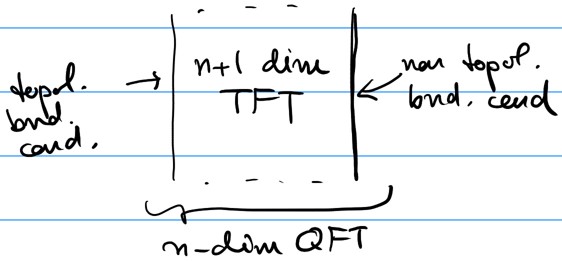
2d CFT via 3d TFT

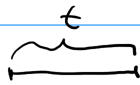
SymTFT **Idea**



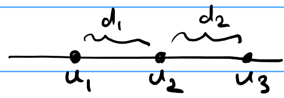
Some properties of n-dim QFT easier to treat via n+1 dim TFT.

Plan Here $n=2$ 2d QFT + top. sym. $\longrightarrow$ 3d TFT & non-top bnd.
specialise to rational CFT, $\longleftarrow$ Point out: The more interesting dir$^{\underline{n}}$ is this

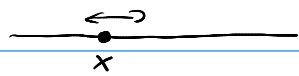
Warm-up: $n=1$ $\mathcal{H}$: state space
e^{-tH} : evolution $\mathcal{H} \longrightarrow \mathcal{H}$



0-dim operator $u \in \mathrm{End}(\mathcal{H})$



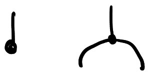
0-dim topol. defect:

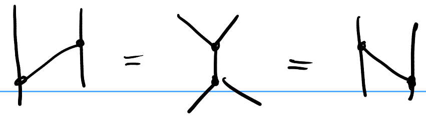


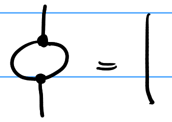
: $x \in \mathrm{End}(\mathcal{H})$
$[H, x] = 0$

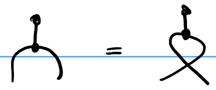
Pick finite collection $x_1, \dots, x_\ell$, close under fusion (= product)
$\rightsquigarrow$ assoc algebra
$$A = \mathrm{span}\{x_i\} \subset \mathrm{End}(\mathcal{H}), \quad [H, A] = \{0\}$$

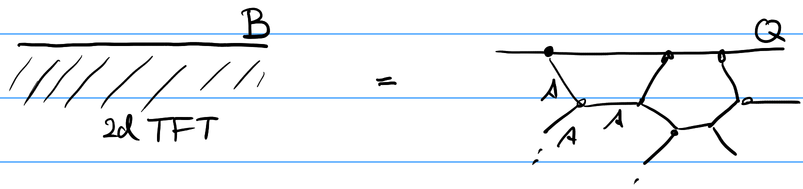
Assume A can be made into Δ-sep. sym. Frob. alg
Have unit $\eta: \mathbb{C} \to A$ **choose** $\varepsilon: A \to \mathbb{C}$ counit
prod. $\mu: A \otimes A \to A$ $\Delta: A \to A \otimes A$ coprod.



s.th.  $\rightsquigarrow$ Frobenius

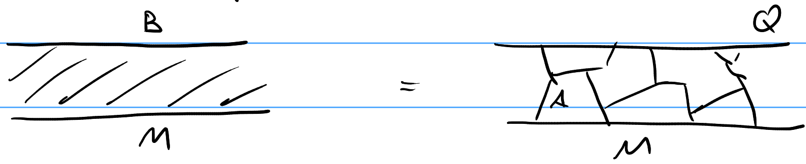
Δ-sep:  ($\Rightarrow A$ ssi)

sym.  ie $\varepsilon(ab) = \varepsilon(ba)$



$A = \bigoplus \mathbb{C} x_i$

topl. bnd $\hat{=}$ fin. dim. A-module



$M = A$: recover Q

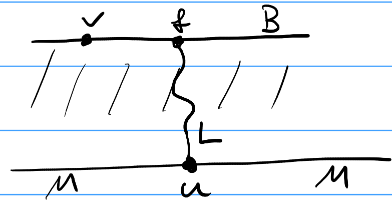
other M: other 1d QFT

Local op. on B: $v \in \mathrm{End}(\mathcal{H})$, $[v, A] = \{0\}$

Non-local op:

