



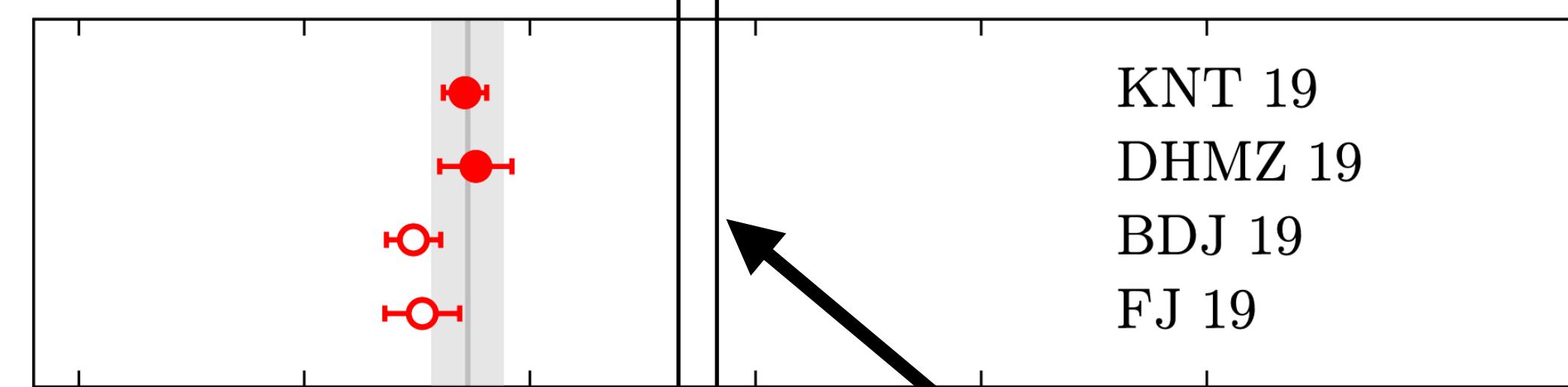
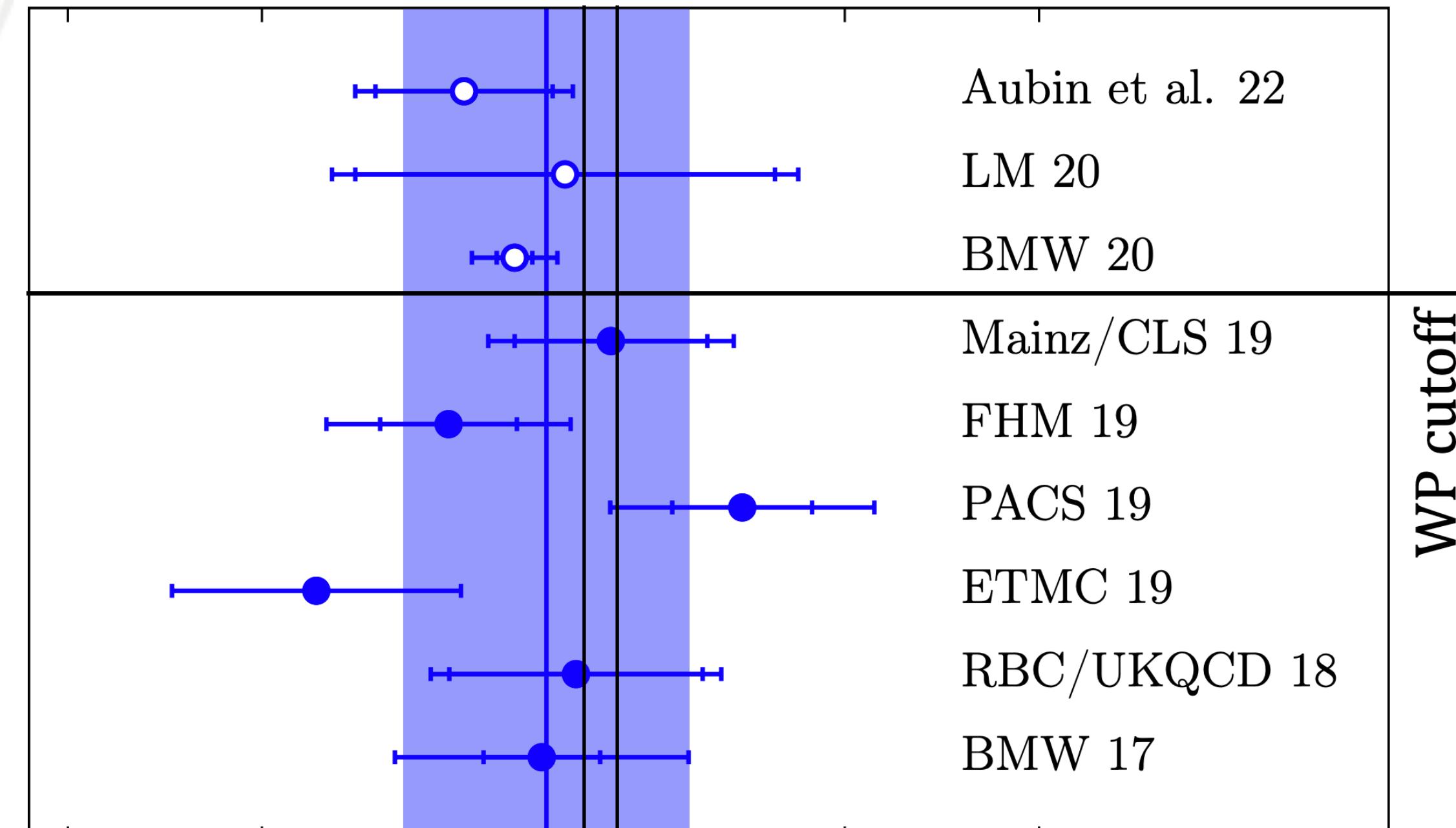
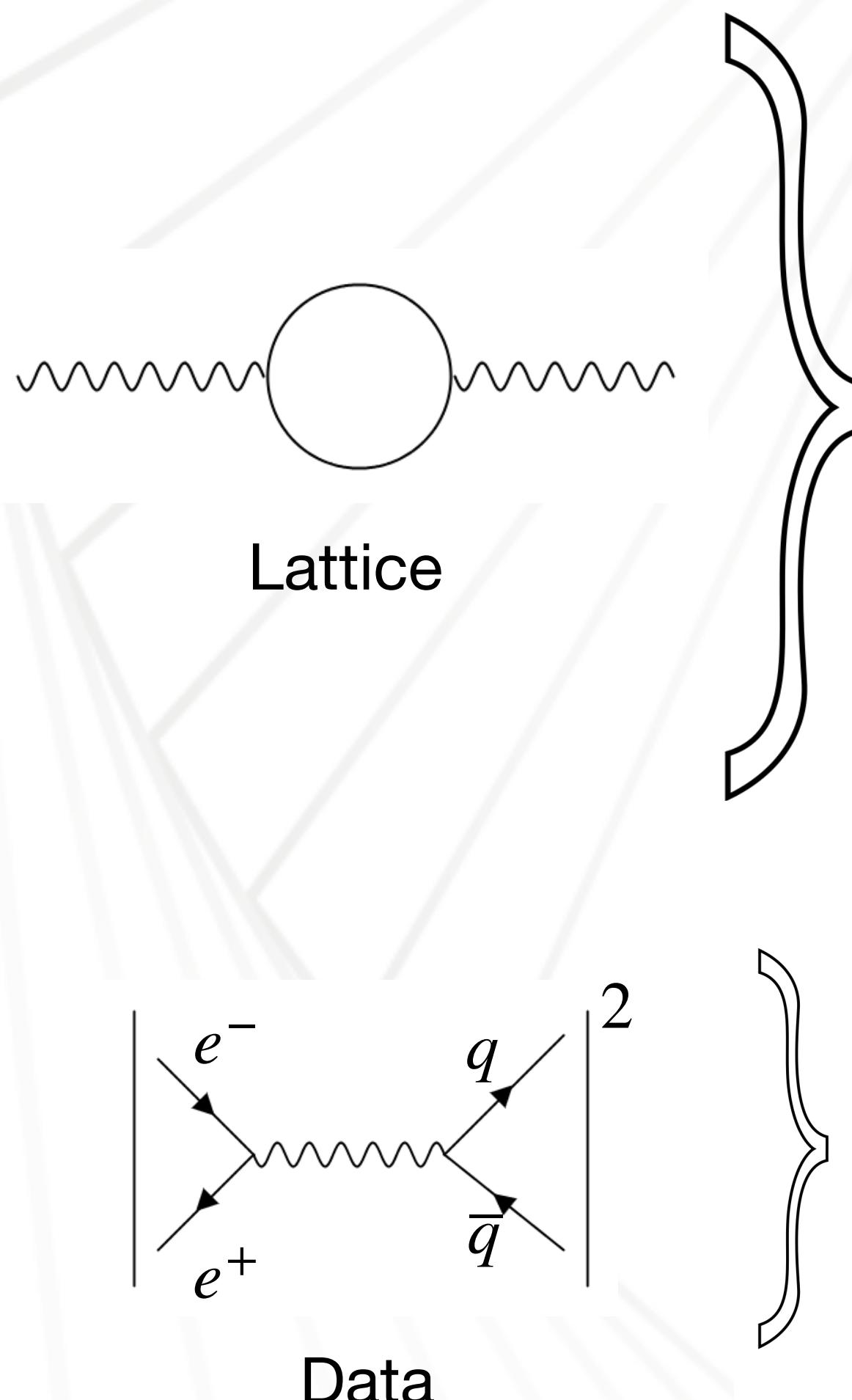
JOHANNES GUTENBERG
UNIVERSITÄT MAINZ

Electroweak Precision Running of $\hat{\alpha}$ and $\sin^2 \hat{\theta}$

Moriond 2024

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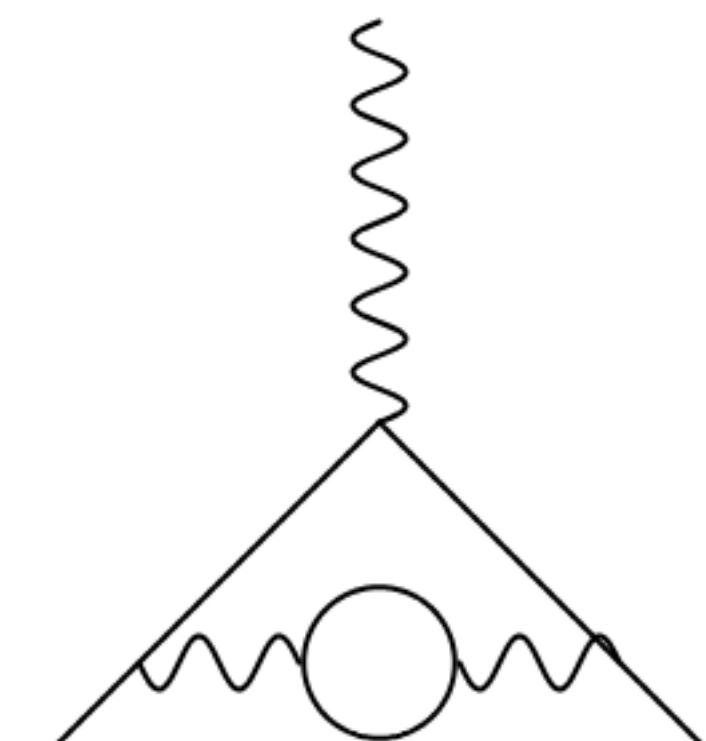
g-2 tension



Wittig 2306.04165

$$a_\mu^{\text{hyp}} \cdot 10^{10}$$

Experimental
measurement
BNL+Fermilab



Quick basics

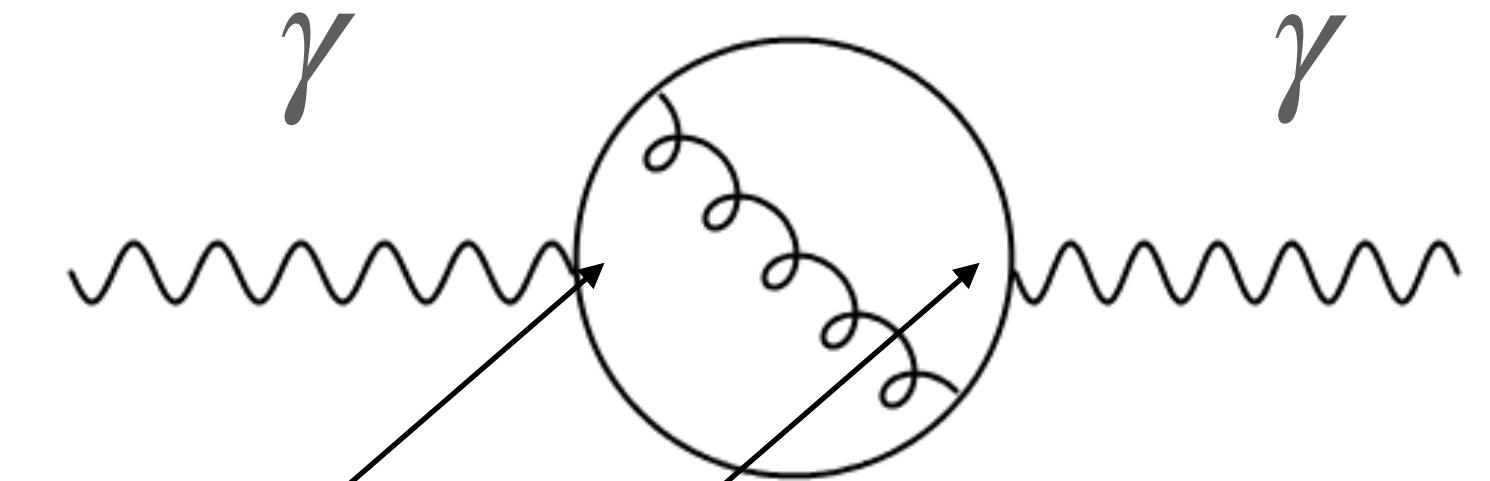
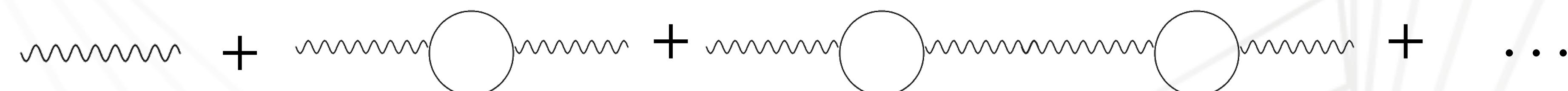
The vacuum polarization function is defined as

$$(-q^2\eta^{\mu\nu} + q^\mu q^\nu) \hat{\Pi}(q^2, \mu^2) = i \int d^4x e^{iqx} \langle 0 | T J_{em}^\mu(x) J_{em}^\nu(0) | 0 \rangle$$

The running coupling $\hat{\alpha}(\mu^2)$ is constructed to absorb the large logarithms that appear in this expression. It is given by:

