\chi_{\mathsf{oct}}(x|y,\xi) = \frac{\cosh y + \cosh \xi}{\cosh y + \cosh(\sqrt{x^2+\xi^2})}$$

- ✔ Flux tube

$$\chi_{\mathsf{flux}}(x) = -\frac{2}{e^x - 1}$$

The coupling constant $g = \sqrt{\lambda}/(4\pi)$ controls the width of the distribution $s \sim g^2$

How to derive the strong coupling expansion of the TW distribution?

Szegő-Akhiezer-Kac formula

- Asymptotic behaviour for sufficiently smooth symbol $\chi(z)$

$$\mathcal{F}_\chi = -gA_0 + B + O(1/g)$$

SAK formula (1915-1966)

$$A_0 = -2\widetilde{\psi}(0)\,, \qquad B = \frac{1}{2}\int_0^\infty dk\, k\big(\widetilde{\psi}(k)\big)^2\,,$$

$$\widetilde{\psi}(k) = \int_0^\infty \frac{dz}{\pi}\cos(kz)\log(1-\chi(z))$$

B diverges for $\chi(z) \sim 1 - z^{2\beta}$ or $\widetilde{\psi}(k) \sim -\beta/k$ at large k *Fisher-Hartwig singularity*

The SAK formula for the Bessel kernel with Fisher-Hartwig singularity has not been derived yet

- Our conjecture [Belitsky,GK]

$$\mathcal{F}_\chi = -gA_0 + A_1 \log g + B' + O(1/g)$$

$$A_1 = \frac{1}{2}\beta^2\,,$$

$$B' = \frac{1}{2}\int_0^\infty dk\left[k\big(\widetilde{\psi}(k)\big)^2 - \beta^2\frac{1-\mathrm{e}^{-k}}{k}\right] + \frac{\beta}{2}\log(2\pi) - \log G(1+\beta)\,,$$

Power suppressed $O(1/g)$ corrections are determined using the *method of differential equations*

Method of differential equations

A powerful method for computing correlators in integrable models [Its,Izergin,Korepin,Slavnov]

'Potential' = logarithmic derivative of the determinant

$$U(g) = -2g\partial_g \mathcal{F}_\chi(g)$$

Satisfies the system of *exact* integro-differential equations [Belitsky,GK]

$$g\partial_g U = -2\int_0^\infty dx\, Q^2(x)\, x\partial_x \chi\left(\frac{\sqrt{x}}{2g}\right),$$

$$(g\partial_g + 2x\partial_x)^2\, Q(x) + (x - g\partial_g U + U)\, Q(x) = 0$$

✔ For $\chi(x) = \theta(1-x)$ reduces to Painlevé V equation for $q(g) = Q(x=(2g)^2)$

$$(g\partial_g)^2\, q(g) + \left(4g^2 - g\partial_g U + U\right) q(g) = 0\,, \qquad g\partial_g U = [q(g)]^2$$

✔ For generic $\chi(x)$ exact solution is not known, WKB solution at large g

$$Q((2gz)^2) = \frac{a(z,g)\sin(2gz) + a(-z,g)\cos(2gz)}{\sqrt{2\pi gz(1-\chi(z))}}\,, \qquad a(z,g) = 1 + \sum_{k\geq 1}\frac{a_k(z)}{g^k}$$

Tracy-Widom distribution at strong coupling

- Strong coupling expansion:

$$\mathcal{F}_\chi(g) = \underbrace{-gA_0 + A_1 \log g + B}_{\text{SAK formula}} + f(g) + \Delta f(g)$$

- The 'perturbative' function $f(g)$ is given by an asymptotic series

$$f(g) = \sum_{k=1}^{\infty} \frac{A_{k+1}}{2k(k+1)} g^{-k}$$

- The expansion coefficients $A_k = A_k(\chi)$ depend on the symbol function (=choice of observable)

  Curious relation between two different observables

$$A_{k+1}(\chi_{\text{free}}) = (-1)^k A_{k+1}(\chi_{\text{oct}}) \qquad \mapsto \qquad f_{\text{free}}(g) = f_{\text{oct}}(-g)$$

- The expansion coefficients grow factorially $A_k \sim k!\, c^{-k}$

  The perturbative series $f(g)$ is plagued with Borel singularities

- Has to be supplemented with the nonperturbative, exponentially small corrections

$$\Delta f(g) \sim e^{-cg}$$

Nonperturbative corrections

We developed a systematic method to compute transseries for $\Delta f(g)$

General form of the symbol function in SYM

$$1-\chi(x) = b x^{2\beta} \prod_{n\ge 1} \frac{1+\frac{x^2}{(2\pi x_n)^2}}{1+\frac{x^2}{(2\pi y_n)^2}}$$

Has an infinite set of poles and zeros located at $x=-2i\pi x_n$ and $x=-2i\pi y_n$, e.g.

$$1-\chi_{\text{oct}}(x|0,0) = \frac{\sinh^2(x/2)}{\cosh^2(x/2)} = \frac{x^2}{4}\prod_{n\ge 1}\left[\frac{1+\frac{x^2}{(2\pi n)^2}}{1+\frac{x^2}{(2\pi(n-\frac{1}{2}))^2}}\right]^2$$