line defects

$\hat{=}$ ${}_A L_A$ fin. dim. bimodule



non-top

top

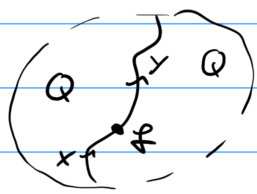
$f: L \to \mathrm{End}(\mathcal{H})$

Point out
- $\{H\}' \subset \mathrm{End}\,\mathcal{H}$ typically too big
- just choosing defects is not enough (here $\varepsilon: A \to \mathbb{C}$)
- new cond. (symm)
- obs = defects conn top & non-top bnd.

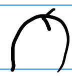
2d QFT with topol. def.

Topol. def. lines of 2d QFT Q $\rightsquigarrow$ pivotal monoidal cat. $\mathcal{E}(Q)$ "endo-defects of Q"

obj $x \in \mathcal{E}(Q)$



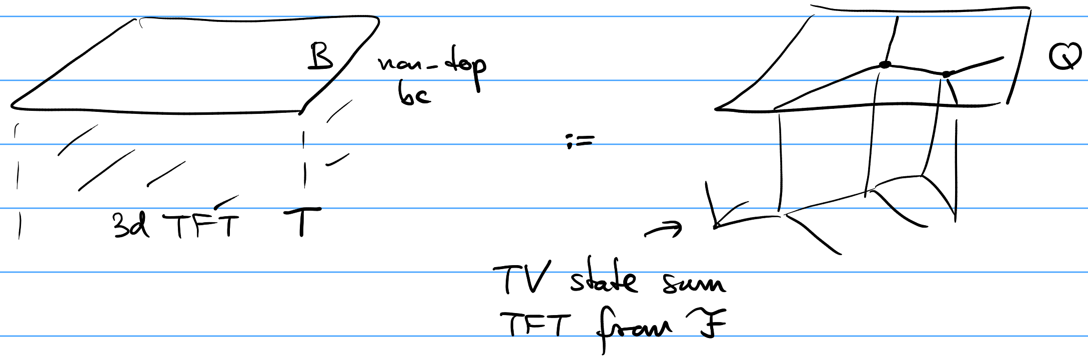
morph $f: x \to y$

$\otimes$: fusion of TDL, pivotal: "  " and $\bigcirc^{*}$

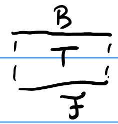
Select finite set $x_1, \dots, x_e \in \mathcal{E}(Q)$, closes under fusion

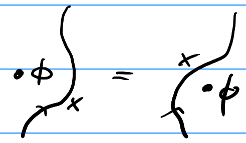
$\mathcal{F} \subset \mathcal{E}(Q)$ subcat, closed under $\otimes$ & duals

<u>Assume</u> $\mathcal{F}$ is spherical fusion category (in part. ssi)



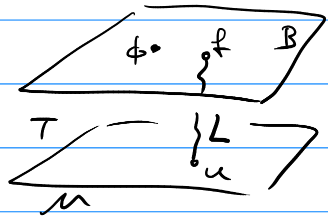
topol. bnd. cond. for T = ssi module category $\mathcal{M}$ for $\mathcal{F}$ with module trace
"$\mathcal{F} \times \mathcal{M} \to \mathcal{M}$ functor s.th..."

take $\mathcal{F}$-module $\mathcal{F}$ ⇝  recovers Q

local op. on B : ϕ field of Q s.th. 

for all $x \in \mathcal{F}$

non-local op.



line op in TV TFT :
$L \in Z(\mathcal{F})$
"Drinfeld centre"

<u>Example</u> Q : 2d Ising CFT $(c = \frac{1}{2})$

Local fields $\mathcal{H} = \underbrace{\overset{Vir}{M_0} \otimes_{\mathbb{C}} \overset{\overline{Vir}}{M_0}}_{\in \mathbb{1}} \oplus \underbrace{M_{\frac{1}{2}} \otimes M_{\frac{1}{2}}}_{\in \varepsilon} \oplus \underbrace{M_{\frac{1}{16}} \otimes M_{\frac{1}{16}}}_{\in \sigma}$