$$\hat{\alpha}(\mu^2) = \frac{\alpha}{1 - \Delta\hat{\alpha}(\mu^2)}$$

$$\Delta\hat{\alpha}(\mu^2) \equiv 4\pi\alpha\hat{\Pi}(0, \mu^2)$$



$\hat{\alpha}$: a key parameter of the SM

Rel error $\sim 10^{-10}$

$$\hat{\alpha}(0)$$

$\hat{\alpha}$: a key parameter of the SM

Rel error $\sim 10^{-10}$

$e^+e^- \rightarrow had$ o lattice + pQCD

$$\hat{\alpha}(0) \xrightarrow{\hspace{30em}} \hat{\alpha}(M_Z^2)$$

$\hat{\alpha}$: a key parameter of the SM

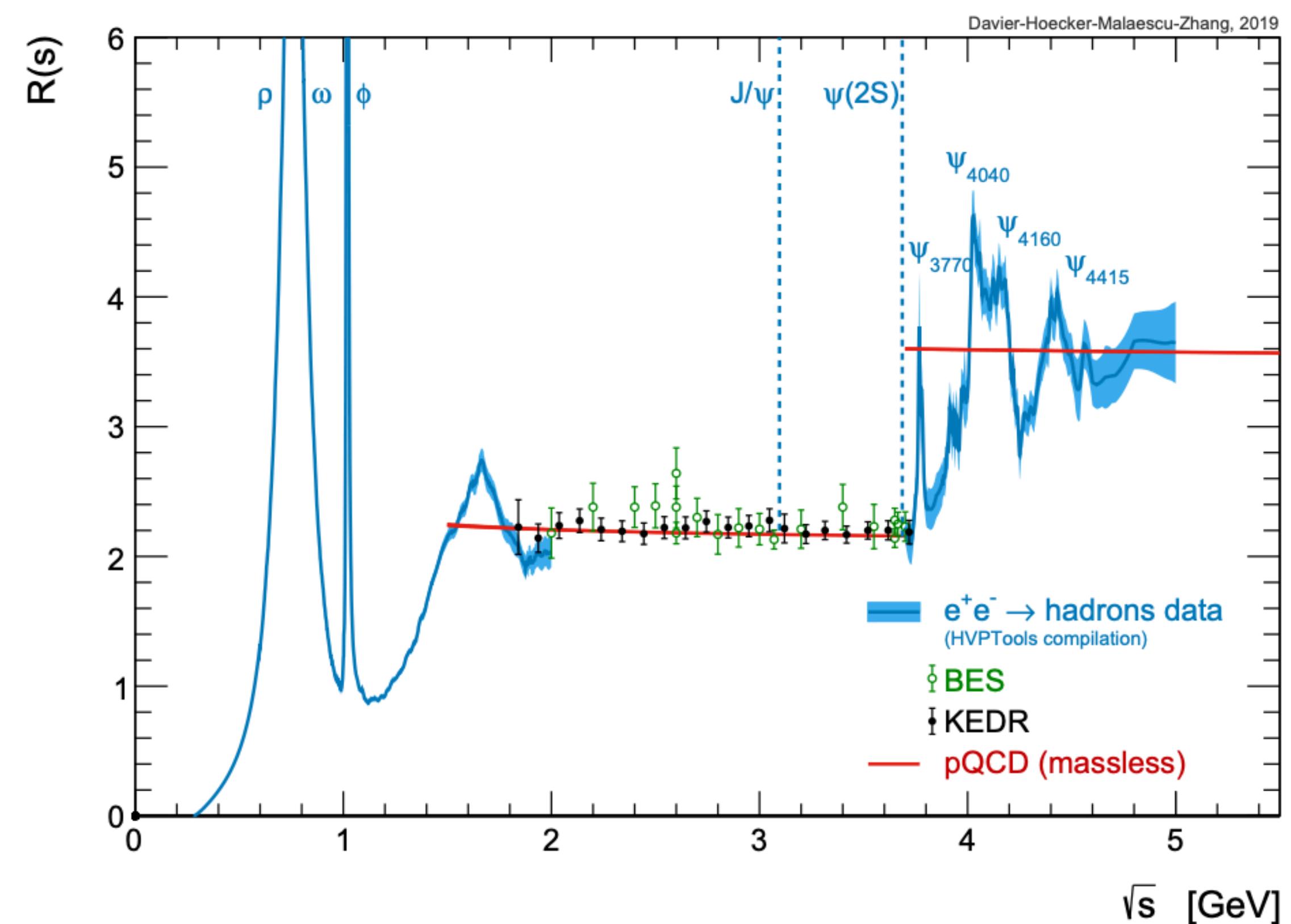
Rel error $\sim 10^{-10}$

$\hat{\alpha}(0)$

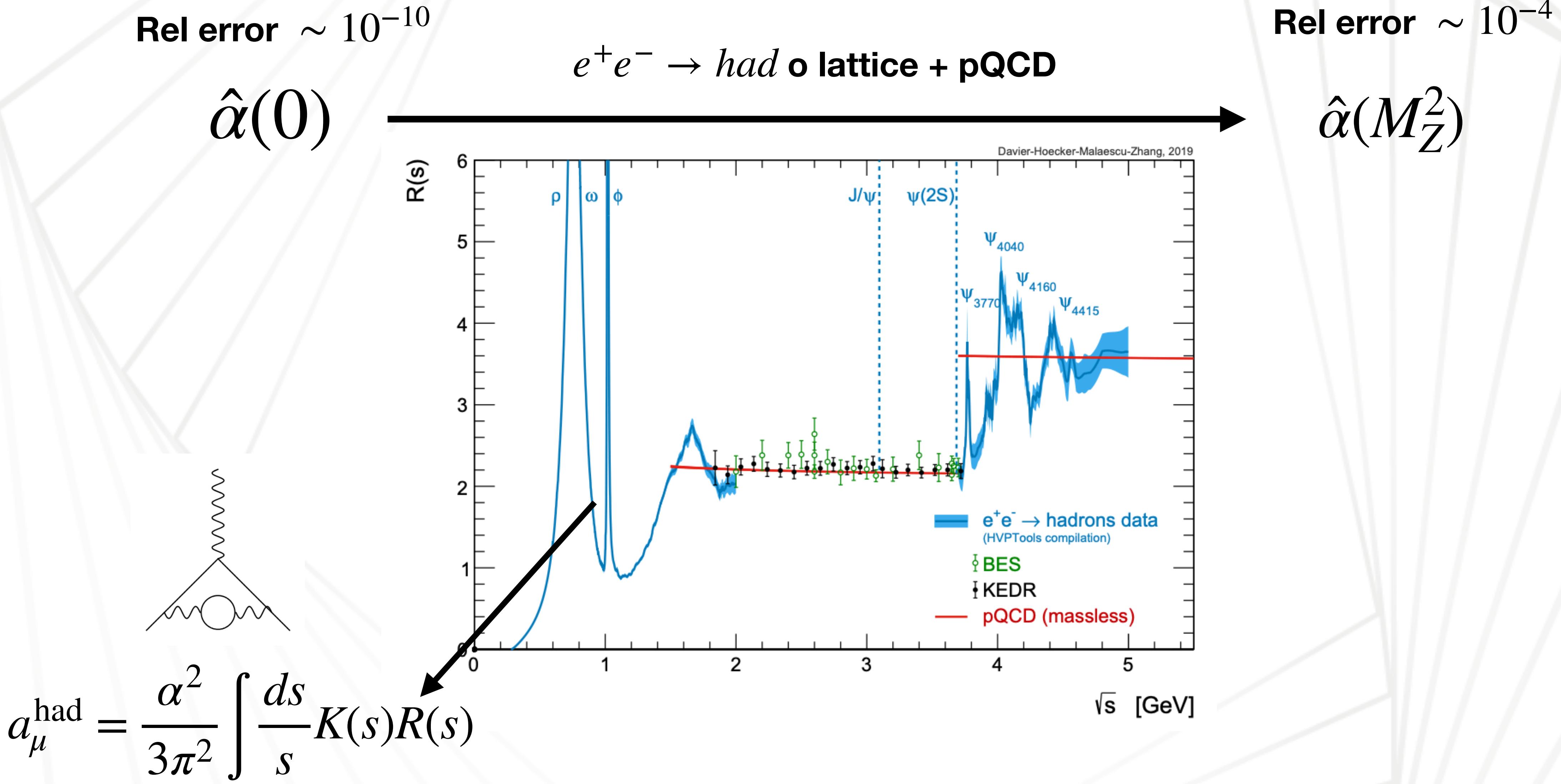
$e^+e^- \rightarrow \text{had}$ o lattice + pQCD

Rel error $\sim 10^{-4}$

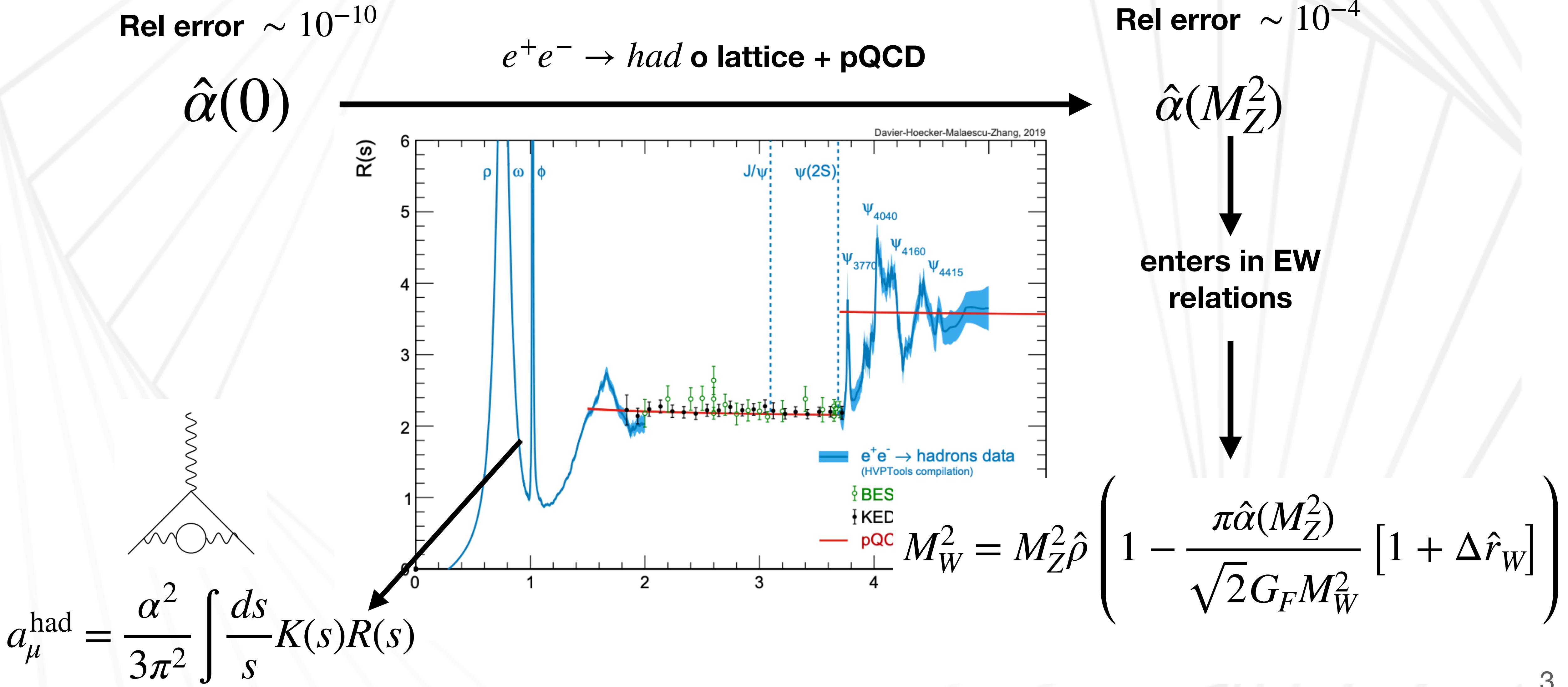
$\hat{\alpha}(M_Z^2)$



$\hat{\alpha}$: a key parameter of the SM

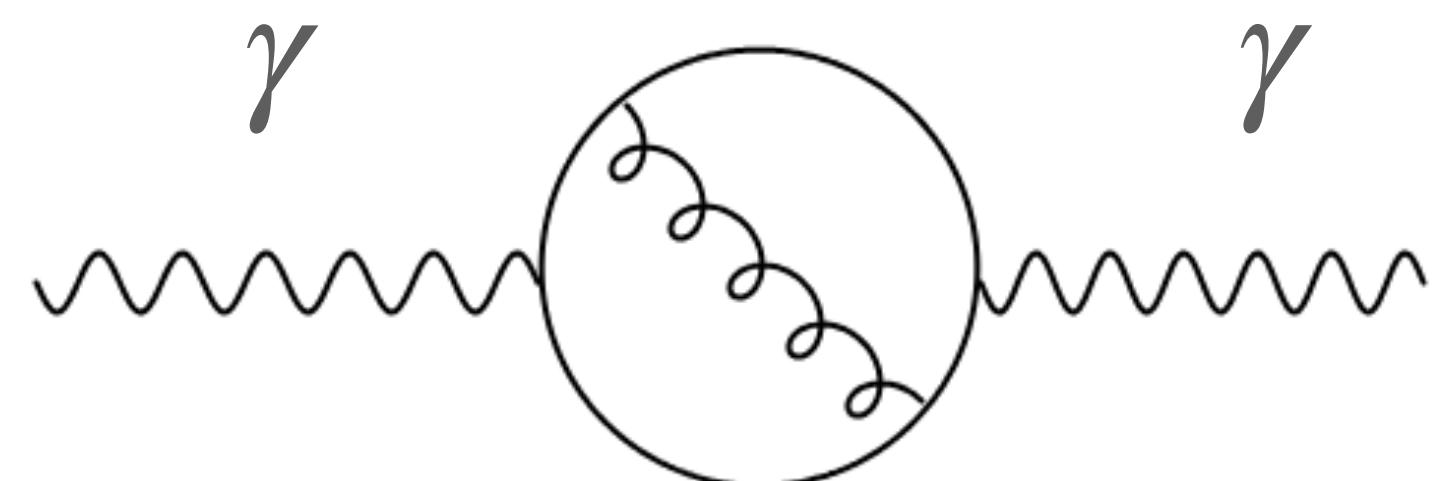
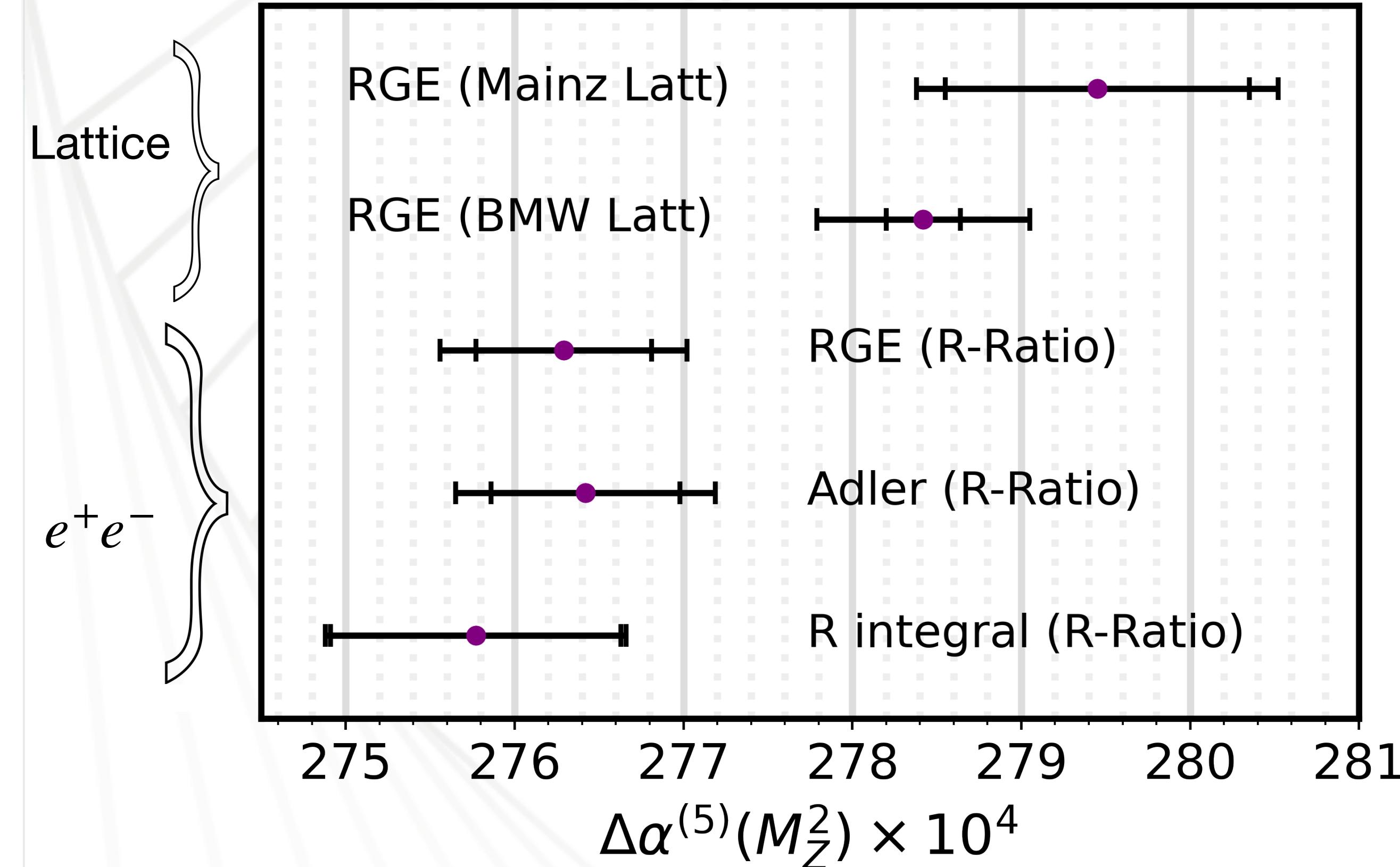


$\hat{\alpha}$: a key parameter of the SM



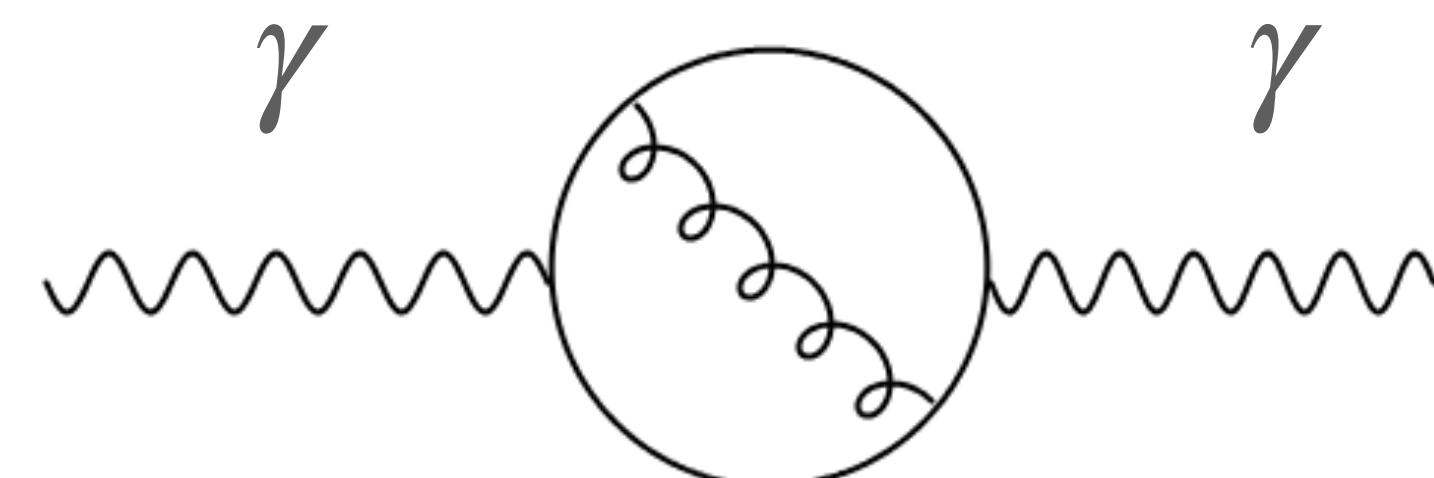
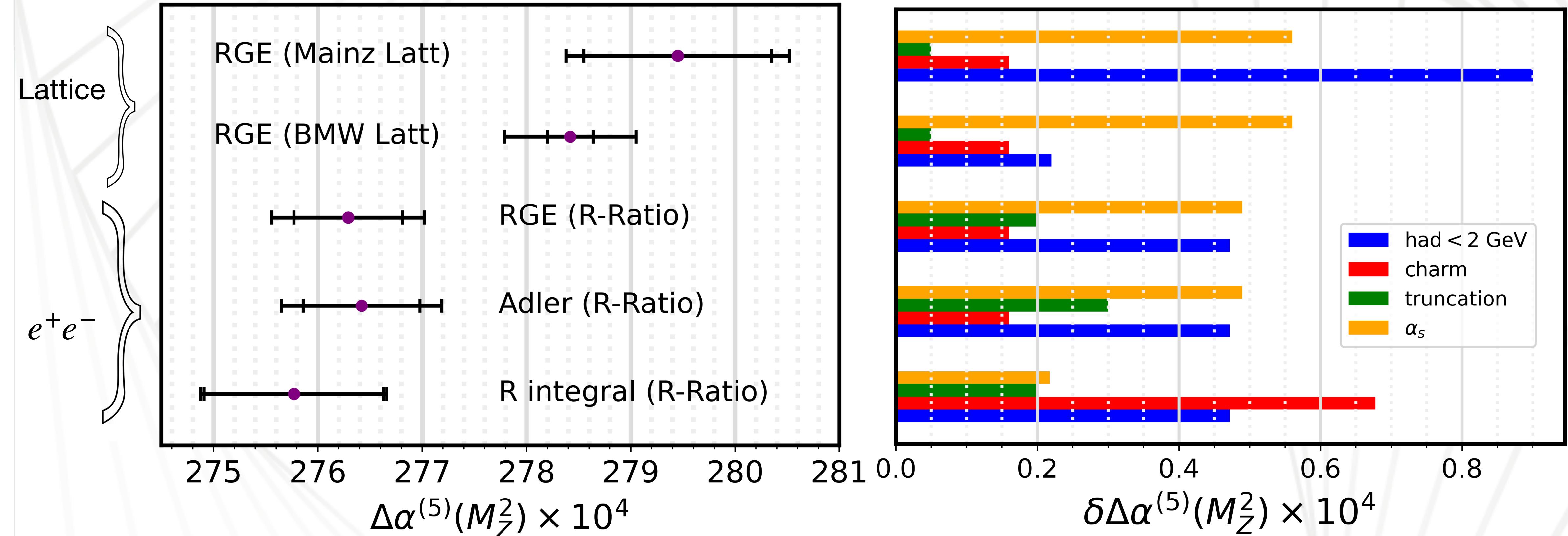
Running $\hat{\alpha}$ comparison

