Logarithmic operators in c=0 bulk CFTs

Yifei He

LPENS, CNRS

Rencontre Claude Itzykson @ IPhT, 12/09/2024

based on 2409.XXXXX

Outline

percolation & SAW ← motivation c=0 CFTs → *review* → recent results on cluster/loop model CFTs

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"$c \to 0$ catastrophe" & resolutions

Jordan blocks, normalizations, …

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→ **Kac operators** →

normalization at generic c

real CFT, c<1 Liouville CFT

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energy density operator at c=0: 2-, 3- and 4-point functions

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Conclusions

Conformal field theories at c=0

despite the amount of knowledge we have of 2d CFTs, a class of 2d CFTs describing physical systems remains almost intractable

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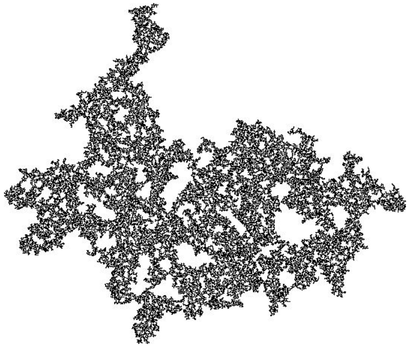
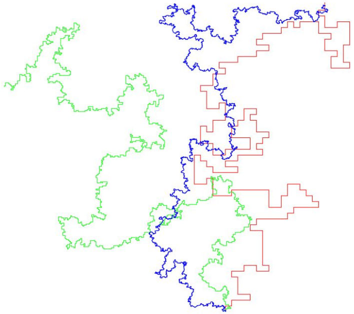
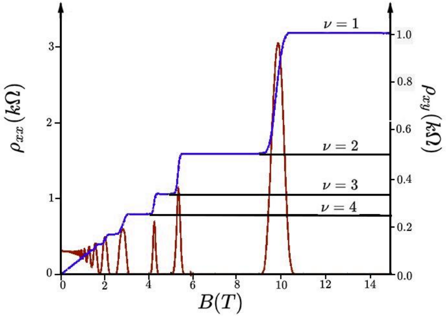
examples:



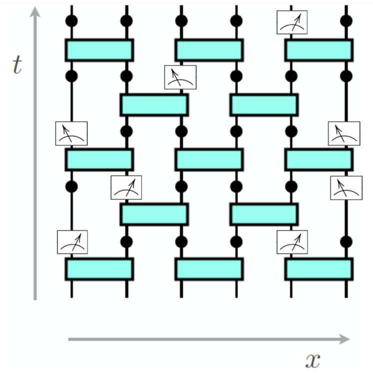
percolation



SAW (self-avoiding walk)



IQHE plateau transition



MIPT

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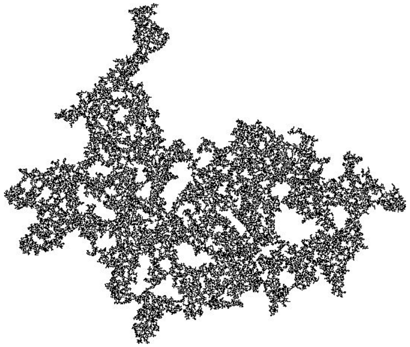

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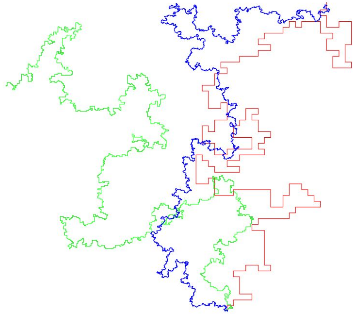

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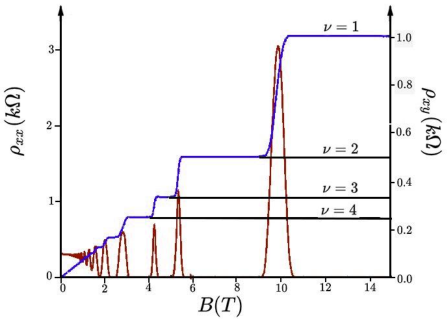

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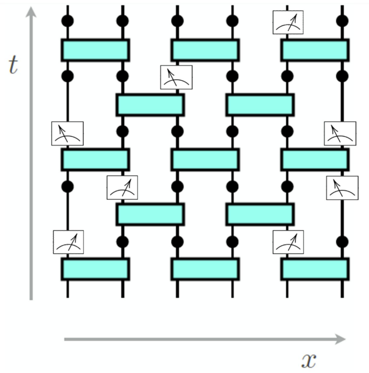

MIPT

c=0 CFTs are notoriously hard to study, non-unitary, logarithmic

[Gurarie, 1993] [Gurarie, Ludwig, 2002, 2004]

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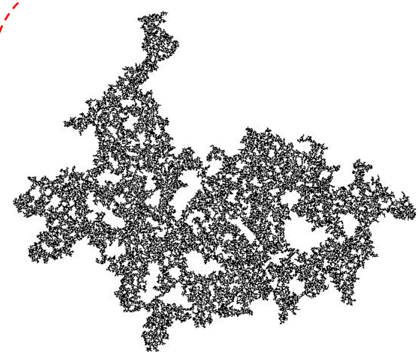
examples: *"simpler", slightly more control*



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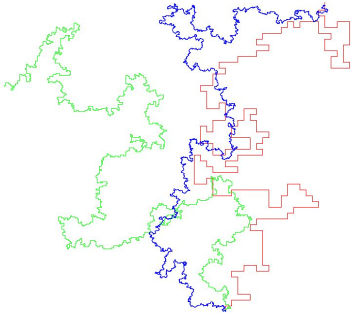

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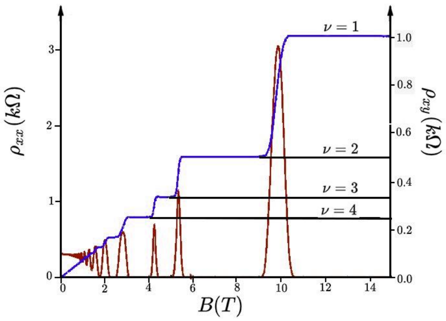

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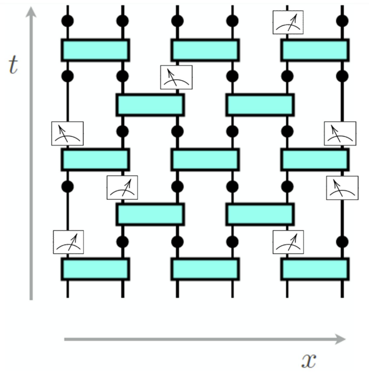

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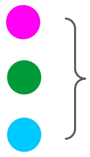
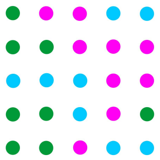
Cluster/loop models from Potts/O(n) spin models

percolation and SAW arise from c→0 limit of cluster/loop models defined through local spin models

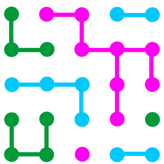
e.g. Potts



Q



$$Z_{\text{spin}} \longrightarrow Z_{\text{cluster}}$$



cluster weight

$Q \in \mathbb{R}$

[Fortuin, Kasteleyn, 1972]

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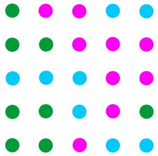
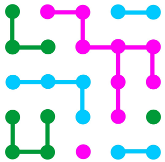
e.g. Potts  $\Big\} Q$  $Z_{\rm spin} \longrightarrow Z_{\rm cluster}$  cluster weight $Q \in \mathbb{R}$ [Fortuin, Kasteleyn, 1972]

critical cluster model $0 < Q < 4$ → cluster model CFT

non-unitary

$$c = 13 - \frac{6}{\beta^2} - 6\beta^2$$

$$-2 < c < 1$$

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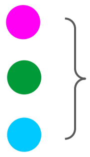
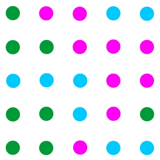
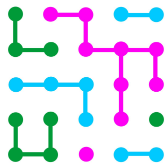
e.g. Potts  Q  $Z_{\text{spin}} \longrightarrow Z_{\text{cluster}}$  cluster weight $Q \in \mathbb{R}$ [Fortuin, Kasteleyn, 1972]

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similarly, O(n) spin model → loop models [De Gennes, 1972]

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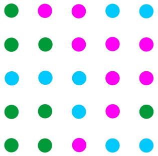
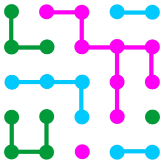
e.g. Potts  $\Big\} Q$  $Z_{\mathrm{spin}} \longrightarrow Z_{\mathrm{cluster}}$ 

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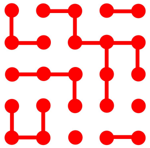
critical cluster model $0 < Q < 4$ → cluster model CFT $Q \to 1$  percolation

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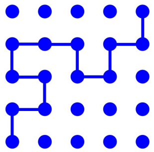
$$c \to 0, \quad \beta^2 \to \frac{2}{3}$$

critical dilute loop model $-2 < n < 2$ → loop model CFT $n \to 0$  SAW

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Physical observables in geometrical models

physical observables: probability type quantities concerning geometrical configurations

e.g. Potts/O(n) spin: order parameter $\sigma \equiv \Phi_{1/2,0}$

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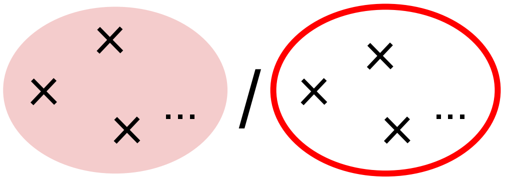

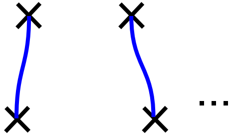

$\langle \sigma\sigma \ldots \sigma \rangle$

CFT correlator of spin operator

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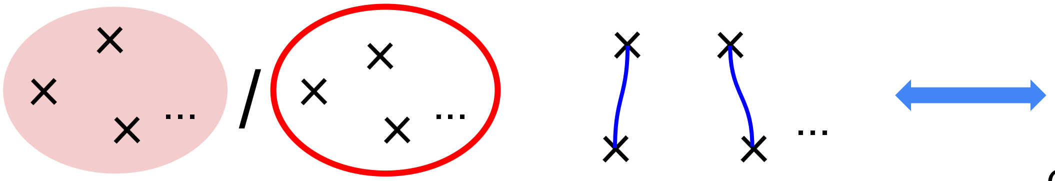

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more complicated types of observables characterized by correlation functions of other operators

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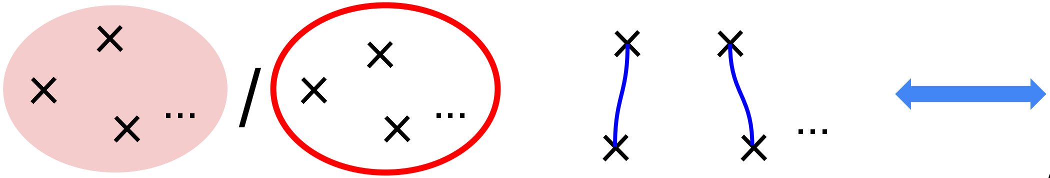

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CFT correlator of spin operator

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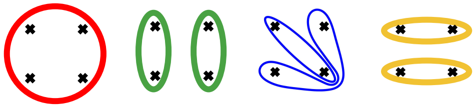
fractional Kac indices: correlation functions not from BPZ

Recent development @ generic c: bootstrap

four-point functions, probe the non-trivial CFT data, e.g. 