"unit" "energy" "spin"

TDLs : $\underset{\text{id}}{\mathbb{1}}$, $\underset{\mathbb{Z}_2\text{-sym}}{\varepsilon}$, $\underset{\text{KW-duality}}{\sigma}$

$\mathcal{F} = \langle \mathbb{1} \rangle$ TFT trivial
local on $\mathcal{B}$: all of $\mathcal{H}$

$\mathcal{F} = \langle \mathbb{1}, \varepsilon \rangle$ TFT : Kitaev's toric code TC
Line-op : $1, e, m, f$ $e \otimes m \cong f$
local on $\mathcal{B}$: $\langle 1, \varepsilon \rangle = M_1 \otimes M_1 \oplus M_\varepsilon \otimes M_\varepsilon$

$\mathcal{F} = \langle \mathbb{1}, \varepsilon, \sigma \rangle$ TFT : TV for Ising fusion cat
$\cong \underbrace{\mathrm{Is} \boxtimes \overline{\mathrm{Is}}}_{9 \text{ line ops}}$, $\mathrm{Is} = \mathrm{Rep}\, V_{\mathrm{vir}, c=1/2}$

local on $\mathcal{B}$: $\underbrace{M_1 \otimes M_1}_{\text{factorises in hol} \otimes \text{antihol}}$

For rational 2d CFT with

- $\underset{\text{hol}}{V} \otimes \underset{\text{antihol}}{V} \subset \mathcal{H}$, V : rational VOA
- unique vacuum ($\dim \mathrm{Hom}_{V\otimes V}(V \otimes V, \mathcal{H}) = 1$)
- non-deg. 2pt corr. on S^2

$\Bigg\}$ (*)

have :

1) For $\mathcal{F}$ = "all TDL transparent to $V \otimes V$" local on $\mathcal{B}$ are $V \otimes V$
$\rightsquigarrow$ factorises !

2) 2d CFTs (*) $\xleftrightarrow{1:1}$ simple bnd. cond. to 3d TFT for $\mathcal{F}$
($\xleftrightarrow{1:1}$ indec. ssi $\mathcal{F}$-module cat. $\mathcal{M}$)

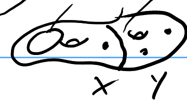
Chiral CFT as bnd of 3d TFT

V : rational VOA

$\mathcal{C} := \operatorname{Rep} V$ rep^{n}s of V, a ssi braided tensor cat (non-deg)
→ modular fusion cat.

RT TFT for $\mathcal{C}$: $Z_{\mathcal{C}} : \mathrm{Bord}^{\chi}_{3,2}(\mathcal{C}) \longrightarrow \mathrm{Vect}$

$M_1 \quad M_2 \in \mathcal{C}\text{-mod}^{tr}$

obj :  $z \in \mathcal{C}$

$x \quad y$

morph : $\Sigma \xrightarrow{M} \Sigma'$

e.g.  : $S^2_{x,y,y^*} \to S^2_{x,M}$

M

$Z_c($  $) \cong$ v.sp. of conformal blocks for V with insertions $X, Y, Z \in \operatorname{Rep} V$

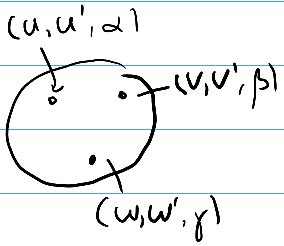
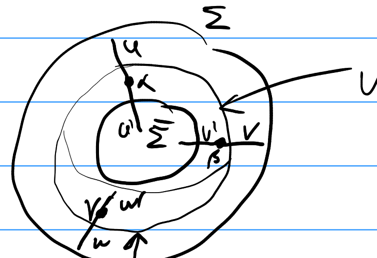
CFT corr. on Σ ($\partial\Sigma = \emptyset$) $\quad \gamma_\Sigma \in$ conf. bl. $(\Sigma \sqcup -\Sigma)$

$\cong$

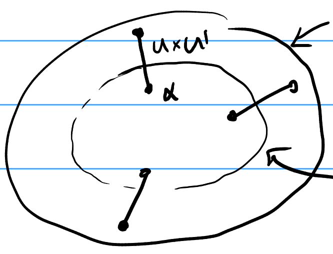
$Z_c(\Sigma \sqcup -\Sigma)$

<u>Idea</u> : $\gamma_\Sigma := Z_c(\emptyset \xrightarrow{M} \Sigma \sqcup -\Sigma) \in$