Erler, Ferro-Hernandez, [10.1007/JHEP12\(2023\)131](https://doi.org/10.1007/JHEP12(2023)131)



Running $\hat{\alpha}$ comparison

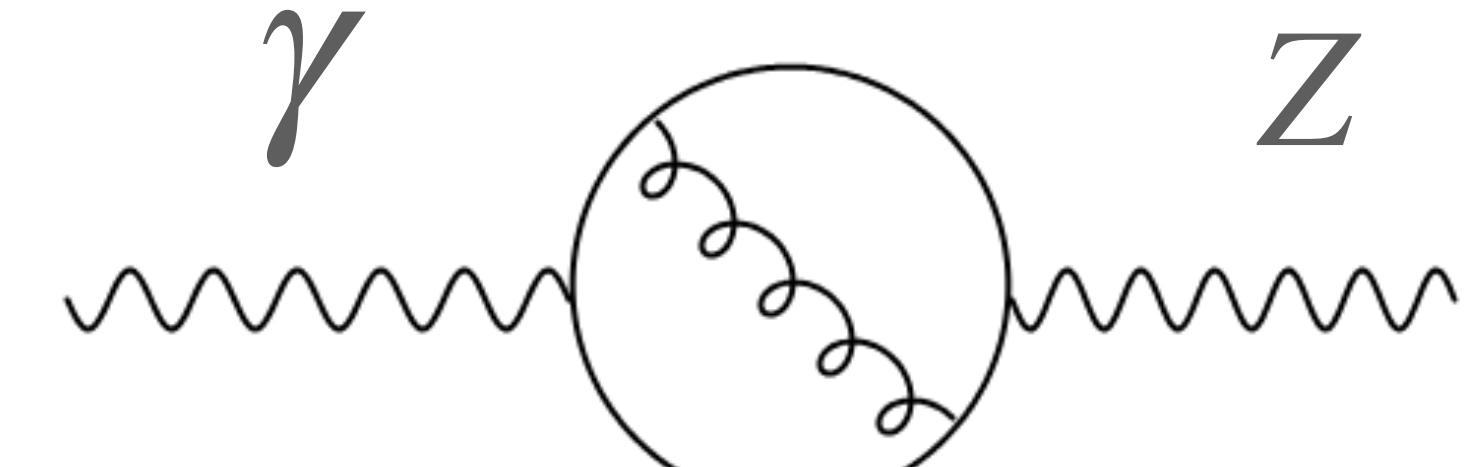
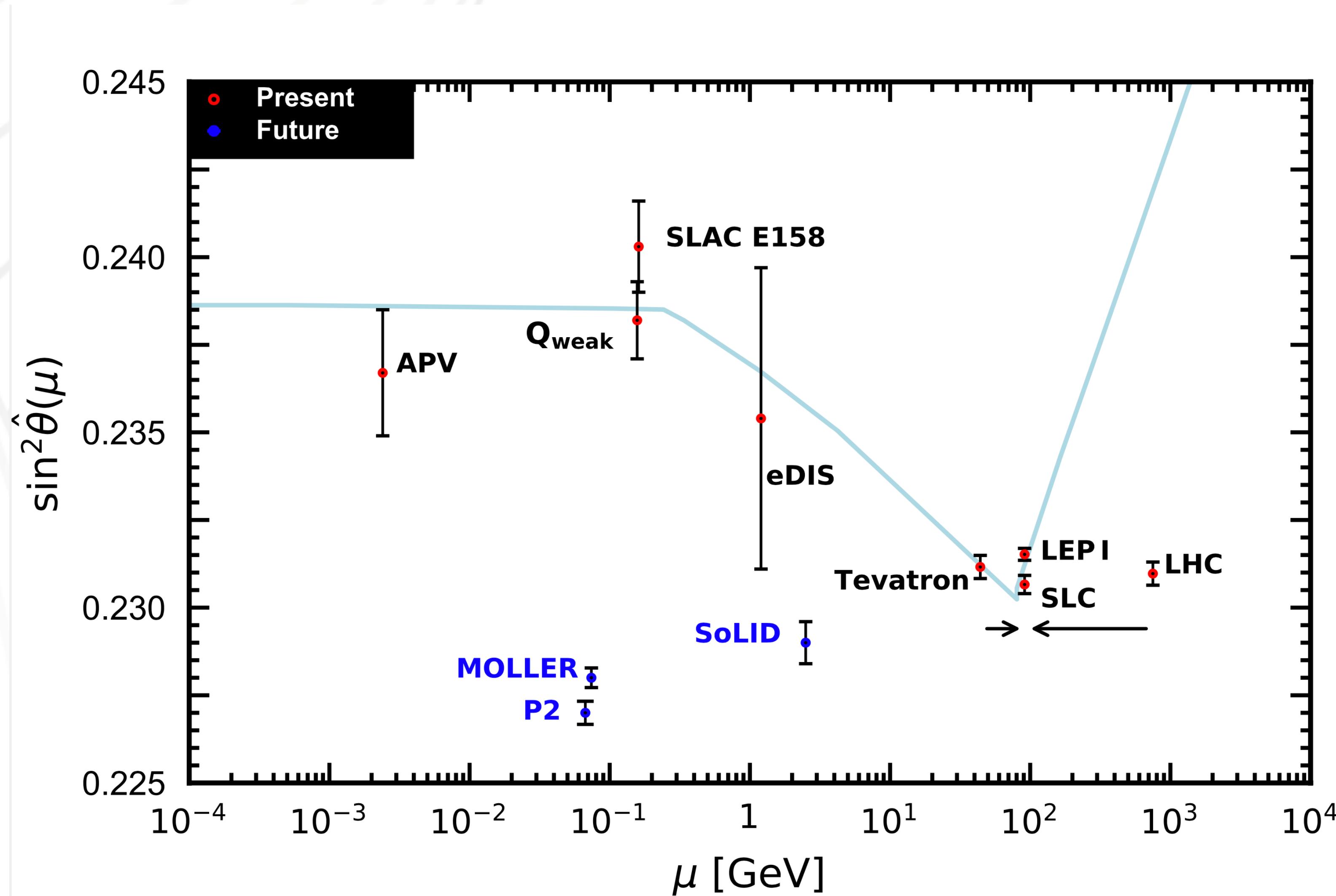
Erler, Ferro-Hernandez, [10.1007/JHEP12\(2023\)131](https://doi.org/10.1007/JHEP12(2023)131)



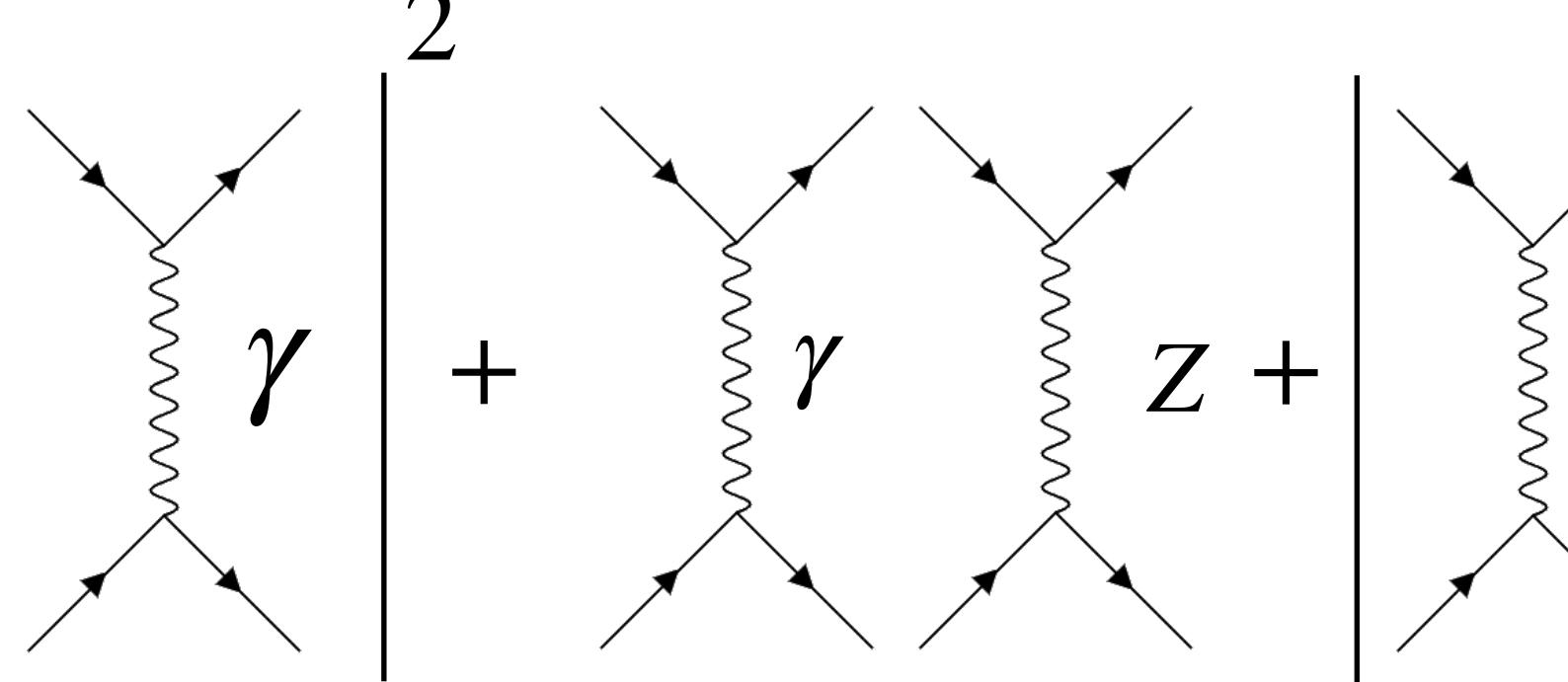
$$\delta\hat{\alpha}_s = 0.0016$$

$$\delta\hat{m}_c = 0.008 \text{ GeV}$$

Low energy Parity Violation



PV asymmetry example: ep scattering

$$A_{PV} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = \left| \gamma^2 + \gamma Z + Z^2 \right|$$


The equation shows the expression for the PV asymmetry in terms of cross sections. The right side of the equation consists of two vertical bars enclosing Feynman diagrams. The first bar contains a diagram of a photon (γ) exchange between an electron and a proton, with arrows indicating fermion flow. The second bar contains a diagram of a Z boson exchange between an electron and a proton, also with fermion flow arrows. The terms are separated by plus signs.

PV asymmetry example: ep scattering

$$A_{PV} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = \left| \begin{array}{c} \text{Parity} \\ \text{Conserving} \end{array} \right|^2 + \left| \begin{array}{c} \gamma \\ Z \end{array} \right|^2 + \left| \begin{array}{c} Z \\ Z \end{array} \right|^2$$

The equation shows the expression for the parity-violating asymmetry A_{PV} as a ratio of left-handed (σ_L) and right-handed (σ_R) cross sections. It is decomposed into three terms: a parity-conserving term (labeled γ), a term involving a virtual photon (γ) and a virtual Z boson (Z), and a term involving two virtual Z bosons (Z). The parity-conserving term is enclosed in vertical bars with a red diagonal line through them, indicating it is zero. The other terms are also enclosed in vertical bars.

PV asymmetry example: ep scattering

$$A_{PV} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = \left| \begin{array}{c} \text{Parity} \\ \text{Conserving} \end{array} \right|^2 + \left| \begin{array}{c} \gamma \\ Z \end{array} \right|^2 + \left| \begin{array}{c} Z \\ Q^2/M_Z^2 \end{array} \right|^2$$

The equation shows the expression for the parity-violating asymmetry A_{PV} as a ratio of left-handed (σ_L) and right-handed (σ_R) cross sections. It is decomposed into three terms: 1) Parity Conserving (γ exchange), 2) Z exchange, and 3) Z exchange suppressed by Q^2/M_Z^2 . The diagrams illustrate the contributions of each term.

Parity Conserving

γ

Z

Q^2/M_Z^2

Suppressed

PV asymmetry example: ep scattering

$$A_{PV} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = \left| \begin{array}{c} \text{Parity} \\ \text{Conserving} \end{array} \right|^2 + \left| \begin{array}{c} \gamma \\ \text{Z} \end{array} \right|^2 + \left| \begin{array}{c} Z \\ Q^2/M_Z^2 \end{array} \right|^2 = \frac{G_F Q^2}{4\pi\alpha\sqrt{2}} [Q_W^p - F(E_i, Q^2)]$$

The equation shows the calculation of the parity-violating asymmetry A_{PV} as a sum of contributions from different channels. The first term is the parity-conserving contribution, which is zero. The second term is the photon (γ) contribution, and the third term is the Z boson (Z) contribution. The Z boson contribution is suppressed by Q^2/M_Z^2 . The final result is proportional to the Fermi coupling constant G_F , the square of the momentum transfer Q^2 , and the function $F(E_i, Q^2)$.

PV asymmetry example: ep scattering

$$A_{PV} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = \left| \text{Parity Conserving} \right|^2 + \left| \text{Suppressed} \right|^2 = \frac{G_F Q^2}{4\pi\alpha\sqrt{2}} [Q_W^p - F(E_i, Q^2)]$$

Diagram illustrating the contribution of different channels to the parity-violating asymmetry:

- Parity Conserving: Represented by a red diagonal line connecting two vertices. The label γ is shown near the vertex on the right.
- Suppressed: Represented by a vertical line connecting two vertices. The label Z is shown near the vertex on the right.

The expression shows the sum of the squares of the amplitudes for these channels, equated to the form factor term $\frac{G_F Q^2}{4\pi\alpha\sqrt{2}} [Q_W^p - F(E_i, Q^2)]$.

↑
Form factors

PV asymmetry example: ep scattering

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At higher orders....

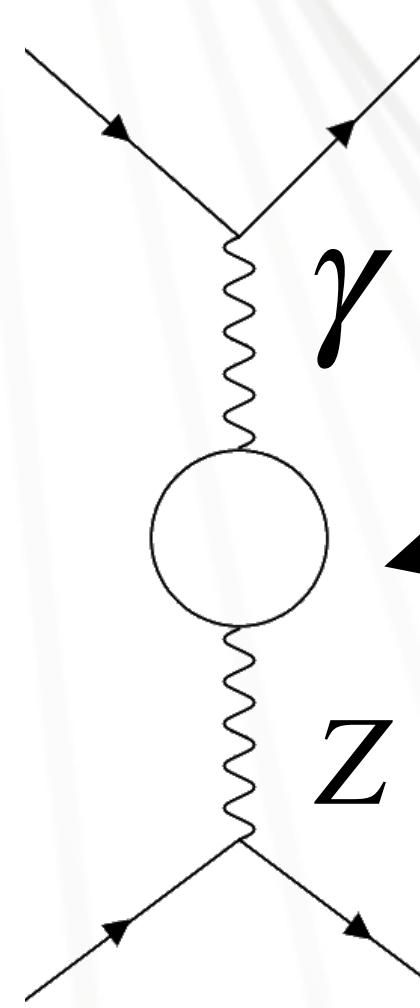
Form factors

The equation shows the calculation of the parity-violating asymmetry A_{PV} as the ratio of left-handed (σ_L) and right-handed (σ_R) cross sections. It is decomposed into two terms: a parity-conserving term involving a virtual photon (γ) exchange and a suppressed term involving a virtual Z boson (Z) exchange. The parity-conserving term is labeled Q^2/M_Z^2 . The suppressed term is labeled Q^2/M_Z^2 . The final result is proportional to the Fermi coupling constant G_F , the square of the fine-structure constant Q^2 , and the ratio of the parity-conserving form factor Q_W^p to the suppressed form factor $F(E_i, Q^2)$.