- bootstrap approach [Picco, Ribault, Santachiara, 2016]
- spectrum [Jacobsen, Saleur, 2018] & bootstrap [YH, Jacobsen, Saleur, 2020] cluster connectivities
- bootstrap O(n) loop model four-point functions

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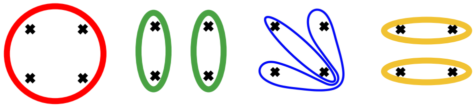

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despite lack of BPZ-type approach, crucial role played again by Kac operator

Kac operators with Virasoro degeneracy exist $\Phi_{2,1}$ or $\Phi_{1,2}$ **but not both (unlike in c<1 Liouville)**

cluster **dilute loop**

analytic bootstrap (using the BPZ from inserting Kac operators) → interchiral blocks

[Teschner, 1995][Zamolodchikov*2, 1995][Estienne, Iklef, 2015][Migliaccio, Ribault, 2017] [YH, Jacobsen, Saleur, 2020]

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Kac operators with Virasoro degeneracy exist $\Phi_{2,1}$ or $\Phi_{1,2}$

another consequence: bulk cluster/loop CFTs *at generic c* are logarithmic

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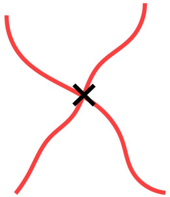

e.g. spin-2 four-leg operator $X : (h_{1,-2}(c), h_{1,2}(c))$

level-2 Virasoro descendants $\bar{A}X$

$$A = L_{-2} - \frac{1}{\beta^2} L_{-1}^2$$

mix into Jordan block $(\Psi, \bar{A}X)$

$$\begin{aligned}
\langle \Psi(z,\bar{z}) \Psi(0,0) \rangle &= \frac{-2b_{1,2}(c) \ln(z\bar{z}) + \theta_{1,2}(c)}{z^{2h(c)} \bar{z}^{2\bar{h}(c)}} \\
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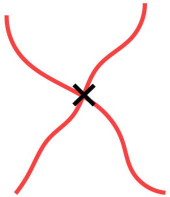

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with these developments, tackle the bulk c=0 CFTs, time-honored strategy: taking c→0 limit

c→0 catastrophe and three resolutions

$$\mathcal{O}(z,\bar{z})\mathcal{O}(0,0) = (z\bar{z})^{-2h_{\mathcal{O}}(c)} B_{\mathcal{O}}(c)\left(1 + \frac{h_{\mathcal{O}}(c)}{c/2}(z^2 T + \bar{z}^2 \bar{T}) + \dots\right)$$

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three resolutions: [Cardy, 2001]

I. … contains another operator X whose contribution cancels the divergence

II. dimension $h_{\mathcal{O}}(c)$ vanishes at c=0

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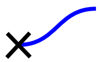
dense loop spin (inserting percolation hull)



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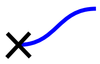

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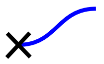

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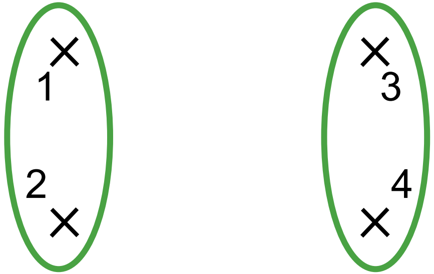
not exactly

Resolution I: spin OPE in cluster model

dilute loop model similar

generic c OPE:

$$\sigma^{\text{cluster}}(z,\bar z)\sigma^{\text{cluster}}(0,0)$$
$$=(z\bar z)^{-2h_\sigma^{\text{cluster}}(c)}\Bigg\{1+\frac{h_\sigma^{\text{cluster}}(c)}{c/2}(z^2T+\bar z^2\bar T)+\frac{\left(h_\sigma^{\text{cluster}}(c)\right)^2}{c^2/4}(z\bar z)^2T\bar T+\dots$$
$$+D^{\text{cluster}}_{\sigma\sigma\Phi_{2,1}}(c)(z\bar z)^{h_{2,1}(c)}\Phi_{2,1}+\dots$$
$$+D^{\text{cluster}}_{\sigma\sigma\Phi_{3,1}}(c)(z\bar z)^{h_{3,1}(c)}\Phi_{3,1}+\dots$$
$$+D^{\text{cluster}}_{\sigma\sigma\Phi_{0,2}}(c)(z\bar z)^{h_{0,2}(c)}\Phi_{0,2}+\dots$$
$$+D^{\text{cluster}}_{\sigma\sigma X}(c)\Big((z\bar z)^{h_{1,2}(c)}(\bar z^2\bar X+z^2X+\dots)+g(c)(z\bar z)^{h_{-1,2}(c)}(\Psi+\ln(z\bar z)\bar A X)+\dots\Big)$$
$$+\dots\Bigg\}$$



$x_1\to x_2,\quad x_3\to x_4$

[YH, Saleur, 2021]

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generic c OPE:

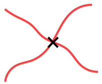
$$\sigma^{\text{cluster}}(z,\bar z)\sigma^{\text{cluster}}(0,0)$$

$$=(z\bar z)^{-2h_\sigma^{\text{cluster}}(c)}\Bigg\{1+\frac{h_\sigma^{\text{cluster}}(c)}{c/2}(z^2T+\bar z^2\bar T)+\frac{\left(h_\sigma^{\text{cluster}}(c)\right)^2}{c^2/4}(z\bar z)^2T\bar T+\ldots$$

$$+D^{\text{cluster}}_{\sigma\sigma\Phi_{2,1}}(c)(z\bar z)^{h_{2,1}(c)}\Phi_{2,1}+\ldots$$ *energy density operator* $\mathcal{E}$

$$+D^{\text{cluster}}_{\sigma\sigma\Phi_{3,1}}(c)(z\bar z)^{h_{3,1}(c)}\Phi_{3,1}+\ldots$$ *second energy density operator* $\mathcal{E}'$

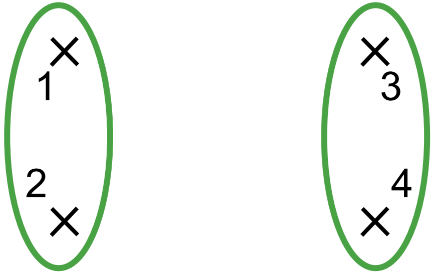
$$+D^{\text{cluster}}_{\sigma\sigma\Phi_{0,2}}(c)(z\bar z)^{h_{0,2}(c)}\Phi_{0,2}+\ldots$$ *four-leg operator*



$$+D^{\text{cluster}}_{\sigma\sigma X}(c)\Big((z\bar z)^{h_{1,2}(c)}(\bar z^2\bar X+z^2X+\ldots)+g(c)(z\bar z)^{h_{-1,2}(c)}(\Psi+\ln(z\bar z)\bar A X)+\ldots\Big)$$

spin-2 four-leg operator and logarithmic conformal family

$$+\ldots\Bigg\}$$



$x_1\to x_2,\quad x_3\to x_4$

[YH, Saleur, 2021]

Resolution I: spin OPE in cluster model

dilute loop model similar

generic c OPE:

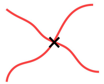
$$\sigma^{\text{cluster}}(z,\bar{z})\sigma^{\text{cluster}}(0,0)$$

$$=(z\bar{z})^{-2h_\sigma^{\text{cluster}}(c)}\Bigg\{1+\frac{h_\sigma^{\text{cluster}}(c)}{c/2}\left(z^2T+\bar{z}^2\bar{T}\right)+\frac{\left(h_\sigma^{\text{cluster}}(c)\right)^2}{c^2/4}(z\bar{z})^2T\bar{T}+\ldots$$

$$+D^{\text{cluster}}_{\sigma\sigma\Phi_{2,1}}(c)(z\bar{z})^{h_{2,1}(c)}\Phi_{2,1}+\ldots$$ *energy density operator* ε

$$+D^{\text{cluster}}_{\sigma\sigma\Phi_{3,1}}(c)(z\bar{z})^{h_{3,1}(c)}\Phi_{3,1}+\ldots$$ *second energy density operator* ε'

$$+D^{\text{cluster}}_{\sigma\sigma\Phi_{0,2}}(c)(z\bar{z})^{h_{0,2}(c)}\Phi_{0,2}+\ldots$$ *four-leg operator*

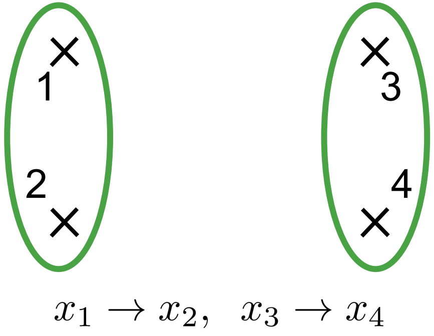


$$+D^{\text{cluster}}_{\sigma\sigma X}(c)\left((z\bar{z})^{h_{1,2}(c)}\left(\bar{z}^2\bar{X}+z^2X+\ldots\right)+g(c)(z\bar{z})^{h_{-1,2}(c)}\left(\Psi+\ln(z\bar{z})\bar{A}X\right)+\ldots\right)$$

$$+\ldots\Bigg\}$$

spin-2 four-leg operator and logarithmic conformal family

$$(h_{1,-2}(c=0),h_{1,2}(c=0))=(2,0)$$ *coincidental dimension*



[YH, Saleur, 2021]

Rank-2 Jordan block

$$\langle T(z)T(0)\rangle = \frac{c/2}{z^4}$$

stress tensor becomes zero-norm state, at risk of being removed from CFT state space rendering the CFT trivial

$\Longrightarrow$ $\langle T|t\rangle = b$

Rank-2 Jordan block

$$\langle T(z)T(0)\rangle = \frac{c/2}{z^4}$$

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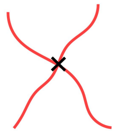
$$\Rightarrow \quad \langle T|t\rangle = b$$

in the OPE: another state with "oppositely behaving norm" to cancel the divergence



$$X$$

$$(h_{1,-2}(c=0), h_{1,2}(c=0)) = (2,0)$$

proper normalization $\langle \hat{X}(z,\bar{z})\hat{X}(0,0)\rangle \overset{c\to 0}{\simeq} \dfrac{-c/2}{z^{2h_{1,-2}}\,\bar{z}^{2h_{1,2}}}$

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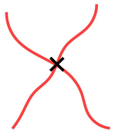