$\mathbb{C} \longrightarrow Z_c(\Sigma \sqcup -\Sigma)$

e.g. $\Sigma =$  $\quad \gamma_\Sigma = Z_c($  $)$

Fold:



Facts : $Z_C(\Sigma_\omega - \Sigma) \simeq Z_{C \boxtimes \bar{C}}(\Sigma) \simeq Z^{TV}_C(\Sigma)$

surf def. in Z_C = $\mathcal{C}$-module (with tr.)

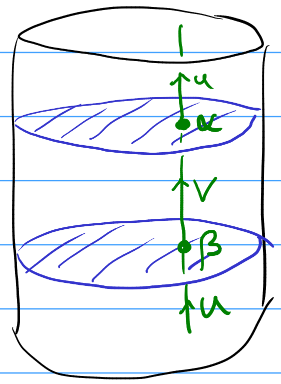
= top. bnd cond in $Z^{TV}_C(\Sigma)$

Check: Only $1 \otimes 1$ commutes with all TDLs

Use invertibility of defect S-matrix.

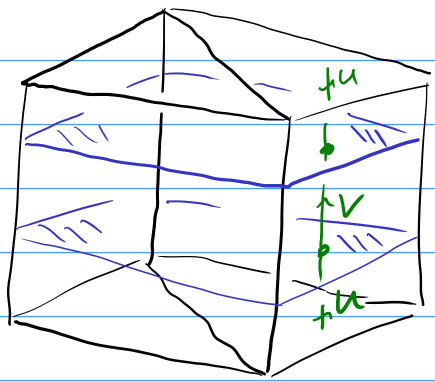
In $S^2 \times S^1$

oppos. orient

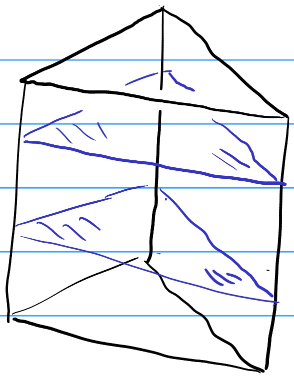


$$= \overbrace{\delta_{u,1}}^{\text{1-pt on } S^2} \delta_{v,1} \overbrace{\delta_{\alpha,1}}^{\text{for simple } A \text{ only 1-dim space anyway}} \delta_{\beta,1} \cdot (\text{same pic})$$

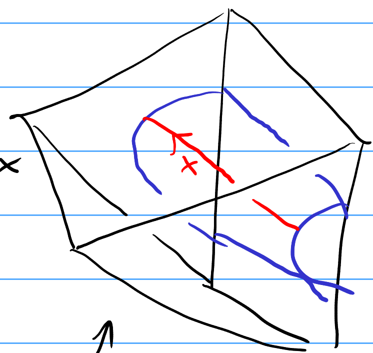
wedge presentation



Claim

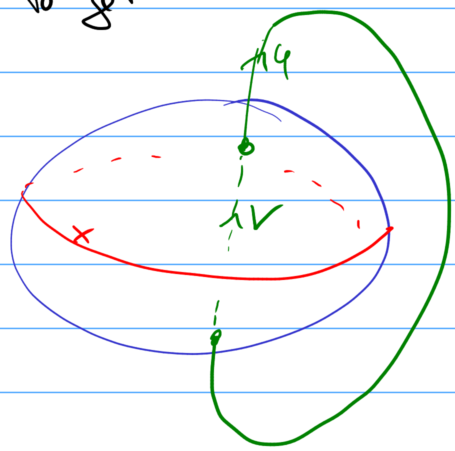


$$= \sum_{x \in F} \dim_{\mathcal{M}} x$$



compose both sides with reflected and use that rhs is δ_{xy}

Replace in top identity to get

$$\delta_{u1}\,\delta_{v1}\,(\text{const}) = \sum_x$$  $\cdot \dim x$

but if (u, v, x) commutes, then ↑ gives the sphere 2pt fun., which is non-deg.