PV asymmetry example: ep scattering

$$A_{PV} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = \left| \begin{array}{c} \text{Parity Conserving} \\ \gamma \end{array} \right|^2 + \left| \begin{array}{c} \text{Suppressed} \\ Z \end{array} \right|^2 = \frac{G_F Q^2}{4\pi\alpha\sqrt{2}} \left[Q_W^p - F(E_i, Q^2) \right]$$

↑
Form factors



$$Q_W^p = 1 - 4 \sin^2 \theta_W(0) + \text{vacuum} + \text{vertex} + \text{box} + \text{higher order}$$

At higher orders....

$$g_{Vf} = T_f - 2Q_f^2 \sin^2 \hat{\theta}$$

Huge effect, 40% reduction of the asymmetry!
Important to resum large logs

Use $\hat{\alpha}$ to compute $\sin^2 \hat{\theta}_W \equiv \hat{s}^2$ ($\overline{\text{MS}}$ scheme)

Erler, *Phys. Rev. D* 72 (2005) 073003

$$\hat{s}^2(\mu) = \hat{s}^2(\mu_0) \frac{\hat{\alpha}(\mu)}{\hat{\alpha}(\mu_0)} + \lambda_1 \left[1 - \frac{\hat{\alpha}(\mu)}{\hat{\alpha}(\mu_0)} \right] + \frac{\hat{\alpha}(\mu)}{\pi} \left[\frac{\lambda_2}{3} \ln \frac{\mu^2}{\mu_0^2} + \frac{3\lambda_3}{4} \ln \frac{\hat{\alpha}(\mu)}{\hat{\alpha}(\mu_0)} + \tilde{\sigma}(\mu_0) - \tilde{\sigma}(\mu) \right]$$

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Requirements

$\hat{\alpha}(M_Z)$ from α

$\sin^2 \hat{\theta}(0)$ from $\sin^2 \hat{\theta}(M_Z)$

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pQCD



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Requirements

$\hat{\alpha}(M_Z)$ from α

pQCD



Total HVP



$\sin^2 \hat{\theta}(0)$ from $\sin^2 \hat{\theta}(M_Z)$



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Requirements

$\hat{\alpha}(M_Z)$ from α



pQCD

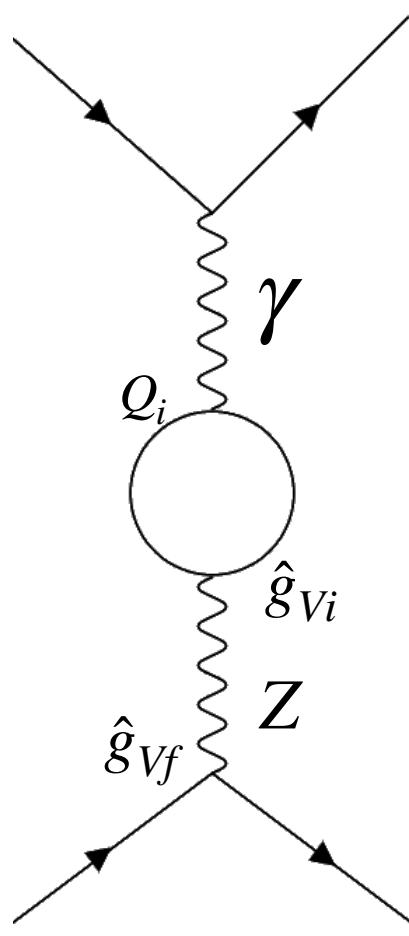


Total HVP



Flavor Separation

$\sin^2 \hat{\theta}(0)$ from $\sin^2 \hat{\theta}(M_Z)$



Use $\hat{\alpha}$ to compute $\sin^2 \hat{\theta}_W \equiv \hat{s}^2$ ($\overline{\text{MS}}$ scheme)

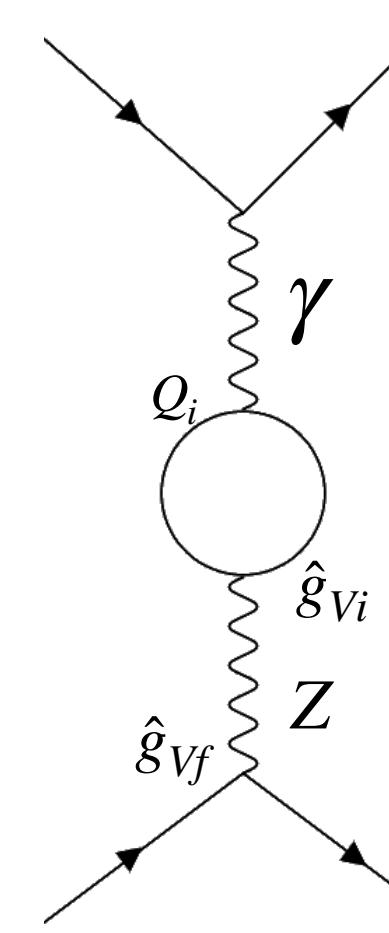
Erler, *Phys. Rev. D* 72 (2005) 073003

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Requirements

	$\hat{\alpha}(M_Z)$ from α
pQCD	✓
Total HVP	✓
Flavor Separation	✗
Using cross section data	Erler, Ferro-Hernandez, 10.1007/JHEP03(2018)196

$\sin^2 \hat{\theta}(0)$ from $\sin^2 \hat{\theta}(M_Z)$



Inferred from “s channels” like the ϕ
Erler, Ferro-Hernandez, [10.1007/JHEP03\(2018\)196](#)

Theory error is negligible compared with future experiments

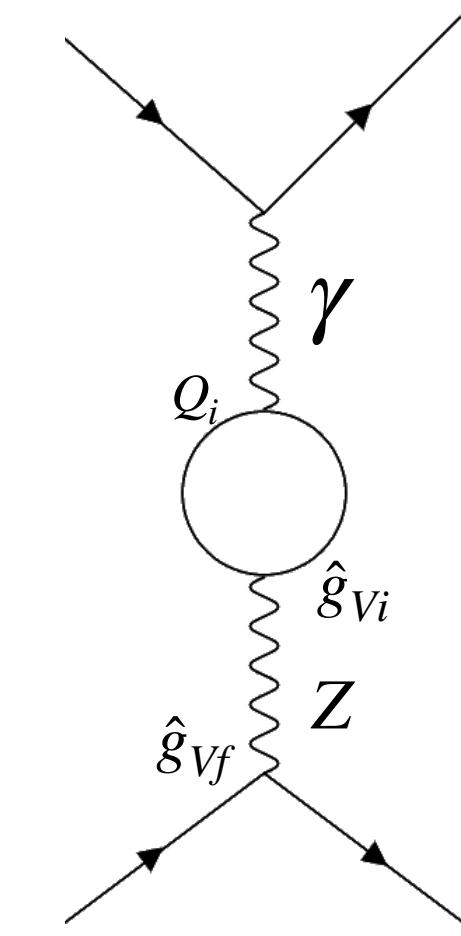
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Requirements

	$\hat{\alpha}(M_Z)$ from α	$\sin^2 \hat{\theta}(0)$ from $\sin^2 \hat{\theta}(M_Z)$
pQCD	✓	✓
Total HVP	✓	✓
Flavor Separation	✗	Lattice can compute the contribution of each flavor
Using cross section data	Erler, Ferro-Hernandez, 10.1007/JHEP03(2018)196	Erler, Ferro-Hernandez, 10.1007/JHEP03(2018)196
Using Lattice	Erler, Ferro-Hernandez, 10.1007/JHEP12(2023)131	To be published soon



Lattice flavor separation

Parameter	Value and error	Π_{disc}	Π_s	Π_{ud}
Π_{disc}	$(-3.7 \pm 1.0) \times 10^{-4}$	1.0	-0.5	-0.6
Π_s	$(83.0 \pm 1.3) \times 10^{-4}$	-0.5	1.0	0.9
Π_{ud}	$(587.8 \pm 8.3) \times 10^{-4}$	-0.6	0.9	1.0

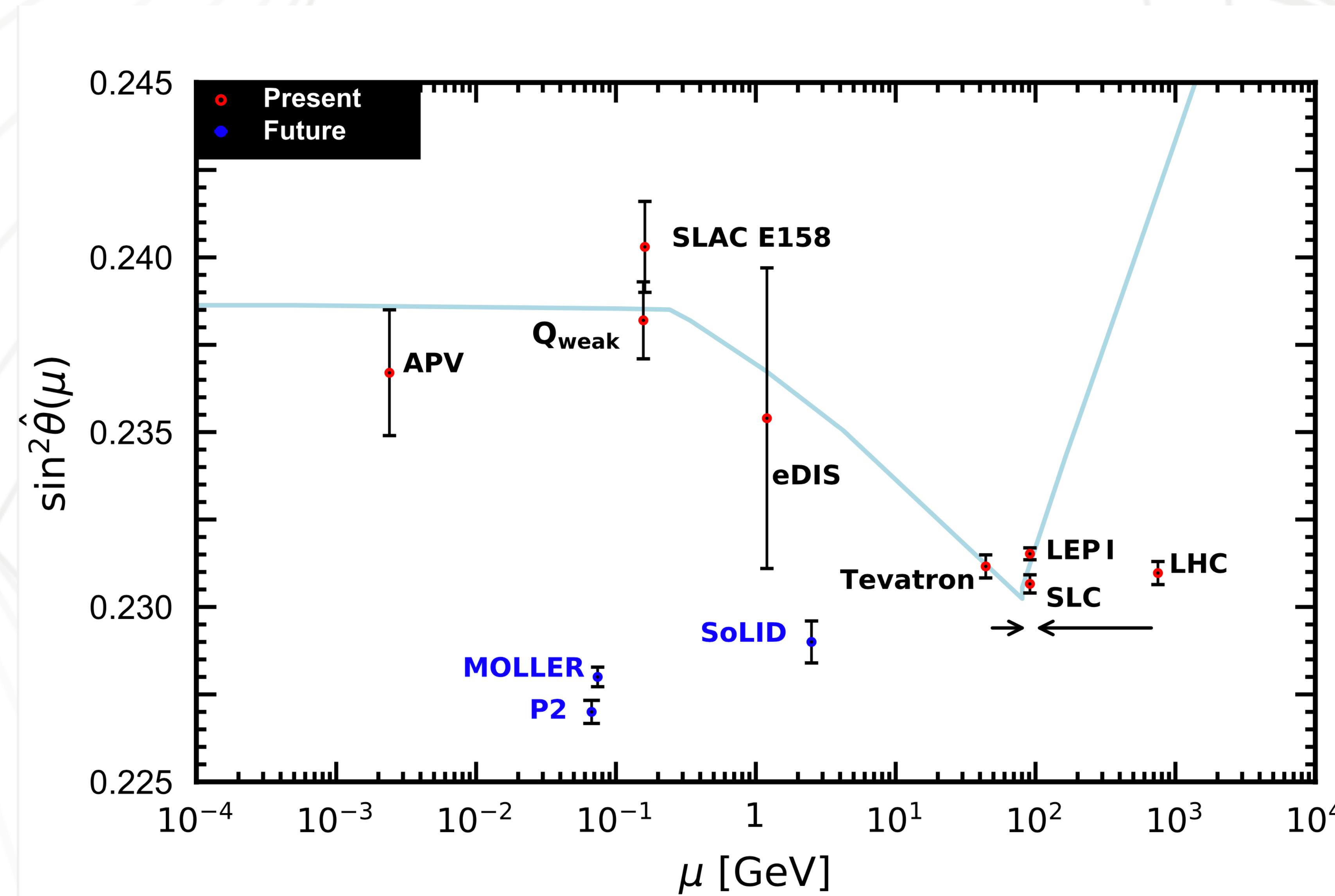
Ce et al [10.1007/JHEP08\(2022\)220](https://doi.org/10.1007/JHEP08(2022)220)

$$\Pi_f(-Q^2) = \hat{\Pi}_f(0, \mu^2) - \hat{\Pi}_f(-Q^2, \mu^2)$$

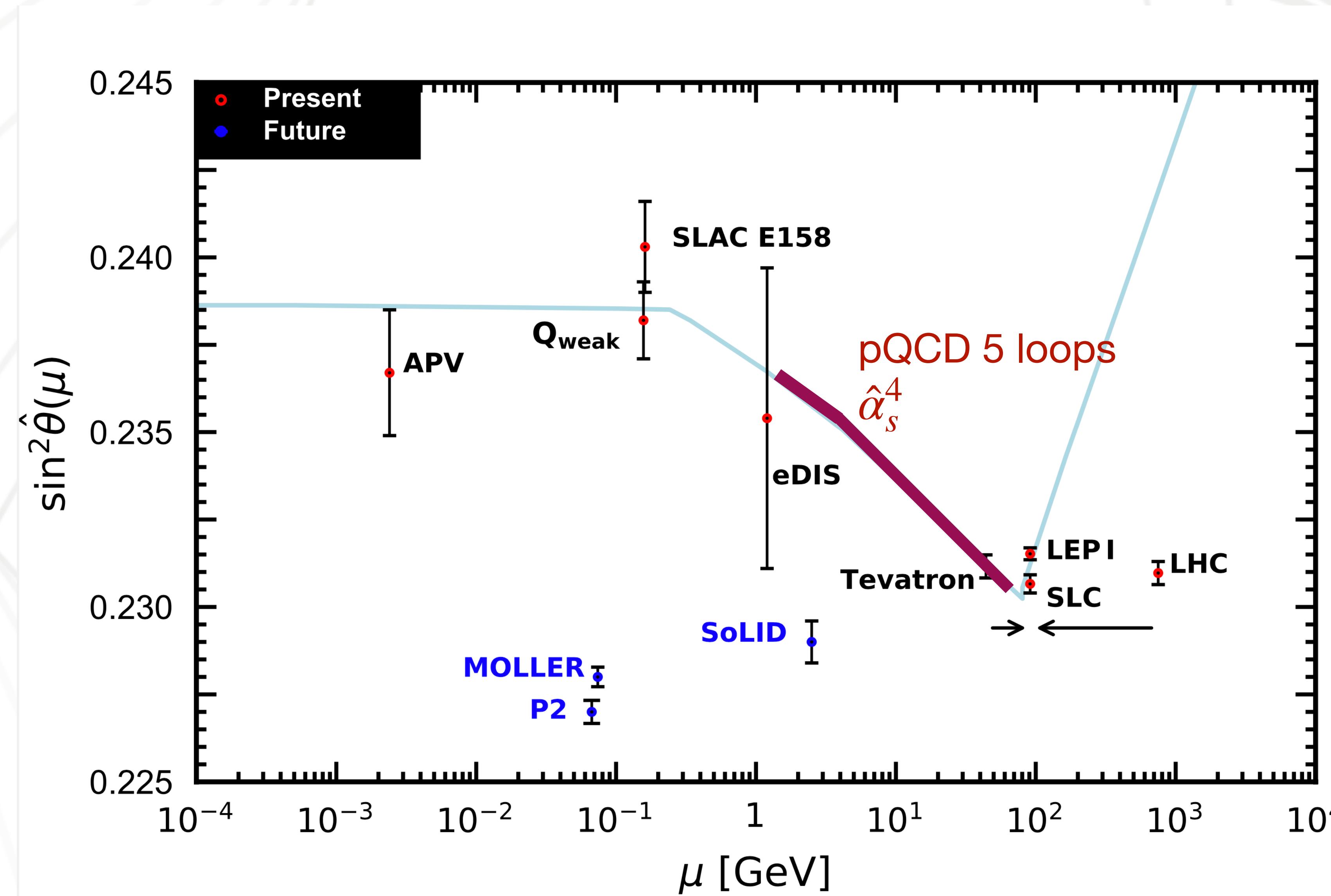
$$\hat{\Pi}_f(-Q^2, Q^2) = \frac{Q_f^2}{4\pi^2} \sum_{n=0}^3 c_n \left(\frac{\hat{\alpha}_s(Q^2)}{\pi} \right)^n$$