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straightforward to construct the "top field" $t = \frac{b}{c/2}(T+\hat{X})$ $\xrightarrow{\text{by definition}}$ $\langle t(z,\bar{z})T(0)\rangle = \frac{b}{z^4}$

logarithmic coupling "b number"

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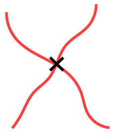

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logarithmic coupling "b number"

$$\langle t(z,\bar{z})t(0,0)\rangle = \frac{4b^2 h'_{1,2}\ln(z\bar{z})+\theta}{z^4} = \frac{-2b\ln(z\bar{z})+\theta}{z^4}$$

conformal Ward identity

$$b = -\frac{1}{2h'_{1,2}} = -5$$

lattice measurement

[Vasseur, Gainutdinov, Jacobsen, Saleur, 2011]

Rank-2 Jordan block

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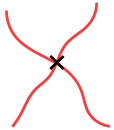

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lattice measurement

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rank-2 Jordan block

top field: t

bottom field: T or $\hat{X}$

Rank-3 Jordan block

$$\langle T\bar{T}(z,\bar{z})T\bar{T}(0,0)\rangle = \frac{c^2/4}{(z\bar{z})^4}$$ double zero

examine the spectrum for coincidental dimensions at c=0 $\Phi_{3,1}, \quad \Psi, \quad \bar{A}X.$ candidates to cancel the divergence

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due to the proper normalization of $\hat{X}$ vanishing at c=0, generic c Jordan block 2-point function acquire first order zero

properly normalized:

$$\langle\hat{\Psi}(z,\bar{z})\hat{\Psi}(0,0)\rangle \overset{c\to 0}{\simeq} \frac{b_{1,2}(c)c\ln(z\bar{z}) + \theta_{1,2}(c)c/2}{(z\bar{z})^{2h_{1,-2}}},$$

$$\langle\hat{\Psi}(z,\bar{z})\bar{A}\hat{X}(0,0)\rangle \overset{c\to 0}{\simeq} -\frac{b_{1,2}(c)c/2}{(z\bar{z})^{2h_{1,-2}}},$$

$$\langle\bar{A}\hat{X}(z,\bar{z})\bar{A}\hat{X}(0,0)\rangle \overset{c\to 0}{\simeq} 0,$$

Rank-3 Jordan block

$$\langle T\bar{T}(z,\bar{z})T\bar{T}(0,0)\rangle = \frac{c^2/4}{(z\bar{z})^4}$$ double zero

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necessary for second energy operator two-point function to behave as $\langle\hat{\Phi}_{3,1}(z,\bar{z})\hat{\Phi}_{3,1}(0,0)\rangle \overset{c\to 0}{\simeq} \frac{-c^2/4}{(z\bar{z})^{2h_{3,1}}}$ to cancel singularities

Rank-3 Jordan block

$$\langle T\bar{T}(z,\bar{z})T\bar{T}(0,0)\rangle = \frac{c^2/4}{(z\bar{z})^4}$$ double zero

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construct a top field: $$\Psi_2 = \frac{a}{c^2/4}\left(T\bar{T}+\hat{\Phi}_{3,1}\right)+\frac{a}{b_{1,2}c/2}\hat{\Psi}$$

bottom field: $$\Psi_0 \equiv -\hat{\Phi}_{3,1},\ T\bar{T},\ \text{or}\ -\bar{A}\hat{X}$$

$$\langle\Psi_2(z,\bar{z})\Psi_0(0,0)\rangle = \frac{a}{(z\bar{z})^4}$$

rank-3 Jordan blocks

a

Rank-3 Jordan block

$$\langle T\bar{T}(z,\bar{z})T\bar{T}(0,0)\rangle = \frac{c^2/4}{(z\bar{z})^4}$$ double zero

examine the spectrum for coincidental dimensions at c=0 $\Phi_{3,1},\ \Psi,\ \bar{A}X.$ candidates to cancel the divergence

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$$\begin{aligned}
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\langle\hat{\Psi}(z,\bar{z})\bar{A}\hat{X}(0,0)\rangle &\overset{c\to 0}{\simeq} -\frac{b_{1,2}(c)c/2}{(z\bar{z})^{2h_{1,-2}}},\\
\langle\bar{A}\hat{X}(z,\bar{z})\bar{A}\hat{X}(0,0)\rangle &\overset{c\to 0}{\simeq} 0,
\end{aligned}$$

necessary for second energy operator two-point function to behave as $\langle\hat{\Phi}_{3,1}(z,\bar{z})\hat{\Phi}_{3,1}(0,0)\rangle \overset{c\to 0}{\simeq} \frac{-c^2/4}{(z\bar{z})^{2h_{3,1}}}$ to cancel singularities

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rank-3 Jordan blocks

$$\begin{aligned}
\langle\Psi_2(z,\bar{z})\Psi_2(0,0)\rangle &= \frac{a_2-2a_1\ln(z\bar{z})+2a\ln^2(z\bar{z})}{(z\bar{z})^4}\\
\langle\Psi_2(z,\bar{z})\Psi_1(0,0)\rangle &= \frac{a_1-2a\ln(z\bar{z})}{(z\bar{z})^4}\\
\langle\Psi_1(z,\bar{z})\Psi_1(0,0)\rangle &= \frac{a}{(z\bar{z})^4}\\
\langle\Psi_2(z,\bar{z})\Psi_0(0,0)\rangle &= \frac{a}{(z\bar{z})^4}\\
\langle\Psi_1(z,\bar{z})\Psi_0(0,0)\rangle &= 0\\
\langle\Psi_0(z,\bar{z})\Psi_0(0,0)\rangle &= 0
\end{aligned}$$

conformal Ward identities

$$a = -\frac{25}{48}$$

Three-point functions with percolation spin

c=0 logarithmic operators written in terms of generic c operators

rank-2 Jordan block (t, T) *with top field*

$$t = \frac{b}{c/2}\big(T + \hat{X}\big)$$

rank-3 Jordan block $\big(\Psi_2, \Psi_1, \Psi_0\big)$ *with top field*

$$\Psi_2 = \frac{a}{c^2/4}\big(T\bar{T} + \hat{\Phi}_{3,1}\big) + \frac{a}{b_{1,2}c/2}\hat{\Psi}$$

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take the c→0 limit of three-point functions

Three-point functions with percolation spin

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rank-3 Jordan block (Ψ_2, Ψ_1, Ψ_0) *with top field*

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take the c→0 limit of three-point functions

$$\langle \sigma(z_1, \bar{z}_1)\sigma(z_2, \bar{z}_2)T(z_3)\rangle^{\mathrm{perco}} = h_\sigma^{\mathrm{perco}}\mathbb{P}_3^0$$

$$\langle \sigma(z_1, \bar{z}_1)\sigma(z_2, \bar{z}_2)t(z_3, \bar{z}_3)\rangle^{\mathrm{perco}} = \left(C_{\sigma\sigma T}^{\mathrm{perco}}\tau_3 + C_{\sigma\sigma t}^{\mathrm{perco}}\right)\mathbb{P}_3^0$$

$$h_\sigma^{\mathrm{perco}} = h_\sigma^{\mathrm{cluster}}(c=0) = \frac{5}{96}$$

logarithmic 3-point functions at c=0

$$\langle \sigma(z_1, \bar{z}_1)\sigma(z_2, \bar{z}_2)\Psi_0(z_3, \bar{z}_3)\rangle^{\mathrm{perco}} = C_{\sigma\sigma\Psi_0}^{\mathrm{perco}}\mathbb{P}_3^0,$$

$$\langle \sigma(z_1, \bar{z}_1)\sigma(z_2, \bar{z}_2)\Psi_1(z_3, \bar{z}_3)\rangle^{\mathrm{perco}} = \left(C_{\sigma\sigma\Psi_1}^{\mathrm{perco}} + C_{\sigma\sigma\Psi_0}^{\mathrm{perco}}\tau_3\right)\mathbb{P}_3^0,$$

$$\langle \sigma(z_1, \bar{z}_1)\sigma(z_2, \bar{z}_2)\Psi_2(z_3, \bar{z}_3)\rangle^{\mathrm{perco}} = \left(C_{\sigma\sigma\Psi_2}^{\mathrm{perco}} + C_{\sigma\sigma\Psi_1}^{\mathrm{perco}}\tau_3 + \frac{1}{2}C_{\sigma\sigma\Psi_0}^{\mathrm{perco}}\tau_3^2\right)\mathbb{P}_3^0,$$

$$C_{\sigma\sigma\Psi_0}^{\mathrm{perco}} = \left(h_\sigma^{\mathrm{perco}}\right)^2$$

consistent with conformal Ward identities

$$\tau_3 = \ln\frac{z_{12}\bar{z}_{12}}{z_{13}\bar{z}_{13}z_{23}\bar{z}_{23}}$$

Resolution I

[YH, Saleur, 2021]

leading log OPE of percolation spin operator:

$$\sigma^{\rm perco}(z,\bar z)\sigma^{\rm perco}(0,0) = (z\bar z)^{-2h_\sigma^{\rm perco}}\Bigg\{1+\ldots+z^2\frac{h_\sigma^{\rm perco}}{b}\left[t+\ln(z\bar z)T+\frac{C_{\sigma\sigma t}^{\rm perco}}{h_\sigma^{\rm perco}}T\right]+c.c.+$$
$$+z\bar z^2\frac{h_\sigma^{\rm perco}}{2b}\partial\bar t+z^2\bar z\frac{h_\sigma^{\rm perco}}{2b}\bar\partial t+(z\bar z)^2\frac{h^{\rm perco}}{4b}(\partial^2\bar t+\bar\partial^2 t)$$
$$+(z\bar z)^2\frac{\left(h_\sigma^{\rm perco}\right)^2}{a}\Bigg[\Psi_2+\ln(z\bar z)\Psi_1+\frac{1}{2}\ln^2(z\bar z)\Psi_0+\left(\frac{C_{\sigma\sigma\Psi_1}^{\rm perco}}{\left(h_\sigma^{\rm perco}\right)^2}-\frac{a_1}{a}\right)\Psi_1$$
$$+\left(\frac{C_{\sigma\sigma\Psi_1}^{\rm perco}}{\left(h_\sigma^{\rm perco}\right)^2}-\frac{a_1}{a}\right)\ln(z\bar z)\Psi_0+\left(\frac{C_{\sigma\sigma\Psi_2}^{\rm perco}}{\left(h_\sigma^{\rm perco}\right)^2}-\frac{a_1C_{\sigma\sigma\Psi_1}^{\rm perco}}{a\left(h_\sigma^{\rm perco}\right)^2}+\frac{a_1^2}{a^2}-\frac{a_2}{a}\right)\Psi_0\Bigg]+\ldots\Bigg\}.$$