Nonperturbative corrections

$$\Delta f(g) = \sum_{n\ge 1}\left(g^{a-1}e^{-8\pi g x_1}\right)^n\left[A_1^{(n)} + \sum_{k=1}^{\infty}\frac{A_{k+1}^{(n)}}{2k(k+1)}g^{-k}\right]$$

The parameter x_1 is a solution to $\chi(2i\pi x_1)=1$ closest to the origin, with degeneracy $a=1,2$

$$\Delta f_{\text{oct}}(g) = \frac{i\pi g'}{4}e^{-8\pi g}\left[1-\frac{\frac{7}{4}}{(4\pi g')}-\frac{\frac{63}{32}}{(4\pi g')^2}\right]+\frac{(\pi g')^2}{32}e^{-16\pi g}\left[1+\frac{\frac{81i}{4}-\frac{7}{2}}{4\pi g'}+\frac{-\frac{1431i}{32}-\frac{3}{4}}{(4\pi g')^2}\right]+\dots$$

The expansion parameter $g' = g+\log(2)/\pi$

Resurgence for the octagon

High precision calculation of the first 400 terms of the perturbative series at strong coupling

$$f_{\text{oct}}(g) = \sum_{n\geq 1} \frac{\alpha_n}{(4\pi g')^n}\,, \qquad \alpha_n \sim \Gamma(n+1)$$

The diagonal Pade approximant for the Borel transform

$$\mathcal{B}_{\text{oct}}(s) = \sum_{n=0}^{\infty} \alpha_n \frac{s^n}{\Gamma(n+1)}$$

has poles which condense on the real axis for $s < -1$ and $s > 2$

Large order behaviour of the coefficients

$$\alpha_n = (-1)^n \sum_{k\geq 0} \left(a_k^{(1)} \Gamma_{n+1-k} + a_k^{(2)} \frac{\Gamma_{n+2-k}}{2^{n+2-k}} + \ldots \right) + \sum_{k\geq 0} \left(b_k^{(1)} \frac{\Gamma_{n+1-k}}{2^{n+1-k}} + b_k^{(2)} \frac{\Gamma_{n+2-k}}{4^{n+2-k}} + \ldots \right)$$

The two sums produce logarithmic cuts of $\mathcal{B}_{\text{oct}}(s)$ at $s = -1, -2, \ldots$ and $s = 2, 4, \ldots,$ respectively.

The cut at $s = 4$ is also generated by

$$b_k^{(1)} = -\frac{2}{\pi} \left(c_0^{(2)} \frac{\Gamma_{k+1}}{2^{k+1}} + c_1^{(2)} \frac{\Gamma_k}{2^k} + c_2^{(2)} \frac{\Gamma_{k-1}}{2^{k-1}} + \ldots \right)$$

Resurgence for the octagon II

The coefficients of the strong coupling expansion

$b_0^{(1)} = -\frac{16}{\pi}$	$b_1^{(1)}/b_0^{(1)} = -\frac{7}{4}$	$b_2^{(1)}/b_0^{(1)} = -\frac{63}{32}$
$b_0^{(2)} = -\frac{256i}{\pi}$	$b_1^{(2)}/b_0^{(2)} = -\frac{7}{2}$	$b_2^{(2)}/b_0^{(2)} = -\frac{3}{4}$
$c_0^{(2)} = 0$	$c_1^{(2)}/b_0^{(2)} = \frac{81i}{4}$	$c_2^{(2)}/b_0^{(2)} = -\frac{1431i}{32}$

The ratio of perturbative coefficients $b_k^{(1)}/b_0^{(1)}$ and $(b_k^{(2)} + c_k^{(2)})/b_0^{(2)}$ coincide with the coefficients of transseries

$$\Delta f_{\text{oct}}(g) = \frac{i\pi g'}{4} e^{-8\pi g}\left[1 - \frac{\frac{7}{4}}{(4\pi g')} - \frac{\frac{63}{32}}{(4\pi g')^2}\right] + \frac{(\pi g')^2}{32} e^{-16\pi g}\left[1 + \frac{\frac{81i}{4} - \frac{7}{2}}{4\pi g'} + \frac{-\frac{1431i}{32} - \frac{3}{4}}{(4\pi g')^2}\right] + \dots$$

The ambiguities generated by Borel singularities at $s > 0$ cancel in the sum $f_{\text{oct}}(g) + \Delta f_{\text{oct}}(g)$

What is a physical meaning of singularities of $\mathcal{B}_{\text{oct}}(s)$ at negative s ?

$$f_{\text{free}}(g) = f_{\text{oct}}(-g) \qquad \mapsto \qquad \mathcal{B}_{\text{loc}}(s) = \mathcal{B}_{\text{oct}}(-s)$$

Discontinuity of $\mathcal{B}_{\text{oct}}(s)$ across the cuts at negative s yields nonperturbative corrections to $\Delta f_{\text{loc}}(g)$

Conclusions and open questions

Various quantities (free energy, correlation functions, Wilson loop, tilted cusp) in *different* 4d super Yang-Mills theories are expressed in terms of the *same* (temperature dependent) Tracy-Widom distribution

This relation is powerful enough to predict the dependence on ’t Hooft coupling

- ✔ Who ordered this universality?
- ✔ What is the reason why the Bessel kernel appears in all cases?
- ✔ How to reproduce the strong coupling expansion from holography?