pQCD

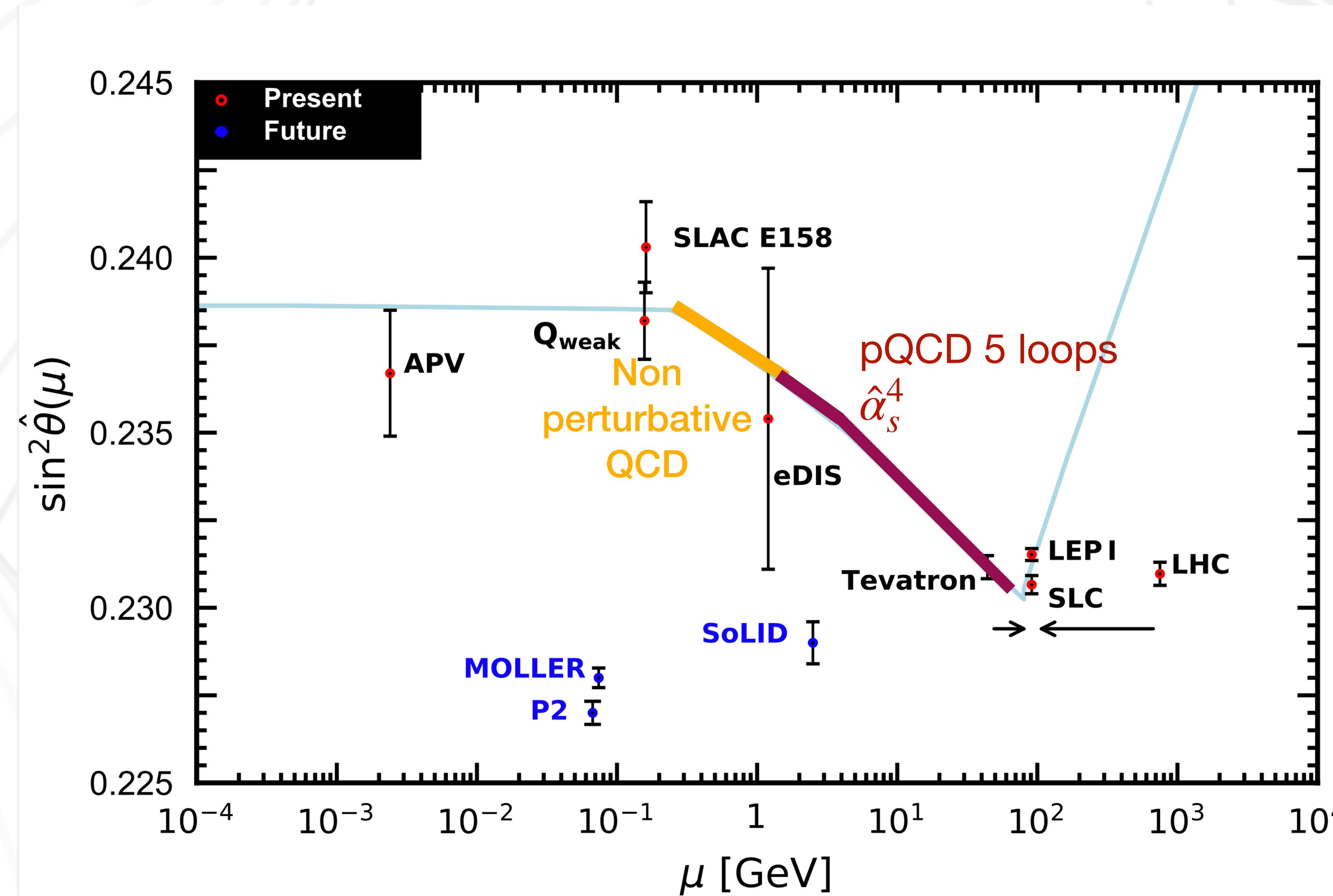
Flavor separation



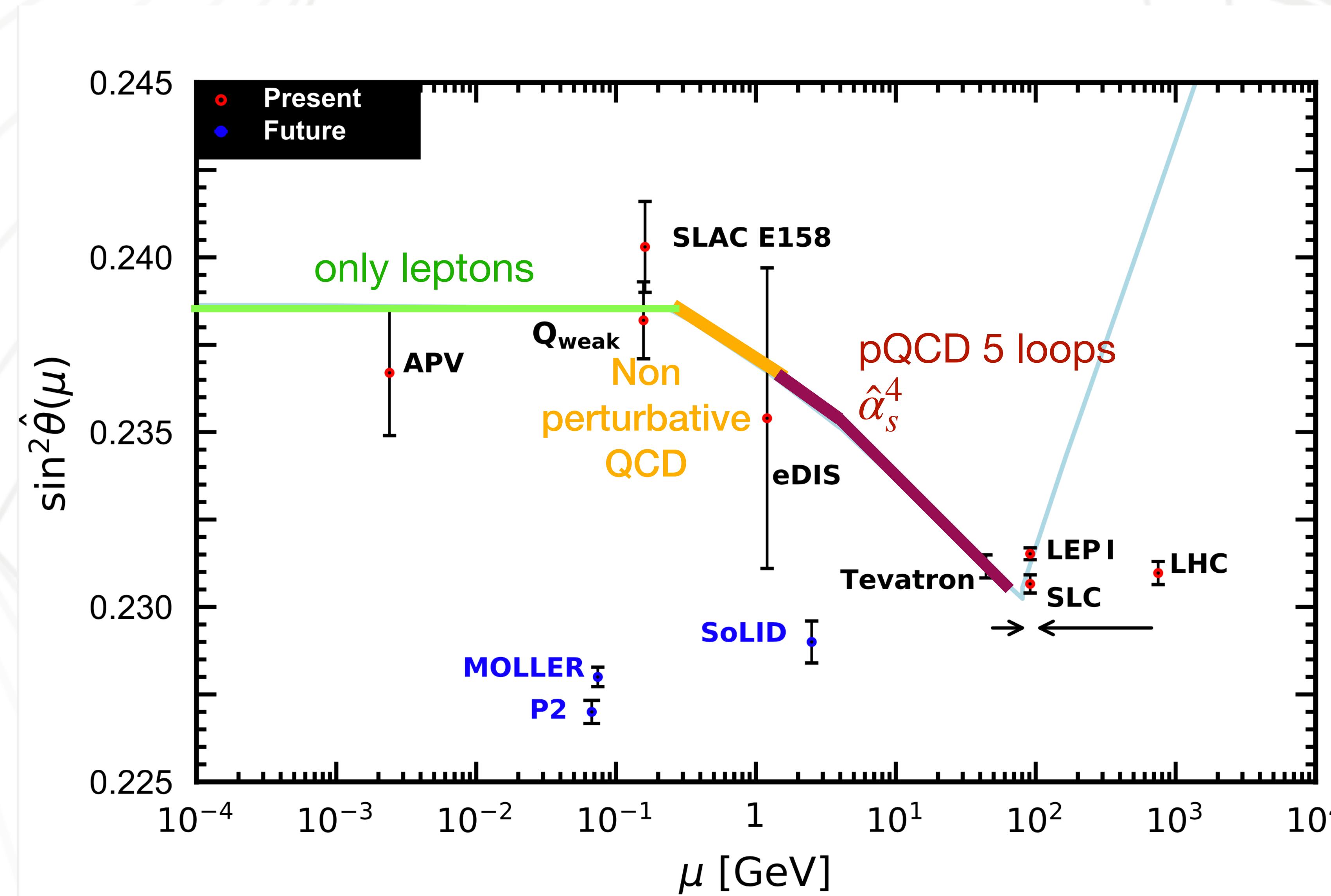
Flavor separation



Flavor separation



Flavor separation



Results

Results

Defining

$$\hat{\kappa}(0) = \frac{\sin^2 \hat{\theta}_W(0)}{\sin^2 \hat{\theta}_W(M_Z)}$$

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We obtain

$$\hat{\kappa}(0)_{\text{lat}} = 1.03233 - 0.42\Delta\hat{s}_Z^2 + 0.030\Delta\hat{\alpha}_s - 0.0012\Delta\hat{m}_c - 0.0003\Delta\hat{m}_b \pm 0.00010,$$

$$\Delta\hat{\alpha}_s \equiv \hat{\alpha}_s(M_Z) - 0.1185 \text{ GeV}$$

$$\Delta\hat{m}_b \equiv \hat{m}_b(\hat{m}_b) - 4.18 \text{ GeV}$$

$$\Delta\hat{m}_c \equiv \hat{m}_c(\hat{m}_c) - 1.274 \text{ GeV}$$

Results

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$$\Delta\hat{m}_c \equiv \hat{m}_c(\hat{m}_c) - 1.274 \text{ GeV}$$

While from cross section data the result is: $\hat{\kappa}(0)_{e^+e^-} = 1.03200 \pm 0.00008$,

[Erler, Ferro-Hernandez, 10.1007/JHEP03\(2018\)196](#)

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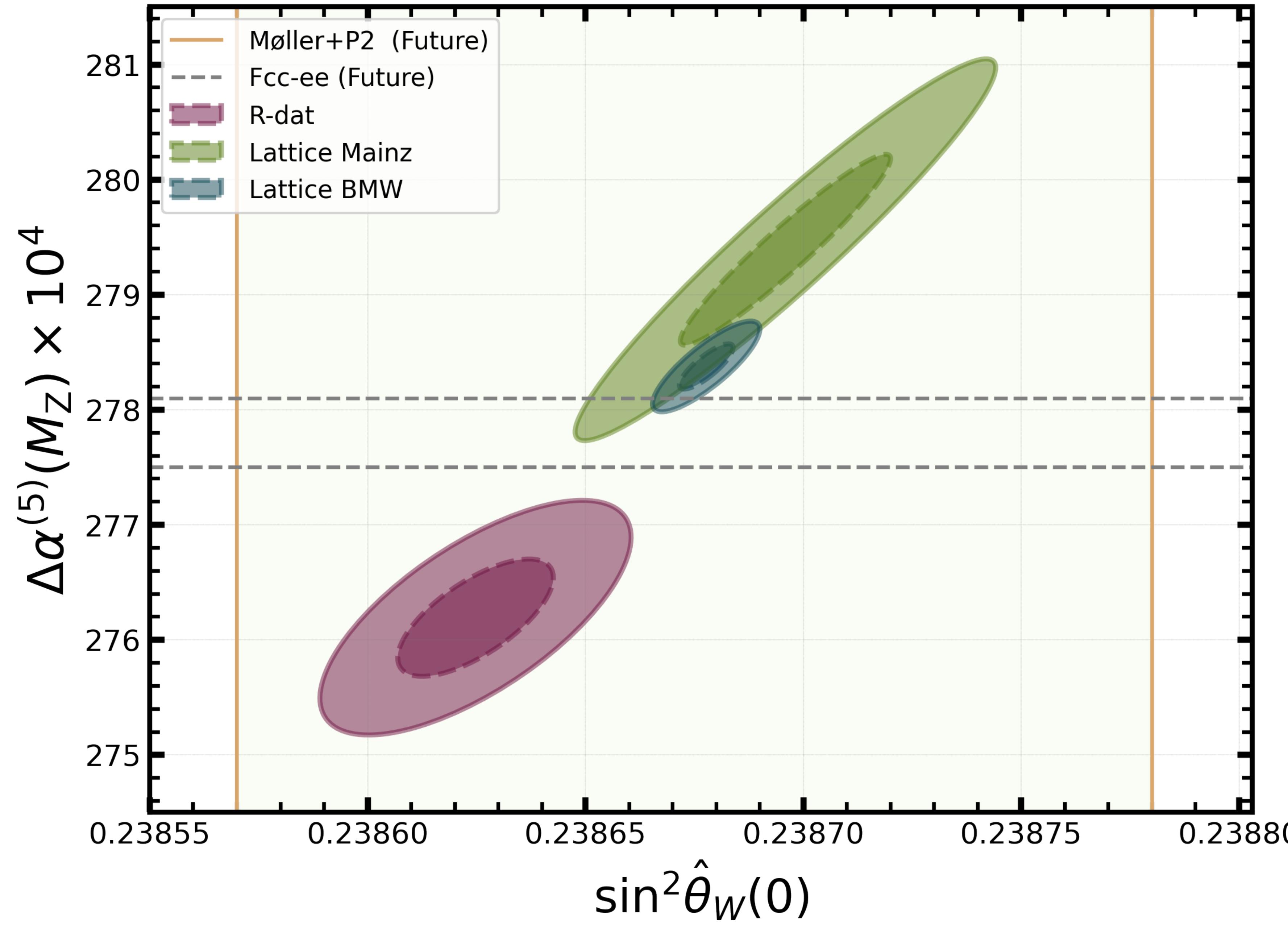
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[Erler, Ferro-Hernandez, 10.1007/JHEP03\(2018\)196](#)

$$\hat{\kappa}(0)_{\text{lat}} - \hat{\kappa}(0)_{e^+e^-} = 0.00033 \pm 0.00013$$

Results

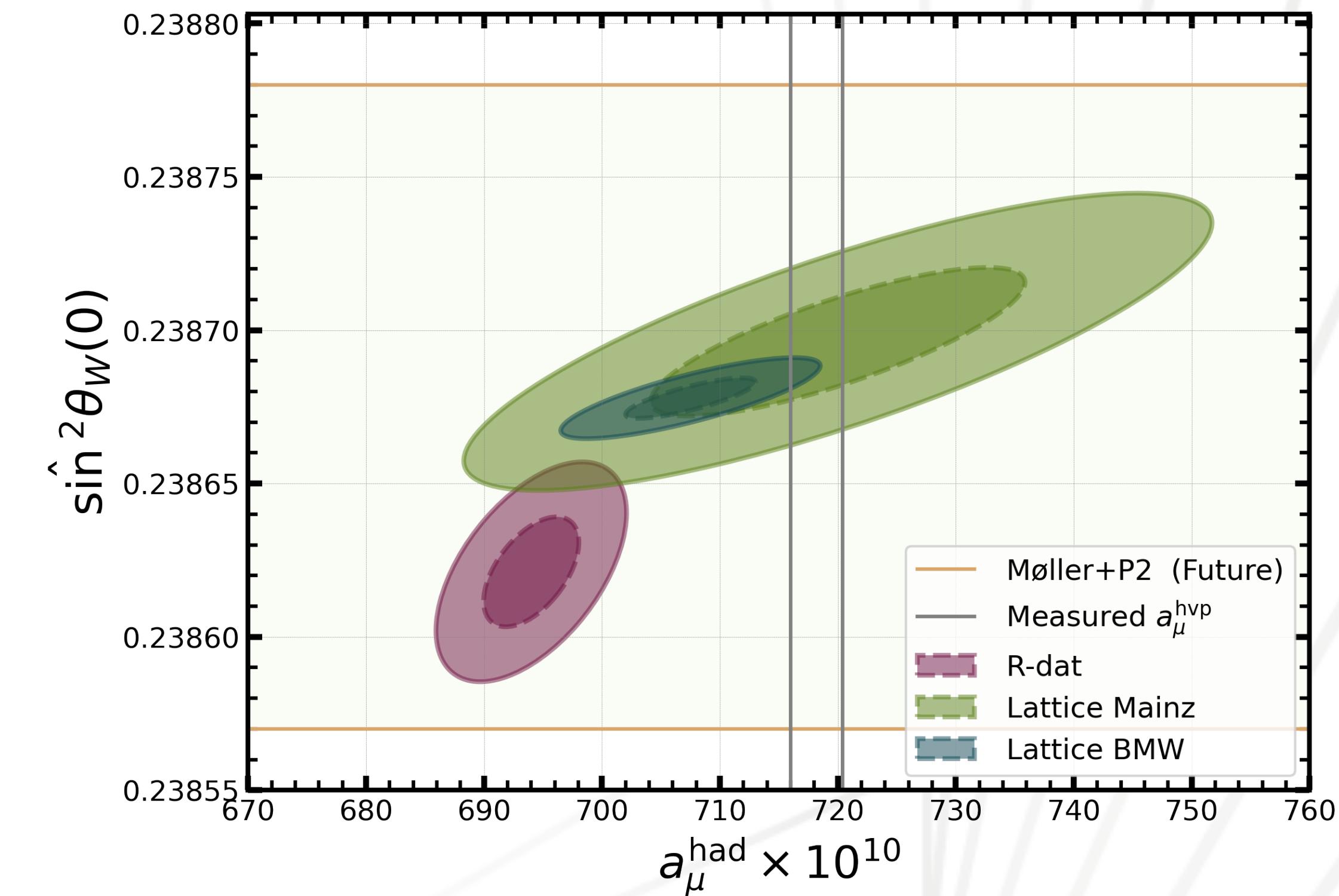
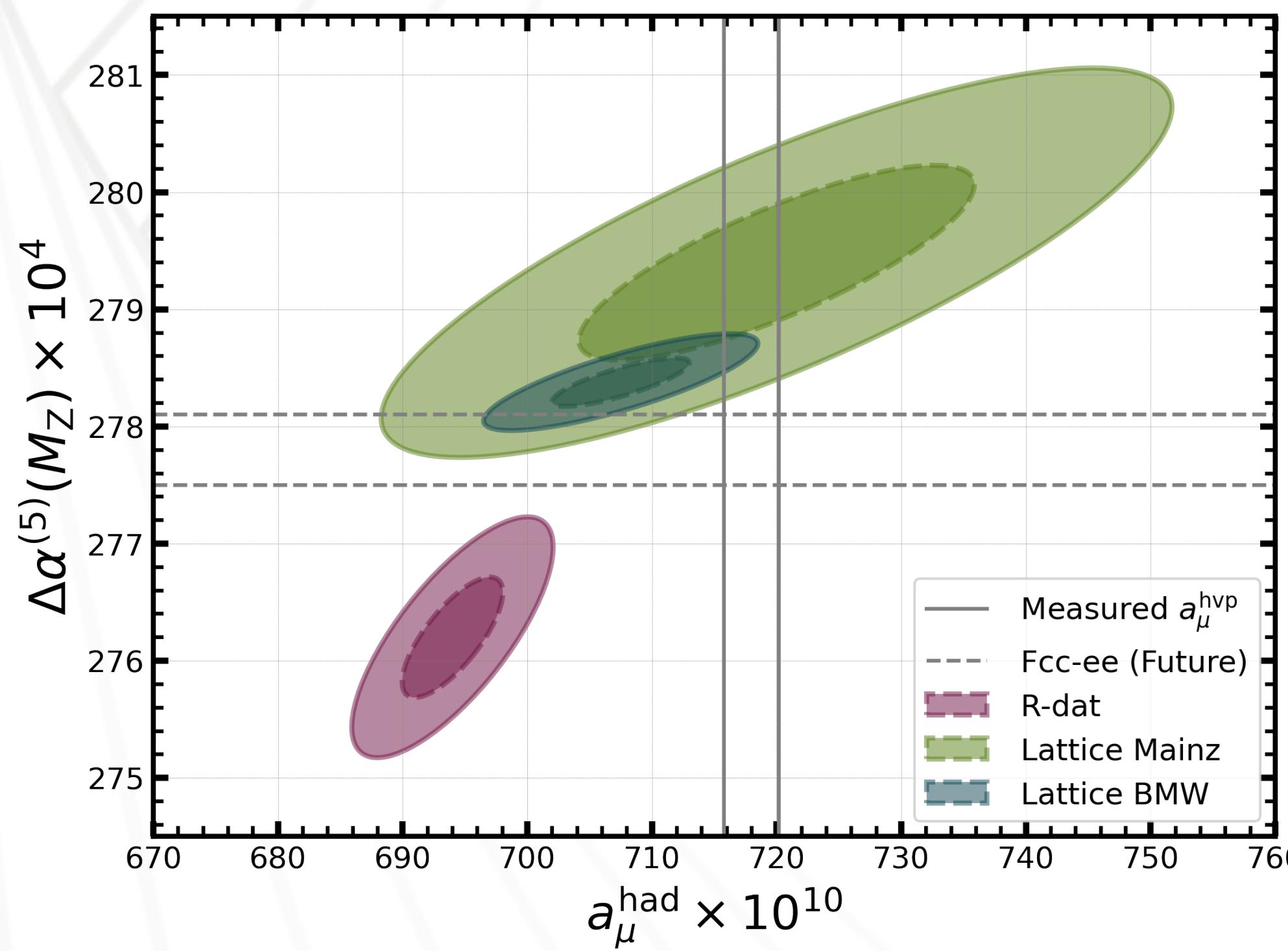


Results

$$f(K^2) = \frac{m_\mu^2 K^2 Z^3 (1 - K^2 Z)}{1 + m_\mu^2 K^2 Z^2}$$

$$Z = -[K^2 - (K^4 + 4m_\mu^2 K^2)^{1/2}]/2m_\mu^2 K^2$$

$$a_\mu^{hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dK^2 f(K^2) \Pi(K^2)$$



Summary

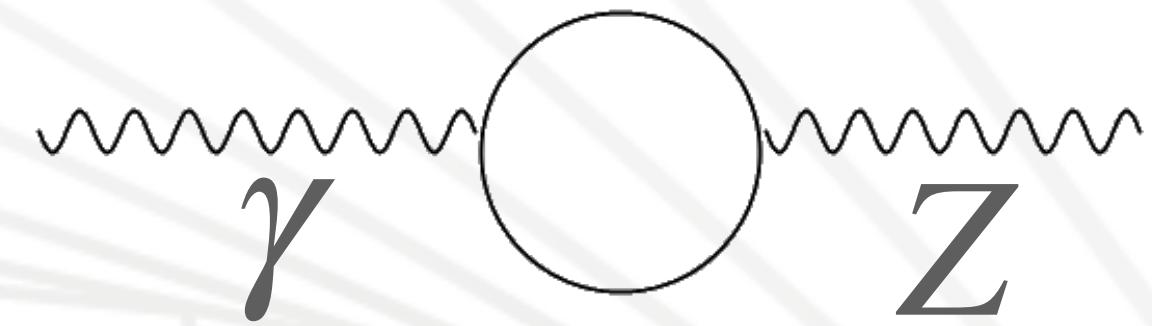
Summary

1. We computed $\sin^2 \hat{\theta}_W(0)$ using lattice QCD as input.
2. We found a $\sim 3\sigma$ tension when compared to the result using e^+e^- cross section data.
3. As expected the tension is in the same direction as the tension in α .
4. Tension smaller than the precision expected in future PV experiments.
5. We computed the correlation of a_μ^{hyp} with both $\hat{\alpha}$ and $\sin^2 \hat{\theta}_W(0)$.
6. There is consistency between the SM prediction and the experimental average of M_W .