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$$+\left(\frac{C_{\sigma\sigma\Psi_1}^{\rm perco}}{\left(h_\sigma^{\rm perco}\right)^2}-\frac{a_1}{a}\right)\ln(z\bar z)\Psi_0+\left(\frac{C_{\sigma\sigma\Psi_2}^{\rm perco}}{\left(h_\sigma^{\rm perco}\right)^2}-\frac{a_1C_{\sigma\sigma\Psi_1}^{\rm perco}}{a\left(h_\sigma^{\rm perco}\right)^2}+\frac{a_1^2}{a^2}-\frac{a_2}{a}\right)\Psi_0\Bigg]+\ldots\Bigg\}.$$

will include contribution from $\Phi_{2,1},\ \ \Phi_{0,2}$ *later*

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[YH, Saleur, 2021]

leading log OPE of percolation spin operator:

$$\sigma^{\text{perco}}(z,\bar z)\sigma^{\text{perco}}(0,0) = (z\bar z)^{-2h_\sigma^{\text{perco}}}\Bigg\{1+\ldots+z^2\frac{h_\sigma^{\text{perco}}}{b}\left[t+\ln(z\bar z)T+\frac{C^{\text{perco}}_{\sigma\sigma t}}{h_\sigma^{\text{perco}}}T\right]+c.c.+$$
$$+z\bar z^2\frac{h_\sigma^{\text{perco}}}{2b}\partial\bar t+z^2\bar z\frac{h_\sigma^{\text{perco}}}{2b}\bar\partial t+(z\bar z)^2\frac{h_\sigma^{\text{perco}}}{4b}(\partial^2\bar t+\bar\partial^2 t)$$
$$+(z\bar z)^2\frac{\left(h_\sigma^{\text{perco}}\right)^2}{a}\Bigg[\Psi_2+\ln(z\bar z)\Psi_1+\frac{1}{2}\ln^2(z\bar z)\Psi_0+\left(\frac{C^{\text{perco}}_{\sigma\sigma\Psi_1}}{\left(h_\sigma^{\text{perco}}\right)^2}-\frac{a_1}{a}\right)\Psi_1$$
$$+\left(\frac{C^{\text{perco}}_{\sigma\sigma\Psi_1}}{\left(h_\sigma^{\text{perco}}\right)^2}-\frac{a_1}{a}\right)\ln(z\bar z)\Psi_0+\left(\frac{C^{\text{perco}}_{\sigma\sigma\Psi_2}}{\left(h_\sigma^{\text{perco}}\right)^2}-\frac{a_1C^{\text{perco}}_{\sigma\sigma\Psi_1}}{a\left(h_\sigma^{\text{perco}}\right)^2}+\frac{a_1^2}{a^2}-\frac{a_2}{a}\right)\Psi_0\Bigg]+\ldots\Bigg\}.$$

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zero-norm state appears at c=0 in a continuous family of CFTs

⇩

study the “proper normalization” of operators

Continuous family of non-unitary CFTs

to study normalization operators, going back to generic c, consider the family of non-unitary CFTs

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle = \sum_{\Phi} A_{\Phi}^{\mathcal{O}} \mathcal{F}_{\Phi}^{\mathcal{O}}$$

amplitude

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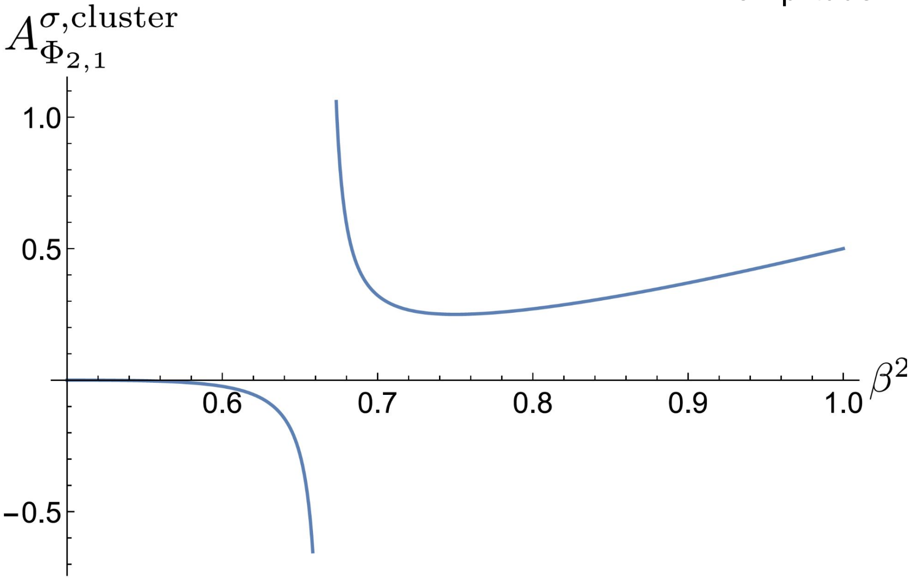
$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle = \sum_{\Phi} A_{\Phi}^{\mathcal{O}} \mathcal{F}_{\Phi}^{\mathcal{O}}$$

amplitude

e.g. amplitude of energy density in cluster four-spin correlator



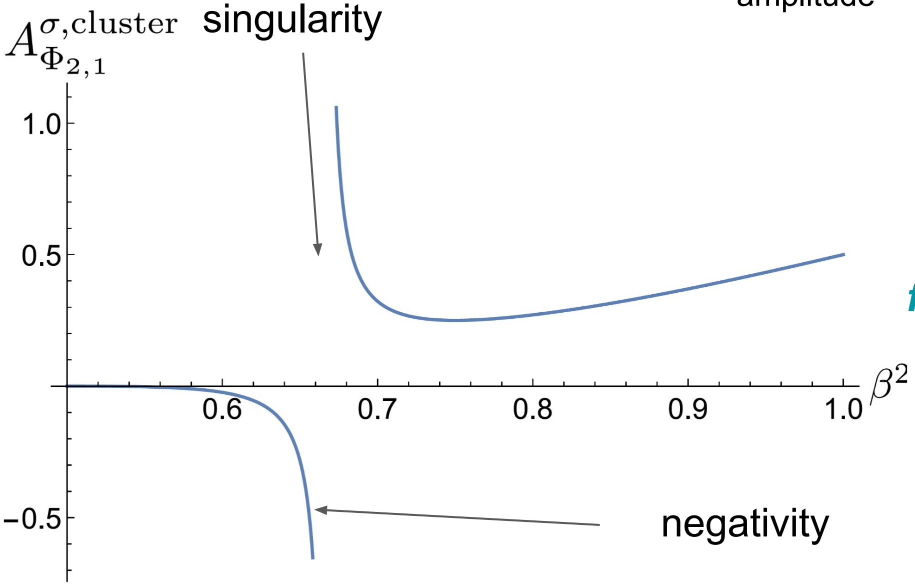
Continuous family of non-unitary CFTs

to study normalization operators, going back to generic c, consider the family of non-unitary CFTs

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle = \sum_{\Phi} A_{\Phi}^{\mathcal{O}} \mathcal{F}_{\Phi}^{\mathcal{O}}$$

amplitude

e.g. amplitude of energy density in cluster four-spin correlator



features of non-unitarity

Conformal data in unitary CFTs

recall how the amplitude arise from conformal data:

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unitarity: $C^2_{\mathcal{O}\mathcal{O}\Phi} \geq 0$ = positive amplitude $\Longrightarrow$ ***positive bootstrap***

Real non-unitary CFTs

in real CFTs, exist real operators, correlations satisfying [Gorbenko, Rychkov, Zan, 2018]

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\mathcal{O}_n(x_n)\rangle^* = \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\mathcal{O}_n(x_n)\rangle$$

three-point function: $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle^* = \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle$

$$C^*_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3} = C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3} \longrightarrow C^2_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3} \geq 0$$

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manifestly real: random objects with real positive measure

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reflection positivity violated [Biskup, 1998]

cluster/loop models are described by real non-unitary CFTs

Operator normalization

to understand the operator normalizations

real CFT

3-point functions describing geometrical observables

three-point constants $C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$ real and finite

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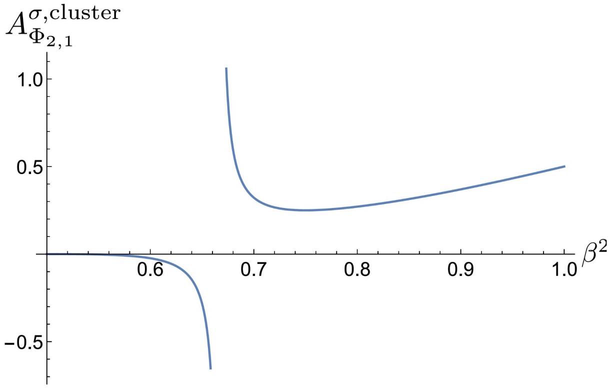

negativity: negative norm states

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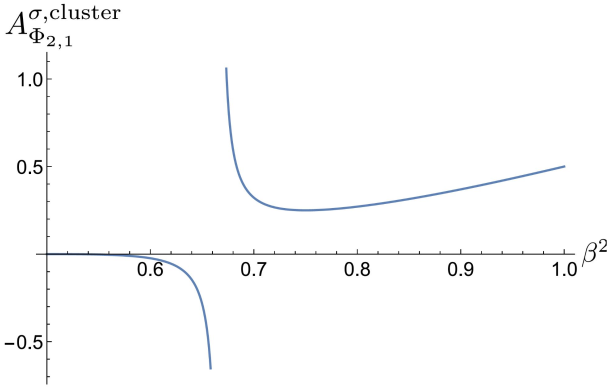

negativity: negative norm states

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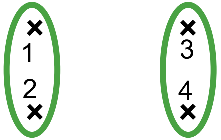
focus on Kac operators in four-spin correlator using analytic bootstrap

Kac operators in cluster model

similar in loop models

$$\text{degenerate } \Phi_{2,1} \to \text{analytic } R^{\text{cluster}}_{i,1} \equiv \frac{A^{\sigma,\text{cluster}}_{\Phi_{i+1,1}}}{A^{\sigma,\text{cluster}}_{\Phi_{i,1}}}$$

[YH, Jacobsen, Saleur, 2020]



$$\langle \sigma\sigma\sigma\sigma \rangle^{\text{cluster}}$$

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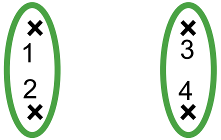


$\langle \sigma\sigma\sigma\sigma \rangle^{\text{cluster}}$

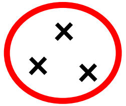
amplitudes of Kac operators fully determined up to an overall constants

$$A^{\sigma,\text{cluster}}_{\Phi_{e,1}} = \prod_{i=1}^{e-1} R^{\text{cluster}}_{i,1}, \quad e = 2, 3, \ldots,$$

$$A^{\sigma,\text{cluster}}(\Phi_{1,1}) = 1$$

tested in bootstrap

consistent with three-point constant from [Delfino, Viti, 2010]



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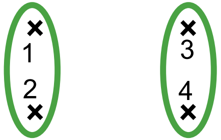

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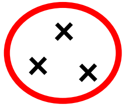

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examine the poles and zeros of the recursion $R^{\text{cluster}}_{i,1}$ in β^2 (equivalently central charge)