Thank you

Backups

$\sin^2 \hat{\theta}$ is analogous to α



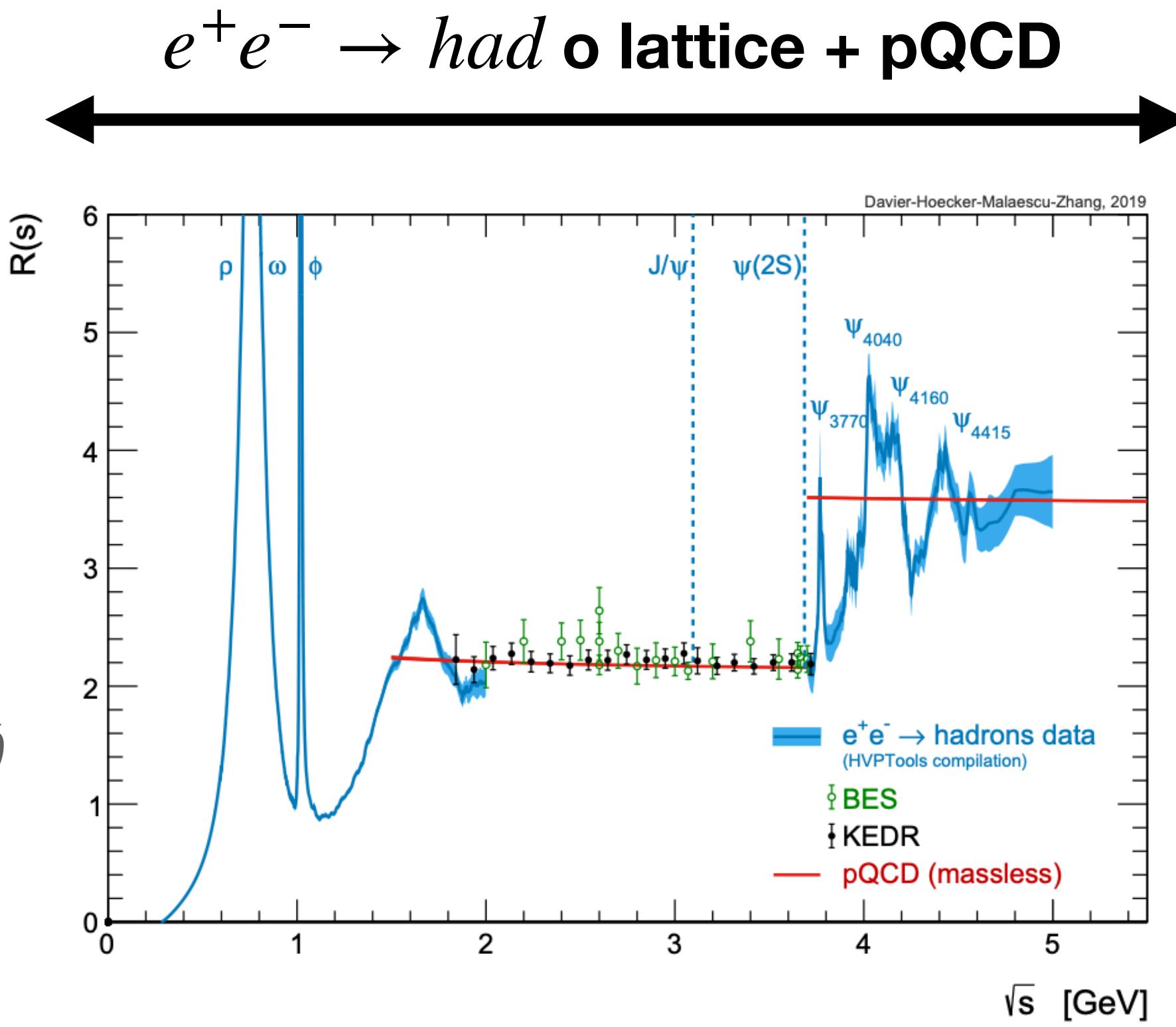
The weak mixing angle is also a key parameter in the Standard Model.

$$\text{Rel error} \sim 10^{-3} (2)$$

$$\sin^2 \hat{\theta}(0)$$

**Low energy Parity
Violation
Experiments**

$$A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} \propto 1 - 4 \sin^2 \hat{\theta}$$



$$\text{Rel error} \sim 10^{-3}$$

$$\sin^2 \hat{\theta}(M_Z)$$

Measured at the Z pole

$$A_f = 2 \frac{g_{Vf} g_{Af}}{g_{Vf}^2 + g_{Af}^2}$$

$$g_{Vf} = T_f - 2Q_f^2 \sin^2 \hat{\theta}$$

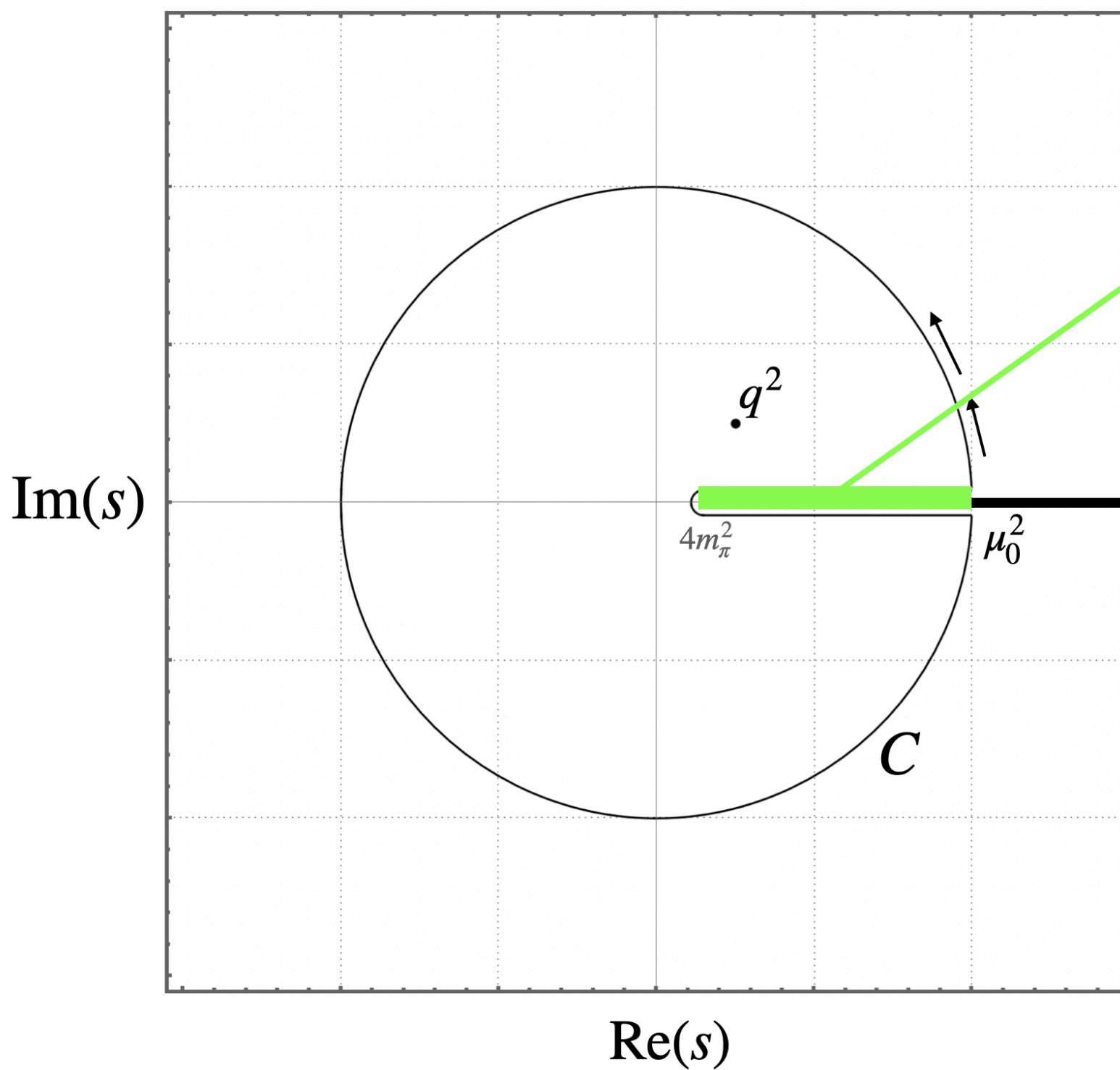
$$g_{Af} = T_f$$

$$A_{FB} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} = \frac{3}{4} A_e A_f$$

$$A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = A_e$$

Explicit integration over \mathbb{R}

Relation with cross section



Picks up $\text{Im } \hat{\Pi}_{\text{had}}(s, \mu^2)$

Use optical theorem

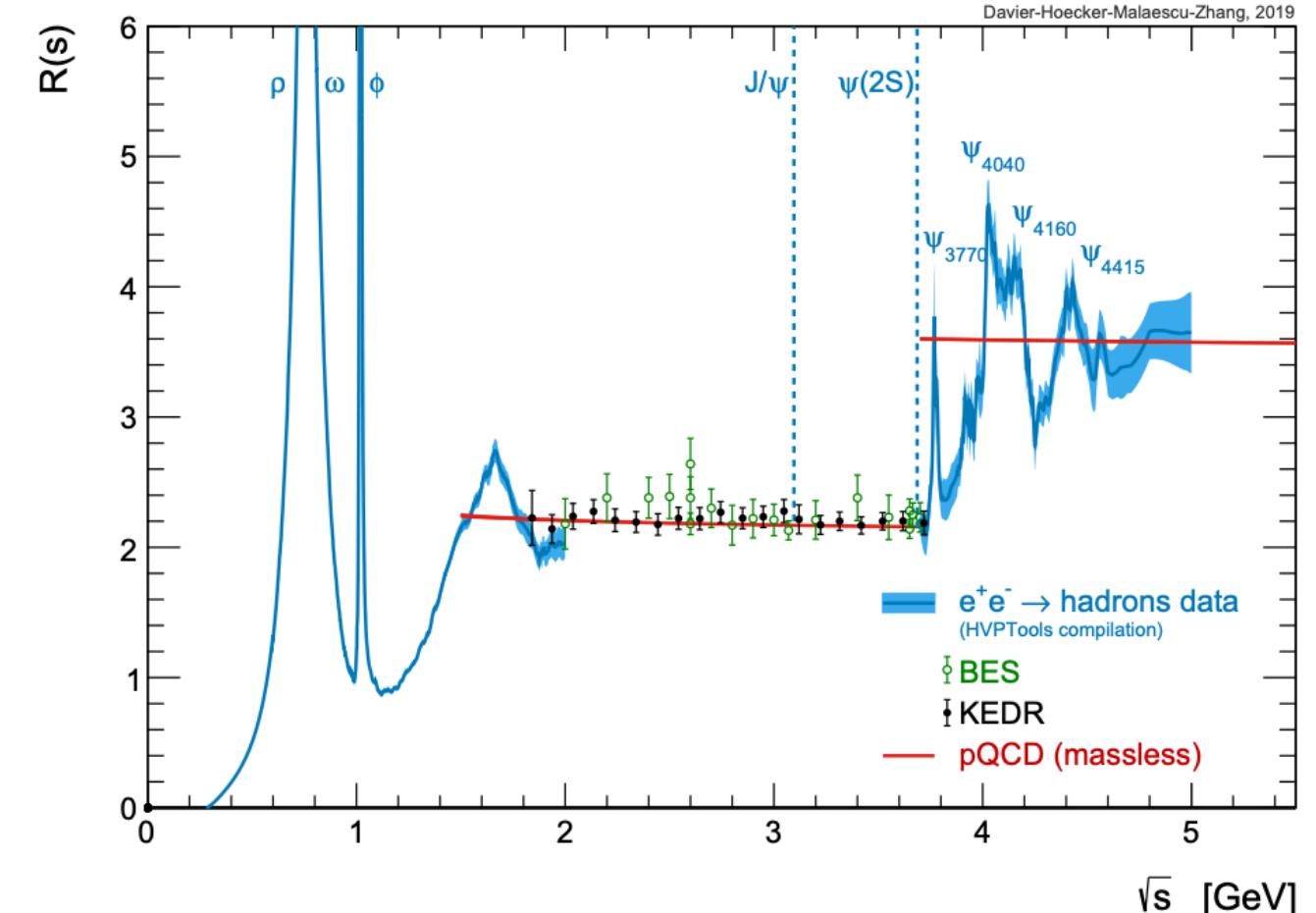
$$R = 12\pi \text{Im } \Pi_{\text{had}}(s)$$

$$R = \frac{\sigma(e^+e^- \rightarrow \text{had})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}$$

$\hat{\Pi}_{\text{had}}(s, \mu^2)$ is analytic in the complex plane of s , except for poles and branch cuts

$$\hat{\Pi}_{\text{had}}(q^2, \mu^2) = \frac{1}{2\pi i} \oint_C \frac{\hat{\Pi}_{\text{had}}(s, \mu^2)}{s - q^2} ds$$

$$\hat{\Pi}_{\text{had}}(q^2, \mu^2) = \frac{1}{12\pi^2} \int_{4m_\pi^2}^{\mu_0^2} \frac{R(s)}{s - q^2} ds + \frac{1}{2\pi i} \int_{|s|=\mu_0^2} \frac{\hat{\Pi}_{\text{had}}(s, \mu^2)}{s - q^2} ds$$



Explicit integral over R

Compute $\hat{\Pi}_{\text{had}}(q^2, \mu^2) - \hat{\Pi}_{\text{had}}(0, \mu^2)$ directly

$$\Delta\alpha_{\text{had}}(q^2) = -\text{Re} \left[\frac{\alpha q^2}{3\pi} \int_{4m_\pi^2}^\infty \frac{R(s)}{s(s-q^2)} ds \right]$$

Very similar to the expression used to compute a_μ !
Collaborations usually quote both results