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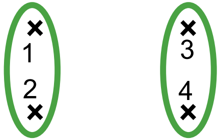

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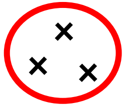

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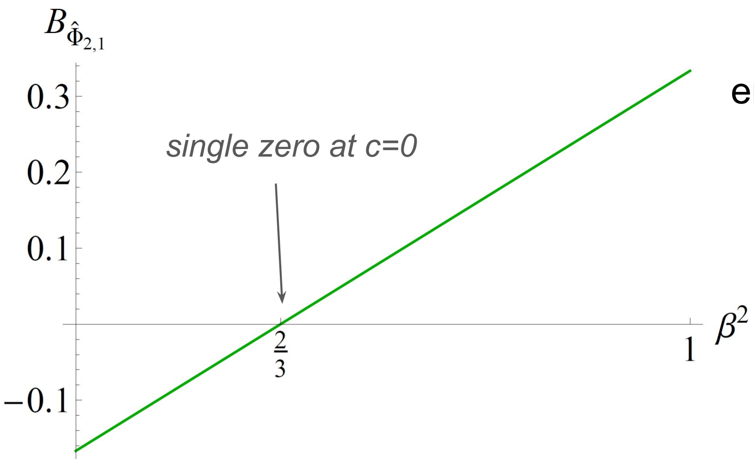
to guarantee real and finite 3-point constants for $\frac{1}{2} < \beta^2 < 1$ *take the norm*

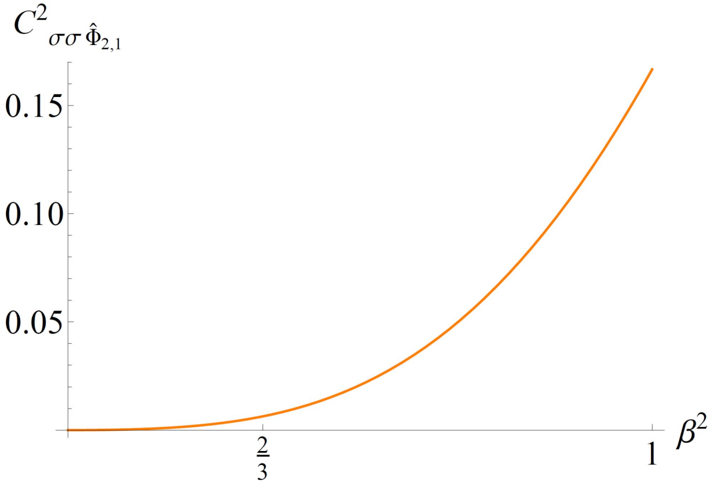
$$B_{\hat{\Phi}_{e,1}} \sim \prod_{i=1}^{e-1} \left(\beta^2 - \beta^2_{\text{poles},i}\right)\left(\beta^2 - \beta^2_{\text{zeros},i}\right)$$

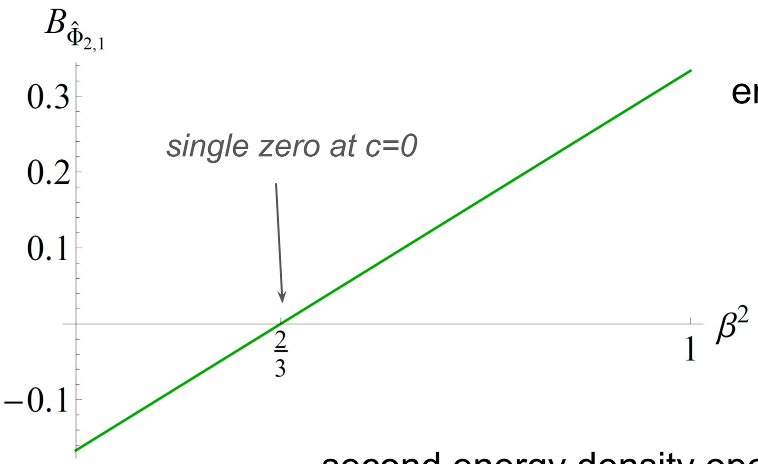


energy density operator

$$\hat{\Phi}_{2,1} \sim \varepsilon$$

$$B_{\hat{\Phi}_{2,1}} \sim \left(\beta^2 - \frac{2}{3}\right)$$





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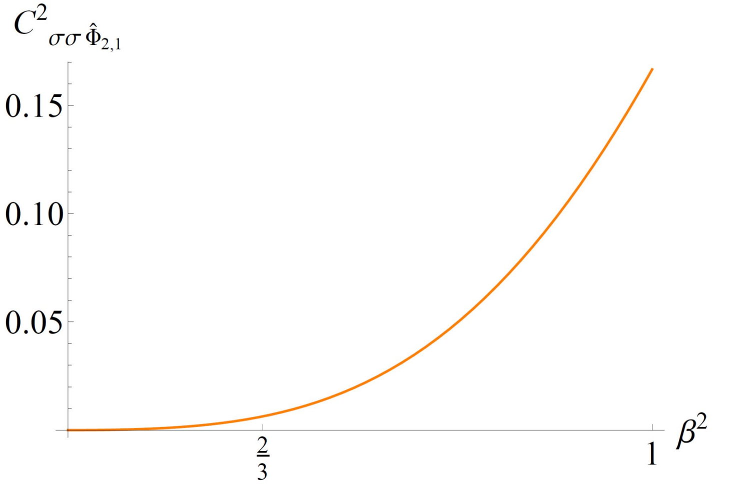

second energy density operator $\hat{\Phi}_{3,1} \sim \varepsilon'$

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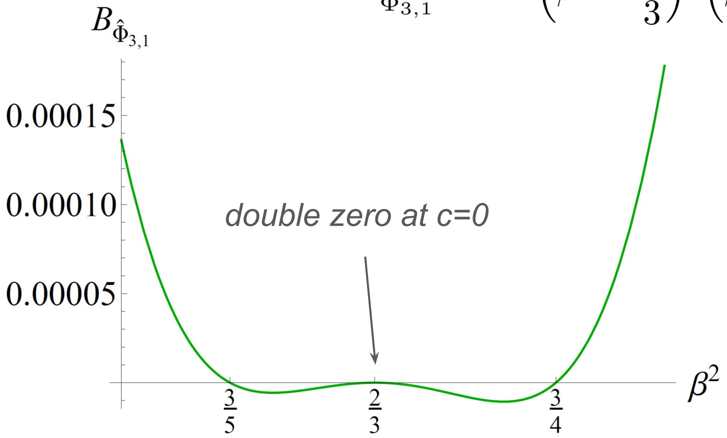

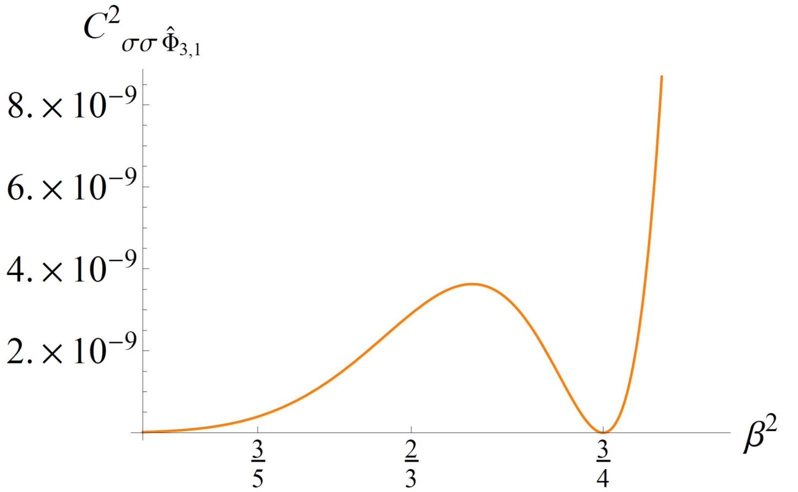

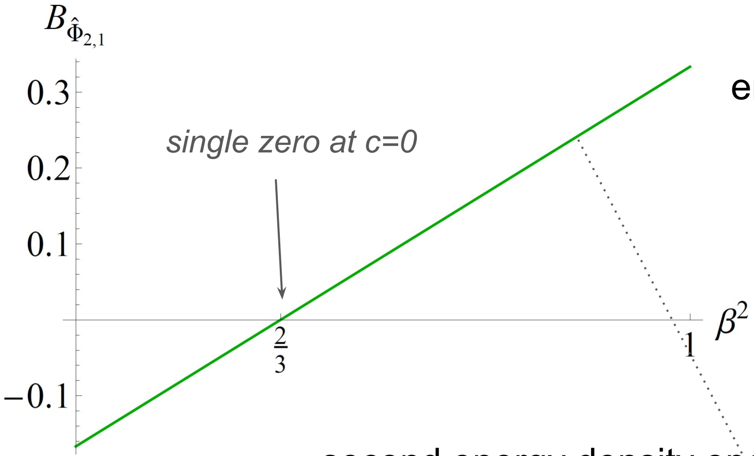

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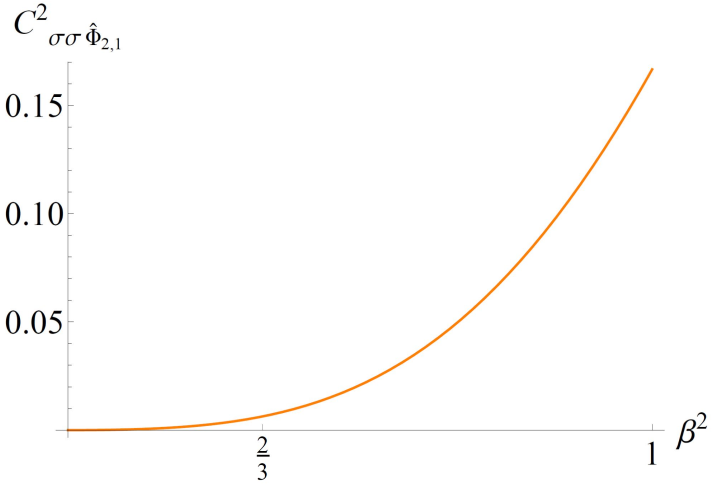

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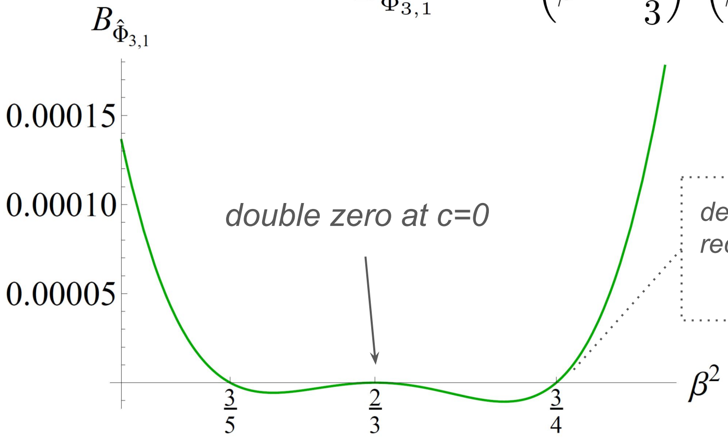
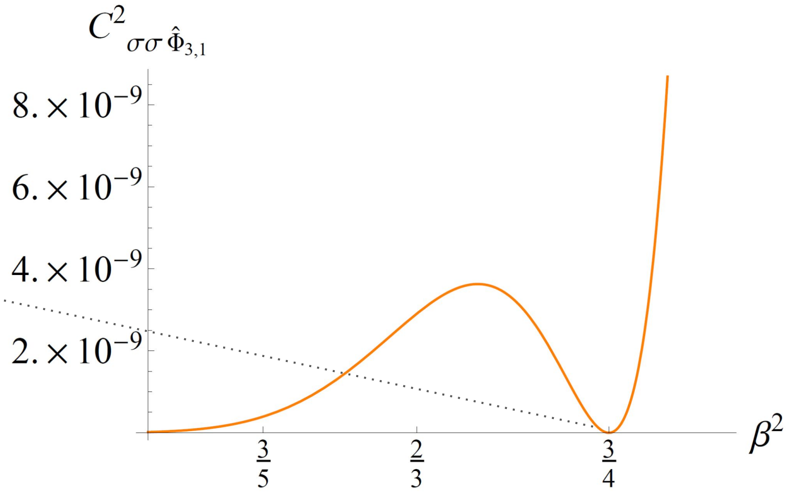


decoupling at Ising point to recover the unitary fusion rule:

$$\sigma \times \sigma \sim \mathbb{I} + \varepsilon$$



Higher Kac operators

$\hat{\Phi}_{4,1}$ $$B_{\hat{\Phi}_{4,1}} \sim \left(\beta^2 - \frac{2}{3}\right)^2 \left(\beta^2 - \frac{3}{5}\right)^2 \left(\beta^2 - \frac{4}{5}\right) \left(\beta^2 - \frac{4}{7}\right) \left(\beta^2 - \frac{3}{4}\right)^2 \left(\beta^2 - \frac{2}{3}\right)$$

from poles

from zeros

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this pattern repeats to higher Kac operators

a pair of pole and zero could be removed together, keeping non-negative $C^2_{\sigma\sigma\hat{\Phi}_{e,1}}(c)$

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norm of $\hat{\Phi}_{4,1}$ at c=0: triple zero or single zero?

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should examine their amplitudes in other four-point functions

⟹ **compare with c<1 Liouville CFT operator normalizations**

Comparison with c<1 Liouville CFT

a family of CFTs closely related to conformal loop ensemble (CLE)

many works from recent probabilistic constructions

analytic solution from bootstrap [Teschner, 1995][Zamolodchikov*2, 1995]

two Kac operators $\Phi_{2,1}$ and $\Phi_{1,2}$ → two sets of recursive amplitudes

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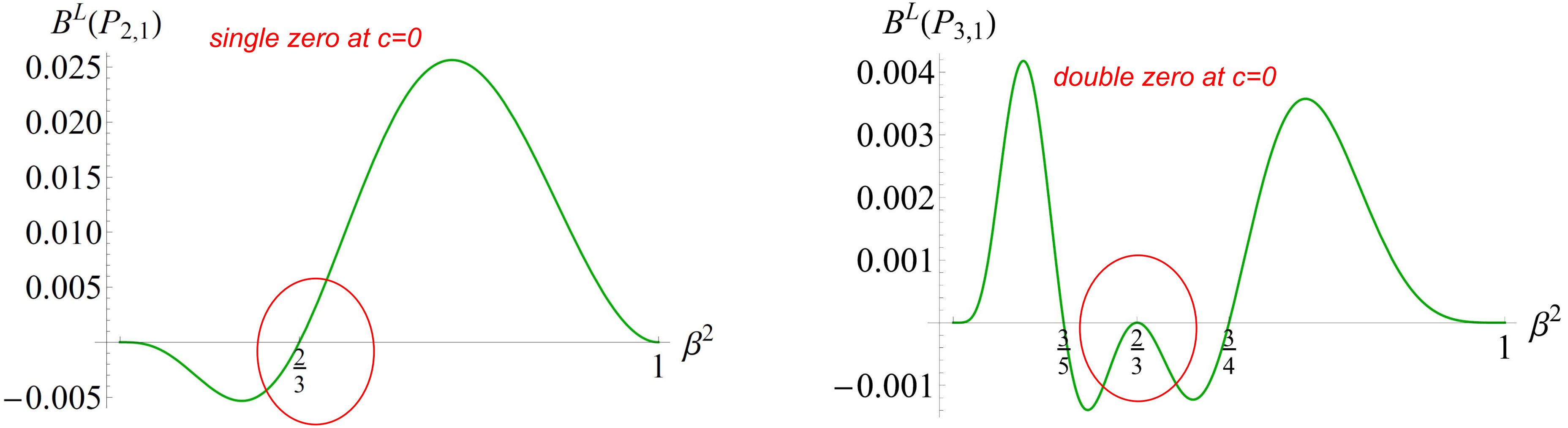
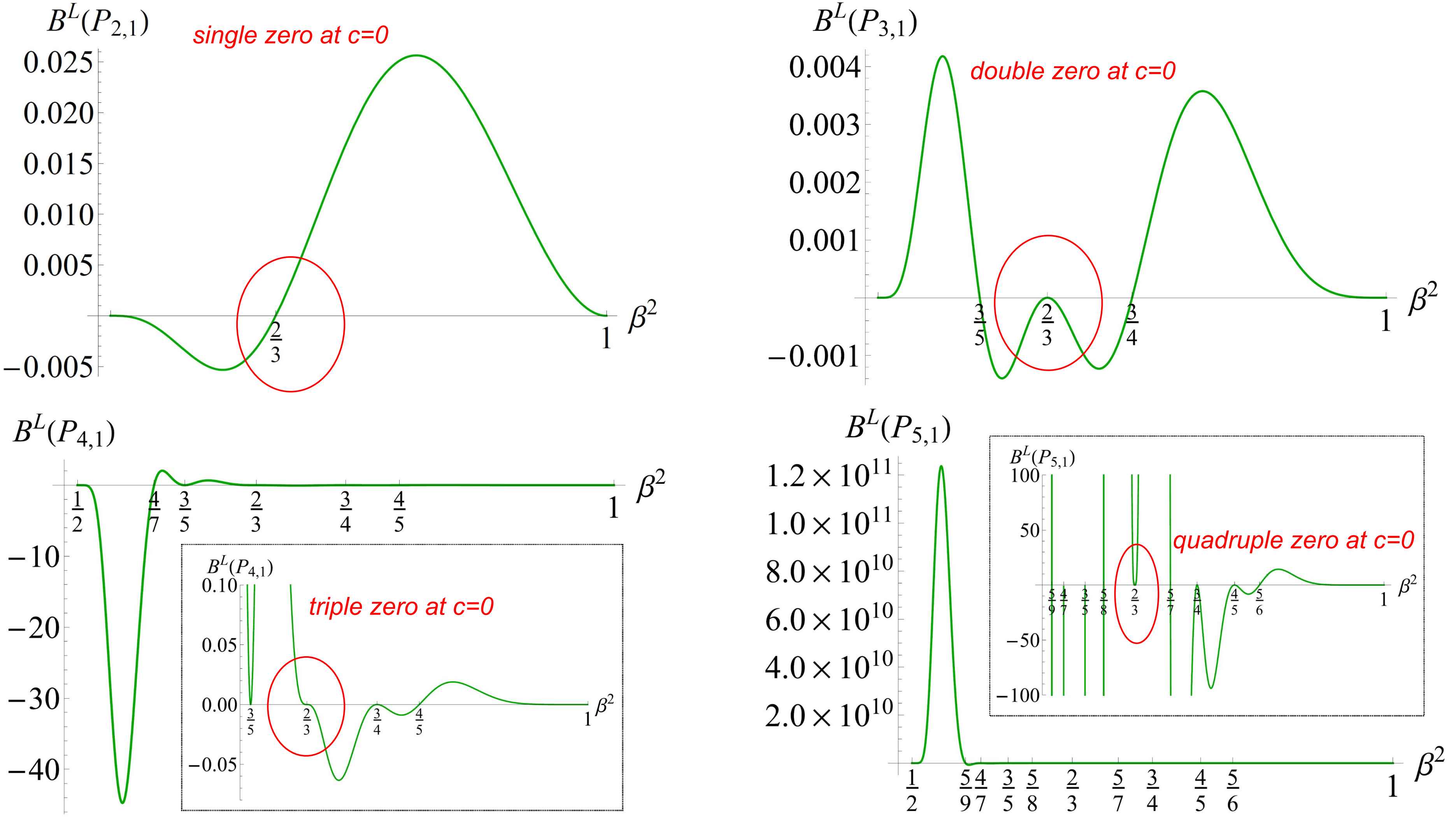
conformal data of loop model diagonal fields formally coincide with c<1 Liouville [Ribault, 2022]

compare the Kac operator normalizations with c<1 Liouville CFT normalization of operator with the same momentum

$$P_{r,s} = \frac{1}{2}\left(\frac{r}{\beta} - s\beta\right) \qquad B_P^L = \prod_{\pm\pm} \frac{1}{\Gamma_\beta(\beta^{\pm 1} \pm 2P)}$$



$B^L(P_{2,1})$
single zero at c=0
0.025
0.020
0.015
0.010
0.005
−0.005
$\frac{2}{3}$
1
β^2
$B^L(P_{3,1})$
double zero at c=0
0.004
0.003
0.002
0.001
−0.001
$\frac{3}{5}$
$\frac{2}{3}$
$\frac{3}{4}$
1
β^2



$B^L(P_{2,1})$
single zero at c=0
0.025
0.020
0.015
0.010
0.005
−0.005
$\frac{2}{3}$
1
β^2
$B^L(P_{3,1})$
double zero at c=0
0.004
0.003
0.002
0.001
−0.001
$\frac{3}{5}$
$\frac{2}{3}$
$\frac{3}{4}$
1
β^2
$B^L(P_{4,1})$
−10
−20
−30
−40
$\frac{1}{2}$
$\frac{4}{7}$
$\frac{3}{5}$
$\frac{2}{3}$
$\frac{3}{4}$
$\frac{4}{5}$
1
β^2
$B^L(P_{4,1})$
triple zero at c=0
0.10
0.05
0.00
−0.05
$\frac{3}{5}$
$\frac{2}{3}$
$\frac{3}{4}$
$\frac{4}{5}$
1
β^2
$B^L(P_{5,1})$
1.2×10^{11}
1.0×10^{11}
8.0×10^{10}
6.0×10^{10}
4.0×10^{10}
2.0×10^{10}
$\frac{1}{2}$
$\frac{5}{9}$
$\frac{4}{7}$
$\frac{3}{5}$
$\frac{5}{8}$
$\frac{2}{3}$
$\frac{5}{7}$
$\frac{3}{4}$
$\frac{4}{5}$
$\frac{5}{6}$
1
β^2
$B^L(P_{5,1})$
quadruple zero at c=0
100
50
−50
−100
$\frac{5}{9}$
$\frac{4}{7}$
$\frac{3}{5}$
$\frac{5}{8}$
$\frac{2}{3}$
$\frac{5}{7}$
$\frac{3}{4}$
$\frac{4}{5}$
$\frac{5}{6}$
1
β^2

if indeed the normalization of Kac operators in cluster/loop coincide with c<1 Liouville CFT

$$B_{\hat{\Phi}_{4,1}} \sim \left(\beta^2 - \frac{2}{3}\right)^3 \left(\beta^2 - \frac{3}{5}\right)^2 \left(\beta^2 - \frac{4}{5}\right) \left(\beta^2 - \frac{4}{7}\right) \left(\beta^2 - \frac{3}{4}\right)^2$$

$$B_{\hat{\Phi}_{5,1}} \sim \left(\beta^2 - \frac{2}{3}\right)^4 \left(\beta^2 - \frac{3}{5}\right)^2 \left(\beta^2 - \frac{4}{5}\right)^2 \left(\beta^2 - \frac{4}{7}\right)^2 \left(\beta^2 - \frac{5}{7}\right) \left(\beta^2 - \frac{5}{9}\right) \left(\beta^2 - \frac{3}{4}\right)^2 \left(\beta^2 - \frac{5}{8}\right) \left(\beta^2 - \frac{5}{6}\right)$$

higher Kac operators have arbitrarily higher order zeros at c=0 in their norms

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interesting implications:

- arbitrarily higher rank Jordan blocks in the c=0 bulk CFTs — Kac operators sit at the bottom

suggested by lattice algebraic representation studies [Gainutdinov, Read, Saleur, Vasseur, 2014]

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- cluster and dense loop model spectra: higher rank Jordan blocks requires more operators with coincidental dimensions at c=0 for mixing → a bigger CFT of both percolation cluster and hull

Constructing Jordan blocks

analyzed the need of operators normalization

rank-2

$$\mathbb{P}_2^c \equiv \frac{1}{z^{2h(c)}\,\bar{z}^{2\bar{h}(c)}}$$

consider a generic construction of Jordan blocks

$$\langle \phi(z,\bar{z})\phi(0,0)\rangle = B_\phi(c)\mathbb{P}_2^c \simeq B'_\phi(c-c_*)\mathbb{P}_2^* + o(c-c_*)$$
$$\langle \psi(z,\bar{z})\psi(0,0)\rangle = B_\psi(c)\mathbb{P}_2^c \simeq B'_\psi(c-c_*)\mathbb{P}_2^* + o(c-c_*)$$

two operators, at c_*
dimensions coincide
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top field: $\mathcal{O}^{(2;2)} = \gamma\left(\frac{\phi}{B_\phi(c)} + \frac{\psi}{B_\psi(c)}\right)$

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logarithmic coupling

bottom field: $\mathcal{O}^{(2;1)} = \phi \text{ or } \psi$

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conformal Ward identities

two point function of top field:

$$\langle\mathcal{O}^{(2;2)}(z,\bar{z})O^{(2;2)}(0,0)\rangle = \left(-2\gamma^2\frac{h'_\phi - h'_\psi}{B'_\phi}\ln(z\bar{z}) + const.\right)\mathbb{P}_2^* = \left(-2\gamma\ln(z\bar{z}) + const.\right)\mathbb{P}_2$$

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very similar construction for higher rank Jordan blocks

logarithmic couplings computed with norms and dimensions, e.g. rank-3

$$a = -\frac{B_\phi^{(2)}}{(2h'_O - h'_\phi - h'_\psi)(h'_\phi - h'_\psi)}$$

Energy density logarithmic pair

energy density operator

$$\varepsilon = \begin{cases} \hat{\Phi}_{2,1}, & \text{cluster}\ , \\ \hat{\Phi}_{1,3}, & \text{dilute loop}\ . \end{cases}$$

norms: single zero at c=0

also in dilute loop

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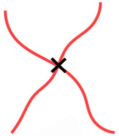
norms: single zero at c=0

also in dilute loop

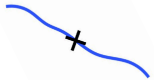
at c=0

$$\text{percolation:} \ \ (h_{2,1}, h_{2,1}) = (h_{0,2}, h_{0,2}) = \left(\frac{5}{8}, \frac{5}{8}\right)$$

$$\text{SAW:} \ \ (h_{1,3}, h_{1,3}) = (h_{1,0}, h_{1,0}) = \left(\frac{1}{3}, \frac{1}{3}\right)$$



two-hull operator



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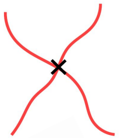

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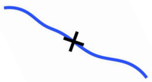

hull operator

top field: $$\tilde{\varepsilon} = \begin{cases} \gamma\left(\frac{\hat{\Phi}_{2,1}}{B_{\hat{\Phi}_{2,1}}(c)} + \frac{\hat{\Phi}_{0,2}}{B_{\hat{\Phi}_{0,2}}(c)}\right), & \text{percolation} \\ \gamma\left(\frac{\hat{\Phi}_{1,3}}{B_{\hat{\Phi}_{1,3}}(c)} + \frac{\hat{\Phi}_{1,0}}{B_{\hat{\Phi}_{1,0}}(c)}\right), & \text{SAW} \end{cases}$$

$$\gamma^{\text{perco}} = \frac{1}{2(h'_{2,1} - h'_{0,2})} = -\frac{5}{4}\ ,$$

$$\gamma^{\text{SAW}} = \frac{1}{2(h'_{1,3} - h'_{1,0})} = \frac{5}{3}\ .$$

Energy density logarithmic pair

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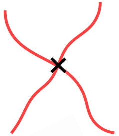

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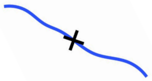

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three-point functions with spin from c→0 limit

$$\langle \sigma(z_1, \bar{z}_1)\sigma(z_2, \bar{z}_2)\varepsilon(z_3, \bar{z}_3)\rangle^{\text{perco}} = C^{\text{perco}}_{\sigma\sigma\varepsilon}\mathbb{P}^0_3$$

$$C^{\text{perco}}_{\sigma\sigma\varepsilon} = \sqrt{\frac{5}{6}}\frac{\Gamma(3/4)}{\Gamma(1/4)}$$

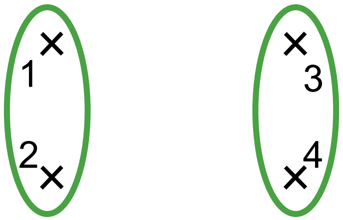
$$\langle \sigma(z_1, \bar{z}_1)\sigma(z_2, \bar{z}_2)\tilde{\varepsilon}(z_3, \bar{z}_3)\rangle^{\text{perco}} = \left(C^{\text{perco}}_{\sigma\sigma\tilde{\varepsilon}} + C^{\text{perco}}_{\sigma\sigma\varepsilon}\tau_3\right)\mathbb{P}^0_3$$

Four-spin correlator in percolation

contribute to spin OPE: $$(z\bar{z})^{-2h_\sigma+h_\varepsilon}\frac{C^{\text{perco}}_{\sigma\sigma\varepsilon}}{\gamma}\left(\tilde{\varepsilon}^{\text{perco}}+\ln(z\bar{z})\varepsilon^{\text{perco}}+\frac{C^{\text{perco}}_{\sigma\sigma\tilde{\varepsilon}}}{C^{\text{perco}}_{\sigma\sigma\varepsilon}}\varepsilon^{\text{perco}}\right)+\dots$$

four-spin correlator in percolation:

$$\begin{aligned}&\langle\sigma(\infty)\sigma(1)\sigma(z,\bar{z})\sigma(0)\rangle^{\text{perco}}\\&=(z\bar{z})^{-2h_\sigma}\left(1+(z\bar{z})^{h_\varepsilon}\frac{C^2_{\sigma\sigma\varepsilon}}{\gamma^2}(\theta_1+\gamma\ln(z\bar{z}))+(z^2+\bar{z}^2)\frac{C^2_{\sigma\sigma T}}{b^2}(\theta+b\ln(z\bar{z}))+\dots\right.\\&\left.\quad+(z\bar{z})^2\frac{C^2_{\sigma\sigma\Psi_0}}{a_0^2}\left(a_2+a_1\ln(z\bar{z})+\frac{a}{2}\ln^2(z\bar{z})\right)+\dots\right)\end{aligned}$$



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consistent with taking the c$\rightarrow$ 0 limit of four-spin correlator [YH, Saleur, 2021]

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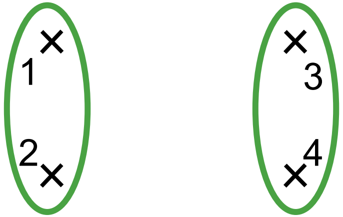

consistent with taking the c→ 0 limit of four-spin correlator [YH, Saleur, 2021]

very interesting recent probabilistic construction [Camia, Feng, 2024]:

a physical interpretation of the logarithm: summing over the probabilities of independent events of the same order at different scales contributing to a certain geometrical configuration

Resolution III

now examine the OPE of the energy density operator itself

back to c⟶0 catastrophe: $\mathcal{O}(z,\bar{z})\mathcal{O}(0,0) = (z\bar{z})^{-2h_{\mathcal{O}}(c)} B_{\mathcal{O}}(c)\left(1 + \frac{h_{\mathcal{O}}(c)}{c/2}(z^2 T + \bar{z}^2 \bar{T}) + \ldots\right)$

$$\Phi_{2,1}(z,\bar{z})\Phi_{2,1}(0,0) = (z\bar{z})^{-2h_{2,1}(c)}\left(1 + \frac{h_{2,1}(c)}{c/2}(z^2 T + \bar{z}^2 \bar{T}) + \frac{h_{2,1}^2(c)}{c^2/4}(z\bar{z})^2 T\bar{T} + \ldots + D_{\Phi_{2,1}\Phi_{2,1}\Phi_{3,1}}(c)(z\bar{z})^{h_{3,1}(c)}\Phi_{3,1} + \ldots\right)$$

Kac operators: Virasoro degeneracy constrains the fusion rules, no X to cancel the singularity

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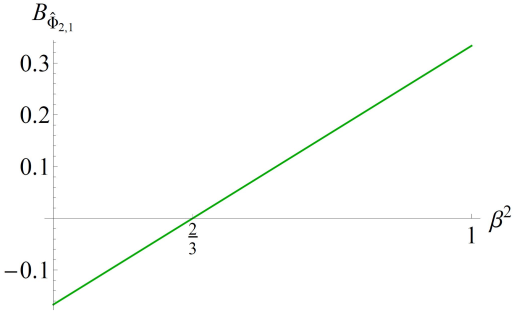
Kac operators: Virasoro degeneracy constrains the fusion rules, no X to cancel the singularity

only resolution: vanishing norm $B_{\hat{\Phi}_{2,1}}(c) \sim c$

indeed what we see from generic c analysis



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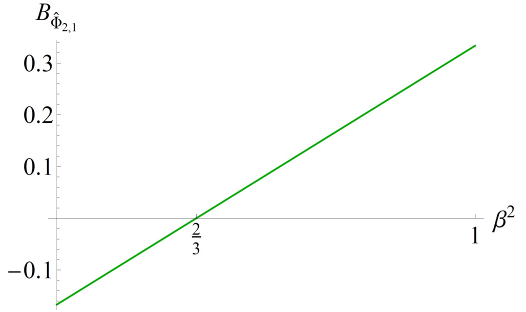

consider the properly normalized energy density operator

$$\varepsilon^{\text{perco}} \equiv \hat{\Phi}_{2,1}$$

Four-energy correlator from BPZ

at generic c, satisfy BPZ $\langle \Phi_{2,1}\Phi_{2,1}\Phi_{2,1}\Phi_{2,1}\rangle = |\mathcal{F}_{\mathbb{I}}(z)|^2 + R(c)|\mathcal{F}_{3,1}(z)|^2$

$$\mathcal{F}_{\mathbb{I}}(z) = (1-z)^{1-\frac{3}{2\beta^2}} z^{1-\frac{3}{2\beta^2}} F\left(2-\frac{3}{\beta^2}, 1-\frac{1}{\beta^2}, 2-\frac{2}{\beta^2}, z\right) \qquad \mathcal{F}_{3,1}(z) = (1-z)^{1-\frac{3}{2\beta^2}} z^{\frac{1}{2\beta^2}} F\left(1-\frac{1}{\beta^2}, \frac{1}{\beta^2}, \frac{2}{\beta^2}, z\right)$$

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for the properly normalized $\hat{\Phi}_{2,1}$

$$\langle \hat{\Phi}_{2,1}\hat{\Phi}_{2,1}\hat{\Phi}_{2,1}\hat{\Phi}_{2,1}\rangle = (z\bar{z})^{-2h_{2,1}(c)}\left\{B^2_{\hat{\Phi}_{2,1}}(c) + \frac{B^2_{\hat{\Phi}_{2,1}}(c)h^2_{2,1}(c)}{c/2}(z^2+\bar{z}^2) + \frac{C^2_{\hat{\Phi}_{2,1}\hat{\Phi}_{2,1}T\bar{T}}(c)}{B_{T\bar{T}}(c)}(z\bar{z})^2 \right.$$
$$\left. + \frac{C^2_{\hat{\Phi}_{2,1}\hat{\Phi}_{2,1}\hat{\Phi}_{3,1}}(c)}{B_{\hat{\Phi}_{3,1}}(c)}(z\bar{z})^{h_{3,1}(c)} + \dots \right\}$$

vanish as c→0

correlation functions of Kac operators vanish at c=0 [Cardy, 2001]

Three-point functions of energy operator

using the results from previous analysis, consider cluster decomposition directly at c=0

generic c three-point functions

$$\langle \Phi_{2,1}(z_1,\bar{z}_1)\Phi_{2,1}(z_2,\bar{z}_2)X(z_3,\bar{z}_3)\rangle = 0$$

$$\langle \Phi_{2,1}(z_1,\bar{z}_1)\Phi_{2,1}(z_2,\bar{z}_2)T(z_3,\bar{z}_3)\rangle = h_{2,1}(c)\mathbb{P}_3^c$$

$$\begin{aligned}
\langle \Phi_{2,1}(z_1,\bar{z}_1)\Phi_{2,1}(z_2,\bar{z}_2)\Psi(z_3,\bar{z}_3)\rangle &= 0\,, \\
\langle \Phi_{2,1}(z_1,\bar{z}_1)\Phi_{2,1}(z_2,\bar{z}_2)T\bar{T}(z_3,\bar{z}_3)\rangle &= h_{2,1}^2(c)\mathbb{P}_3^c\,, \\
\langle \Phi_{2,1}(z_1,\bar{z}_1)\Phi_{2,1}(z_2,\bar{z}_2)\Phi_{3,1}(z_3,\bar{z}_3)\rangle &= C_{\Phi_{2,1}\Phi_{2,1}\Phi_{3,1}}(c)\mathbb{P}_3^c\,,
\end{aligned}$$

c=0 operators

$$\varepsilon^{\text{perco}} \equiv \hat{\Phi}_{2,1} \qquad t = b\left(\frac{T}{c/2} + \frac{\hat{X}}{B_{\hat{X}}(c)}\right) \qquad \Psi_2 = a\left(\frac{\hat{\Phi}_{3,1}}{B_{\hat{\Phi}_{3,1}}(c)} + \frac{T\bar{T}}{c^2/4} + \frac{\Psi}{\gamma_\Psi(c)}\right)$$

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$$\begin{aligned}\langle\Phi_{2,1}(z_1,\bar z_1)\Phi_{2,1}(z_2,\bar z_2)\Psi(z_3,\bar z_3)\rangle &= 0\,,\\ \langle\Phi_{2,1}(z_1,\bar z_1)\Phi_{2,1}(z_2,\bar z_2)T\bar T(z_3,\bar z_3)\rangle &= h_{2,1}^2(c)\mathbb{P}_3^c\,,\\ \langle\Phi_{2,1}(z_1,\bar z_1)\Phi_{2,1}(z_2,\bar z_2)\Phi_{3,1}(z_3,\bar z_3)\rangle &= C_{\Phi_{2,1}\Phi_{2,1}\Phi_{3,1}}(c)\mathbb{P}_3^c\,,\end{aligned}$$

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c→0:

$$\langle\varepsilon(z_1,\bar z_1)\varepsilon(z_2,\bar z_2)T(z_3)\rangle^{\text{perco}} = 0$$

$$\langle\varepsilon(z_1,\bar z_1)\varepsilon(z_2,\bar z_2)t(z_3,\bar z_3)\rangle^{\text{perco}} = C_{\varepsilon\varepsilon t}^{\text{perco}}\mathbb{P}_3^0$$

$$\begin{aligned}\langle\varepsilon(z_1,\bar z_1)\varepsilon(z_2,\bar z_2)\Psi_2(z_3,\bar z_3)\rangle^{\text{perco}} &= (C_{\varepsilon\varepsilon\Psi_2}^{\text{perco}}+C_{\varepsilon\varepsilon\Psi_1}^{\text{perco}}\tau_3)\mathbb{P}_3^0\,,\\ \langle\varepsilon(z_1,\bar z_1)\varepsilon(z_2,\bar z_2)\Psi_0(z_3,\bar z_3)\rangle^{\text{perco}} &= 0\,,\end{aligned}$$

$$C_{\varepsilon\varepsilon t}^{\text{perco}} = bh_\varepsilon^{\text{perco}} = -\frac{25}{8} \qquad C_{\varepsilon\varepsilon\Psi_1}^{\text{perco}} = \frac{a\left(h_\varepsilon^{\text{perco}}\right)^2}{b_{1,2}^{\text{perco}}} = -\frac{125}{512}$$

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vanishing three-point constants: reduced logarithm

simple OPE:

$$\varepsilon^{\text{perco}}(z,\bar z)\varepsilon^{\text{perco}}(0,0) = (z\bar z)^{-2h_\varepsilon^{\text{perco}}}\left\{z^2\frac{C_{\varepsilon\varepsilon t}^{\text{perco}}}{b}T + c.c. + \ldots + (z\bar z)^2\left(\frac{C_{\varepsilon\varepsilon\Psi_1}^{\text{perco}}}{a}\Psi_1 + \frac{C_{\varepsilon\varepsilon\Psi_1}^{\text{perco}}}{a}\ln(z\bar z)\Psi_0 + \frac{aC_{\varepsilon\varepsilon\Psi_2}^{\text{perco}} - a_1 C_{\varepsilon\varepsilon\Psi_1}^{\text{perco}}}{a^2}\Psi_0\right) + \ldots\right\}$$

Four-point function of energy operator

four-point function

$$
\begin{aligned}
&\langle \varepsilon(\infty,\infty)\varepsilon(1,1)\varepsilon(z,\bar{z})\varepsilon(0,0)\rangle^{\text{perco}} \\
&= \lim_{z_1,\bar{z}_1\to\infty} (z_1\bar{z}_1)^{2h_\varepsilon^{\text{perco}}} \langle \varepsilon(z_1,\bar{z}_1)\varepsilon(1,1)\varepsilon(z,\bar{z})\varepsilon(0,0)\rangle^{\text{perco}} \\
&= \sum_{\{\psi\}} \langle \varepsilon^{\text{perco}}|\varepsilon^{\text{perco}}(1,1)|\psi\rangle G^{-1} \langle\psi|\varepsilon^{\text{perco}}(z,\bar{z})|\varepsilon^{\text{perco}}\rangle
\end{aligned}
$$

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non-vanishing contribution

$$\langle \varepsilon^{\rm perco}|\varepsilon^{\rm perco}(1,1)|\Psi_1\rangle \frac{1}{a} \langle \Psi_1|\varepsilon^{\rm perco}(z,\bar{z})|\varepsilon^{\rm perco}\rangle$$

$$\langle \varepsilon(\infty)\varepsilon(1)\varepsilon(z,\bar{z})\varepsilon(0)\rangle^{\rm perco} = (z\bar{z})^{-2h_\varepsilon^{\rm perco}} \left(\frac{\left(C^{\rm perco}_{\varepsilon\varepsilon\Psi_1}\right)^2}{a} (z\bar{z})^2 + \dots \right)$$
$$= (z\bar{z})^{-2h_\varepsilon^{\rm perco}} \left\{ \frac{a\left(h_\varepsilon^{\rm perco}\right)^4}{\left(b_{1,2}^{\rm perco}\right)^2} (z\bar{z})^2 + \dots \right\}$$

- *non-vanishing*
- *non-logarithmic*

similarly for energy density in SAW

$$\langle \varepsilon(\infty)\varepsilon(1)\varepsilon(z,\bar{z})\varepsilon(0)\rangle^{\rm SAW} = (z\bar{z})^{-2h_\varepsilon^{\rm SAW}} \left(\frac{\left(C^{\rm SAW}_{\varepsilon\varepsilon\Psi_1}\right)^2}{a} (z\bar{z})^2 + \dots \right)$$

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cluster decomposition of Kac operators at c=0 (bottom of Jordan blocks) are not obtained by taking c→0 limit of generic c

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- algebraic aspect: what happens to the Virasoro degeneracy of energy density at $c=0$?

Conclusions

- “c→0 catastrophe” and resolutions
- OPE of spin operator probes rank-2 and rank-3 Jordan blocks, logarithmic four-spin correlators

[YH, Saleur, 2021]

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  analytic bootstrap ***real non-unitary CFTs*** ***Kac operators acquire zero norm at c=0***

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  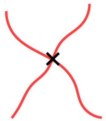

  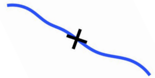

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- non-vanishing four-energy correlator

to understand: cluster decomposition of zero-norm operators & Virasoro degeneracy at c=0

Thank you !