$$a_\mu^{\text{had}} = \frac{\alpha^2}{3\pi^2} \int \frac{ds}{s} K(s) R(s)$$

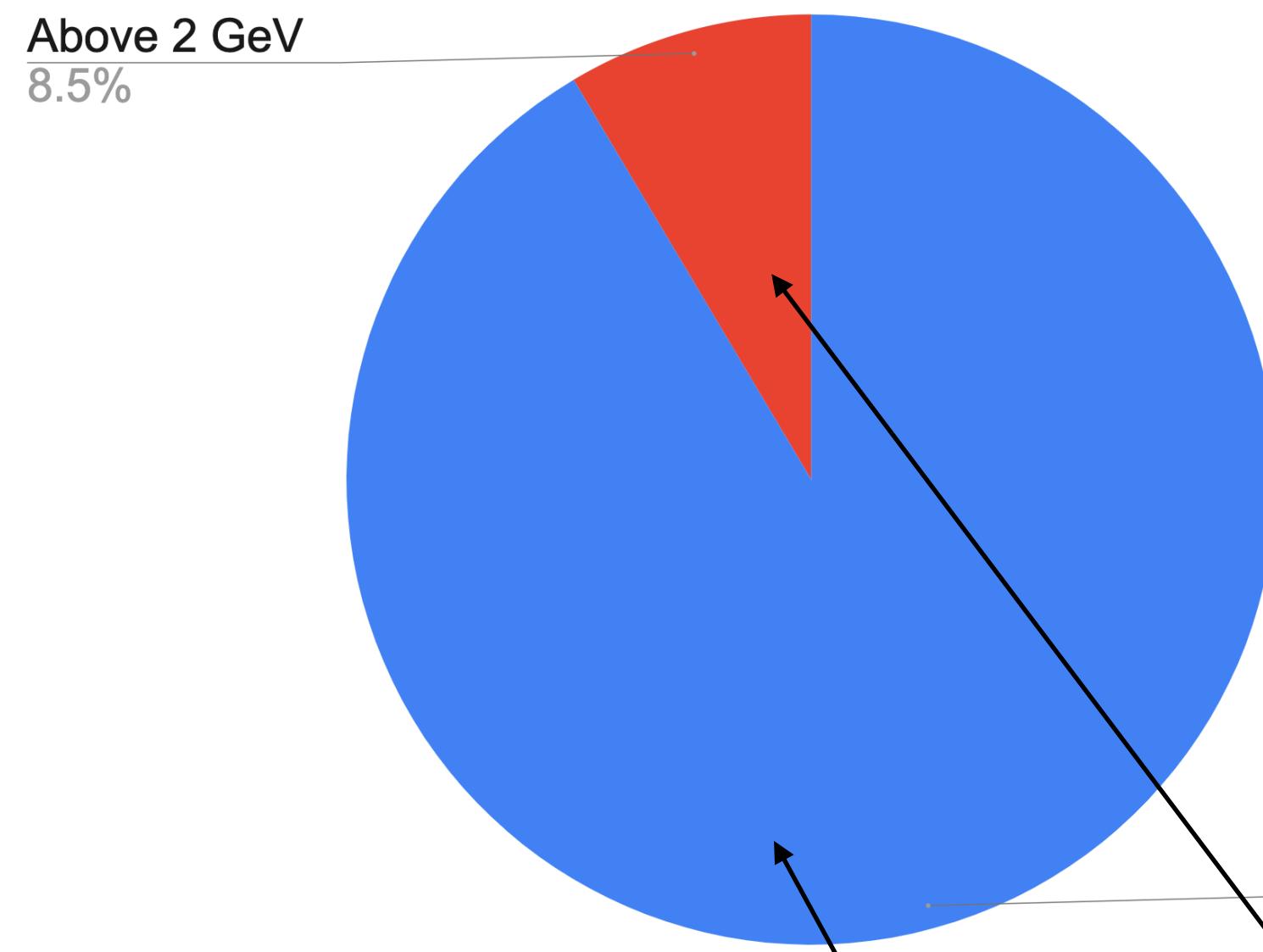
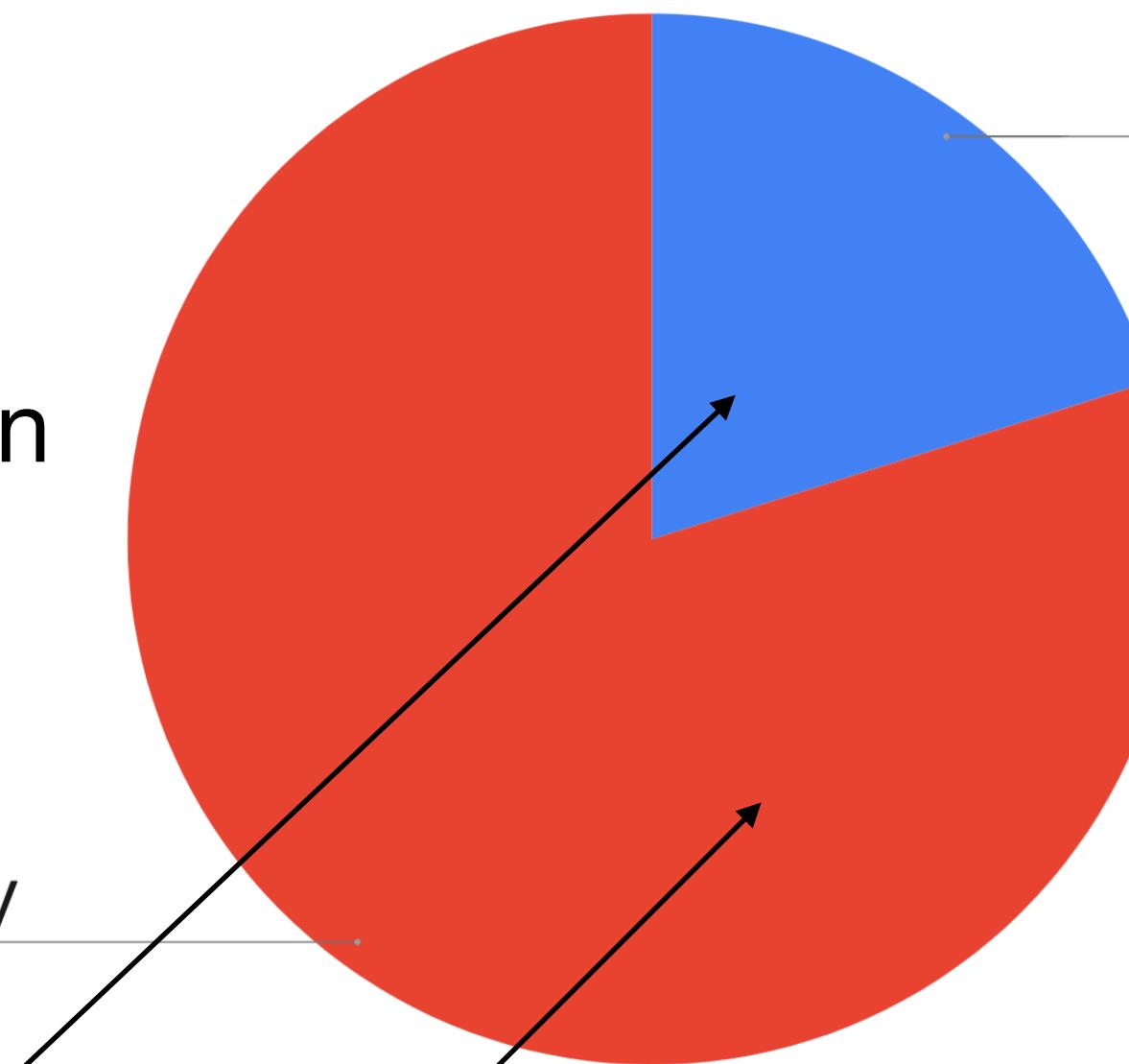
$$K(s) \sim \frac{1}{s}$$

(F. Jegerlehner :[arXiv:1905.05078](https://arxiv.org/abs/1905.05078))

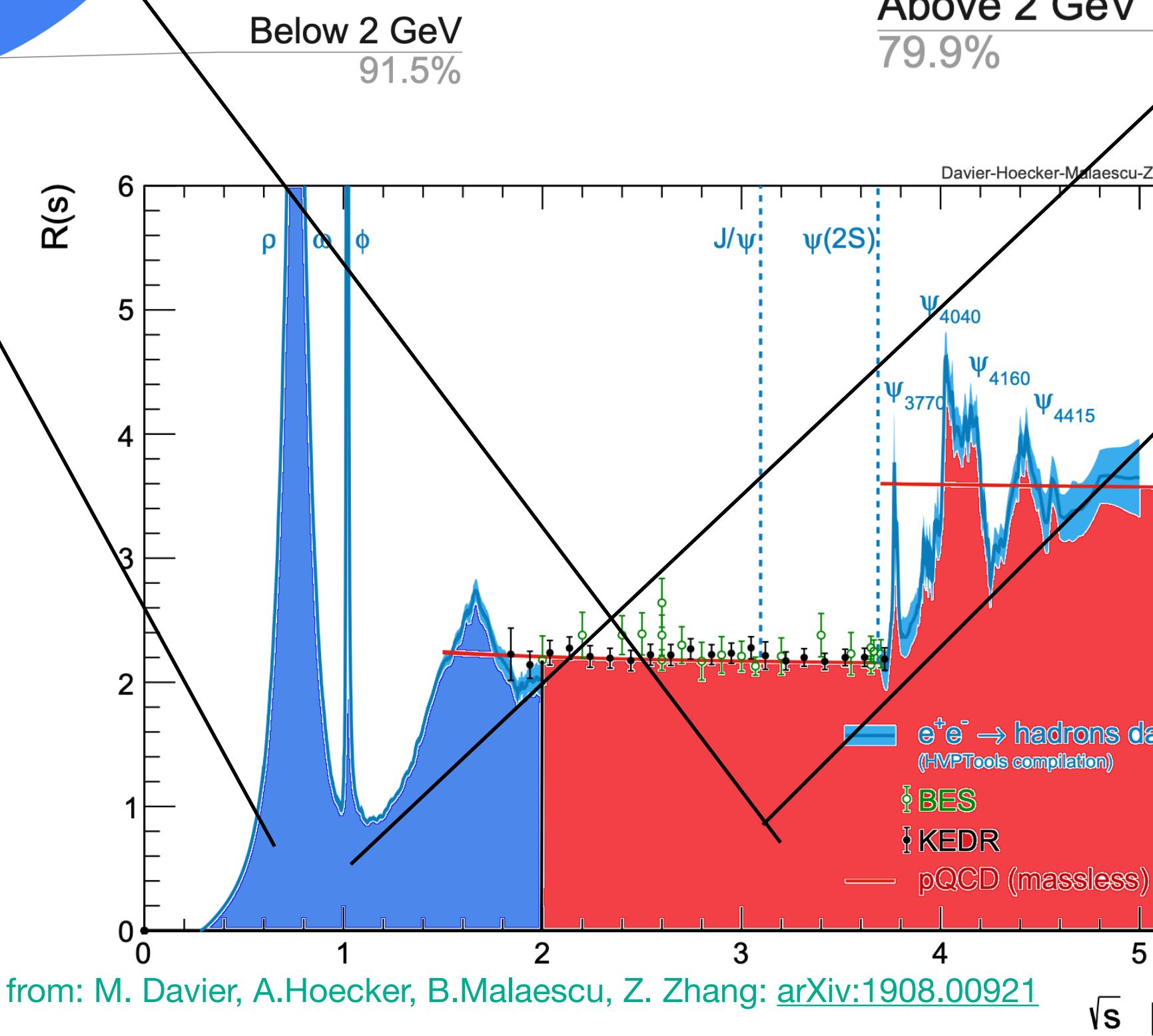
(M. Davier, A. Hoecker, B. Malaescu, Z. Zhang: [arXiv:1908.00921](https://arxiv.org/abs/1908.00921))

(A. Keshavarzi, D. Nomura and Thomas Teubner: [arXiv:1911.00367](https://arxiv.org/abs/1911.00367))

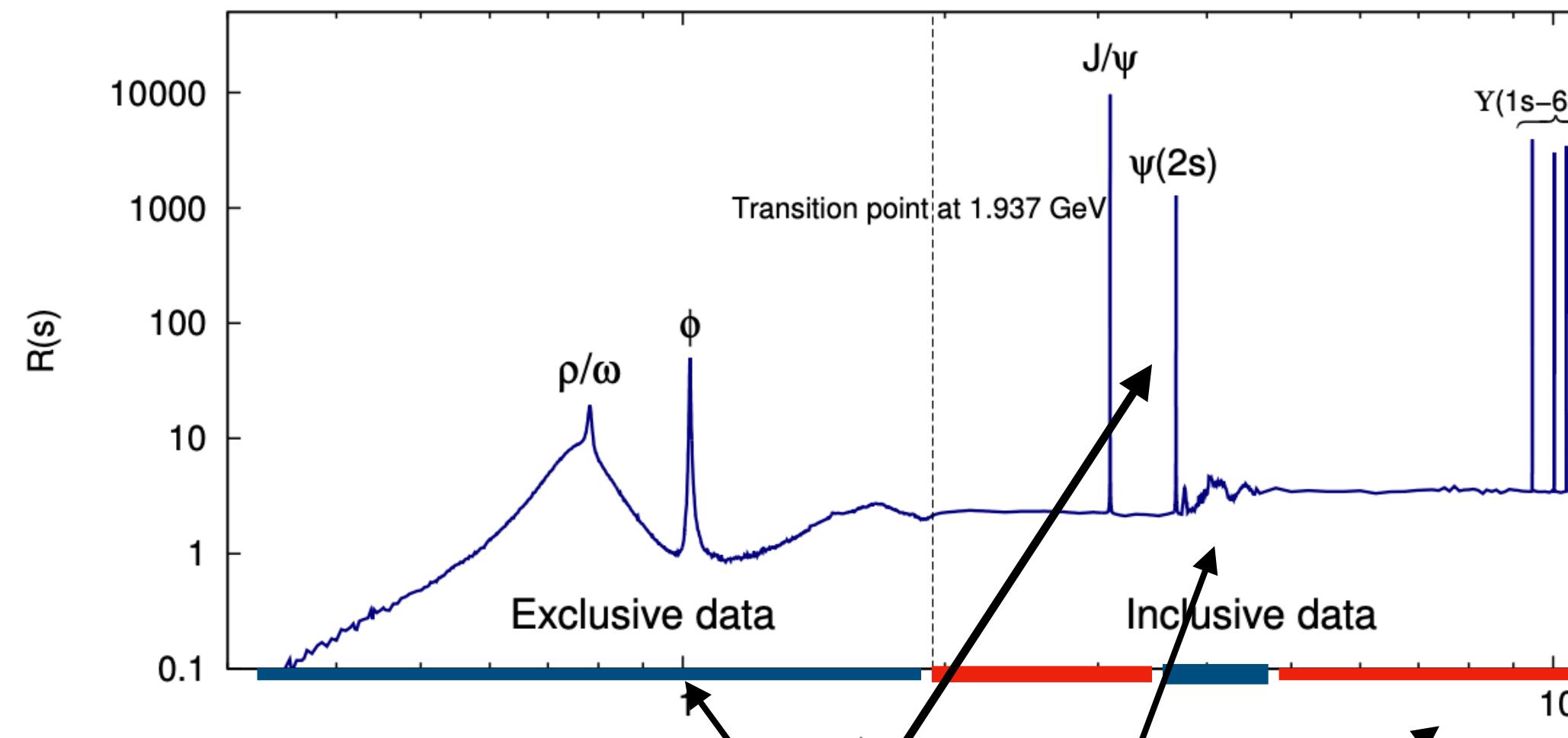
But, sensitivity to different regions of the integral is different.....

a_μ^{had}  $\Delta\alpha(M_Z^2)$ 

Total contribution per region



Integral over R

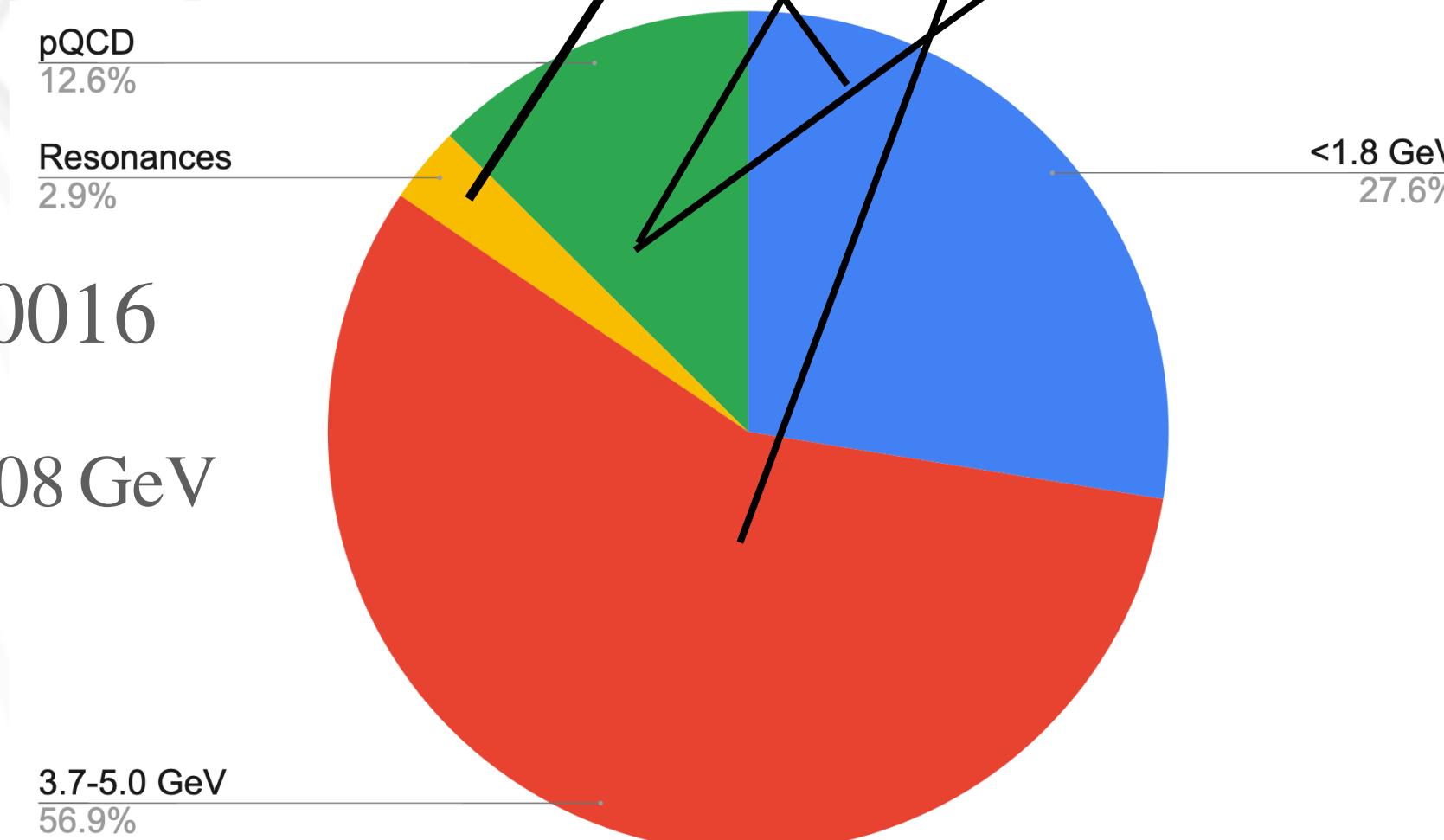


Region where pQCD is used

$$R(s) = 3 \sum_n Q_n^2 \left[1 + \frac{\alpha_s(s)}{\pi} + \dots \right] + \text{mass corrections}$$

Region data is used

(A. Keshavarzi, et al uses data up to 11 GeV (plot taken from them))



$$\delta \hat{\alpha}_s = 0.0016$$

$$\delta \hat{m}_c = 0.008 \text{ GeV}$$

$$\Delta \alpha^{(5)}(M_Z) = [275.77 + 141 \delta \hat{\alpha}_s + 0.7 \delta \hat{m}_c - 1.3 \delta \hat{m}_b \pm 0.67_{\text{c-thr}} \pm 0.19_{\text{trunc}} \pm 0.28_{\text{dual}} \pm 0.38_{\text{dat}<1.8 \text{ GeV}} \pm 0.15_{J/\psi}] \times 10^{-4} \quad (\text{Timelike method, R input})$$

Main error associated to the charm quark

$$\Delta \alpha^{(c)}(M_Z^2) = [78.72 + 27 \delta \hat{\alpha}_s + 0.7 \delta \hat{m}_c \pm 0.02_{\text{trunc}} \pm 0.67_{\text{c-thr}} \pm 0.13_{J/\psi} \pm 0.08_{\psi}] \times 10^4.$$

RGE method

RGE method

Steps:

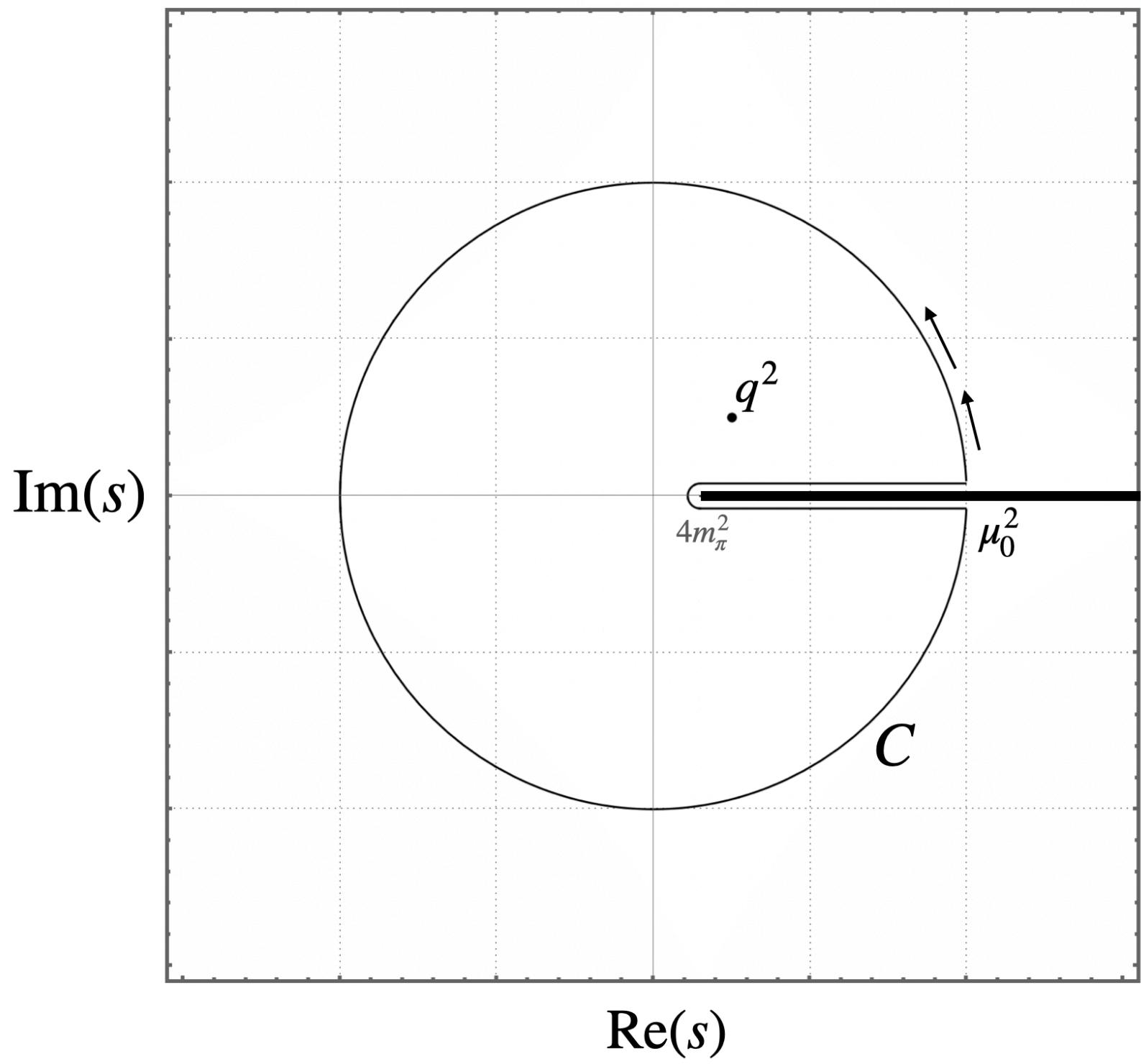
1. Determine light quark contributions to $\hat{\Pi}(0, \mu^2)$ at a low energy scale.
2. Match the charm quark contribution.
3. Run to bottom quark, match it and run.
4. Convert back to the on-shell scheme (effective coupling).

RGE method

Steps:

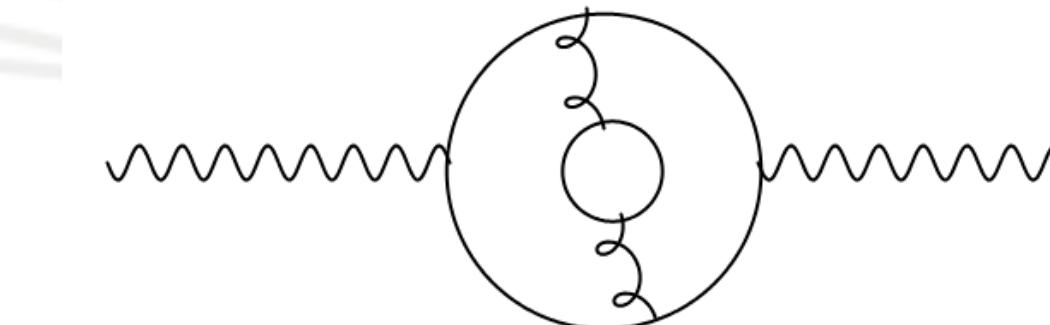
1. Determine light quark contributions to $\hat{\Pi}(0, \mu^2)$ at a low energy scale.
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RGE method: light quark contribution



When need

$$\hat{\alpha}(\mu^2) = \frac{\alpha}{1 - \Delta\hat{\alpha}(\mu^2)}$$



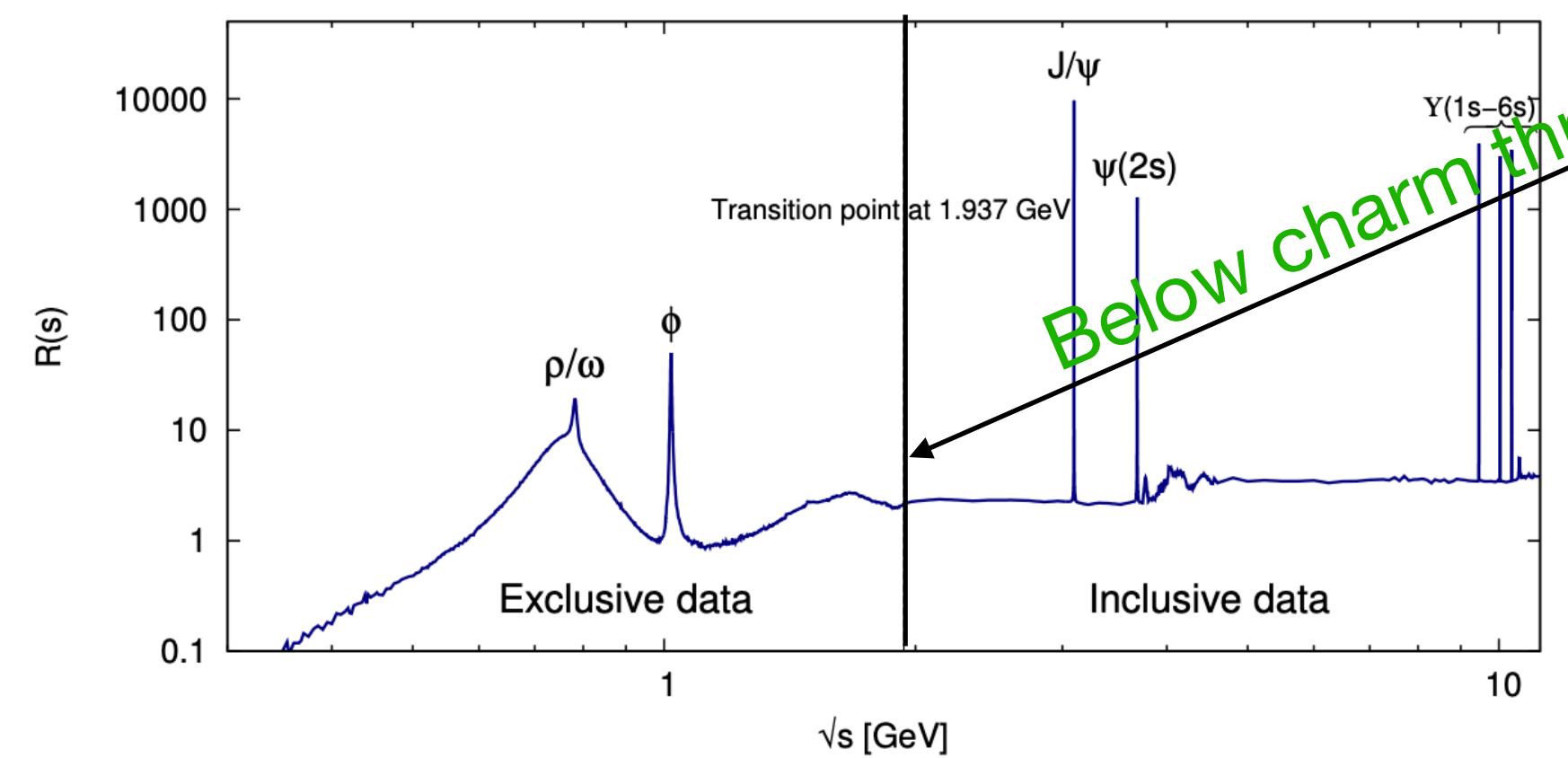
$$\Delta\hat{\alpha}(\mu^2) \equiv 4\pi\alpha\hat{\Pi}(0, \mu^2)$$

So lets go back to our contour and set $q^2 = 0$

(J. Erler. hep-ph/9803453)

$$\hat{\Pi}_{\text{had}}(0, \mu^2) = \frac{1}{12\pi^2} \int_{4m_\pi^2}^{\mu_0^2} \frac{R(s)}{s} ds + \frac{1}{2\pi i} \int_{|s|=\mu_0^2} \frac{\hat{\Pi}_{\text{had}}^{(3)}(s, \mu^2)}{s} ds$$

Gives 3 light quarks contribution



Use data for the low part, and pQCD for the circle integral

$\hat{\Pi}_{\text{had}}(s, \mu^2)$ is known up to order $\hat{\alpha}_s^3$ for massless quarks

Condensate effects are suppressed by two powers of $\hat{\alpha}_s$

RGE method

Steps:

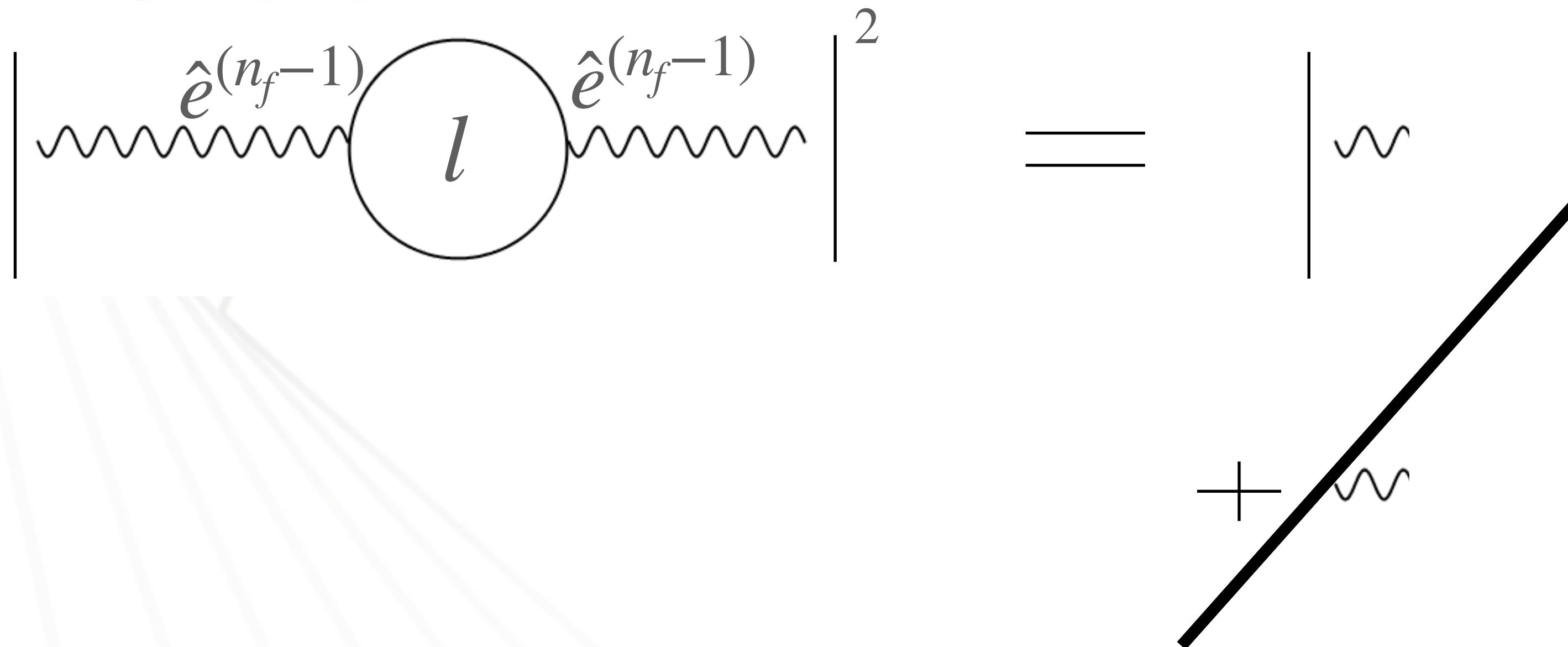
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3. Run to bottom quark, match it and run.
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RGE method: matching

Go from a theory with n_l quarks

Cross sections at low q^2 must be equivalent

$$\hat{\alpha}^{(n_f-1)} = \xi \hat{\alpha}^{(n)}$$



The matching conditions are known to order $\hat{\alpha}_s^3$.

Chetyrkin et al. hep-ph/9708255
Sturm hep-ph/1404.3433

$$\begin{aligned}
\Delta \hat{\alpha}^{(n_f-1)}(\hat{m}_f^2) &= \Delta \hat{\alpha}^{(n_f)}(\hat{m}_f^2) - \frac{15}{16} N_c Q_f^4 a^2 \left(1 + \Delta \hat{\alpha}^{(n_f)}(\hat{m}_f) \right) - a Q_f^2 \left\{ \hat{a}_s^{(n_f)} \frac{13}{12} \right. \\
&\quad + \hat{a}_s^{(n_f)2} \left[\frac{361}{1296} n_f + \frac{655}{144} \zeta_3 - \frac{3847}{864} \right] \\
&\quad + \hat{a}_s^{(n_f)3} \left[-\frac{85637 a_4}{1620} - \frac{656 a_5}{27} - \frac{928399 \zeta_2^2}{129600} - \frac{1289}{135} \zeta_2^2 l_2 - \frac{164}{81} \zeta_2 l_2^3 \right. \\
&\quad + \frac{85637 \zeta_2 l_2^2}{6480} - \frac{49 \zeta_5}{32} + \frac{42223463 \zeta_3}{604800} - \frac{321165301}{21772800} + \frac{82 l_2^5}{405} - \frac{85637 l_2^4}{38880} \\
&\quad + n_f \left(-\frac{17 a_4}{27} + \frac{4487 \zeta_2^2}{2160} + \frac{17}{108} \zeta_2 l_2^2 - \frac{21379 \zeta(3)}{5184} - \frac{86101}{62208} - \frac{17 l_2^4}{648} \right) \\
&\quad \left. + n_f^2 \left(\frac{17897}{93312} - \frac{31}{216} \zeta_3 \right) \right\} - a \sum_{l \neq f} Q_l^2 \left\{ \hat{a}_s^{(n_f)2} \frac{295}{1296} \right. \\
&\quad + \hat{a}_s^{(n_f)3} \left[\frac{67}{360} \zeta_2^2 + \frac{1}{9} \zeta_2 l_2^2 + \frac{163}{162} \zeta_3 - \frac{86369}{186624} - \frac{l_2^4}{54} - \frac{4 a_4}{9} \right. \\
&\quad \left. + \left(\frac{6625}{46656} - \frac{11 \zeta_3}{108} \right) n_f \right\} - a Q_f^2 \hat{a}_s^{(n_f)3} \left\{ \frac{2411}{6048} - \frac{365 a_4}{36} \right. \\
&\quad + \frac{2189 \zeta_2^2}{576} + \frac{365}{144} \zeta_2 l_2^2 - \frac{25 \zeta_5}{72} - \frac{6779 \zeta_3}{1344} - \frac{365 l_2^4}{864} \left. \right\} \\
&\quad - a \sum_{l \neq f} Q_f Q_l \hat{a}_s^{(n_f)3} \left\{ -\frac{\zeta_2^2}{6} - \frac{25 \zeta_5}{36} + \frac{655 \zeta_3}{432} + \frac{515}{1296} \right\},
\end{aligned}$$

RGE method

Steps:

1. Determine light quark contributions to $\hat{\Pi}(0, \mu^2)$ at a low energy scale.
2. Match the charm quark contribution.
3. Run to bottom quark, match it and run.
4. Convert back to the on-shell scheme (effective coupling).

RGE method: Running

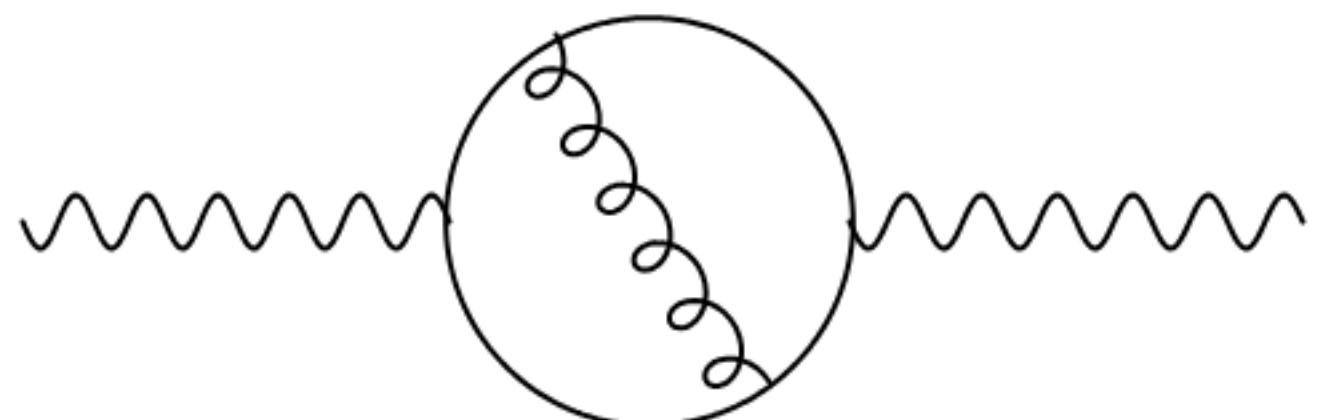
$$\mu^2 \frac{d\hat{\alpha}}{d\mu^2} = -\frac{\hat{\alpha}^2}{\pi} \beta \longrightarrow \mu^2 \frac{d}{d\mu^2} \Delta\hat{\alpha} = -\frac{\alpha}{\pi} \beta$$

β known to 5 loops

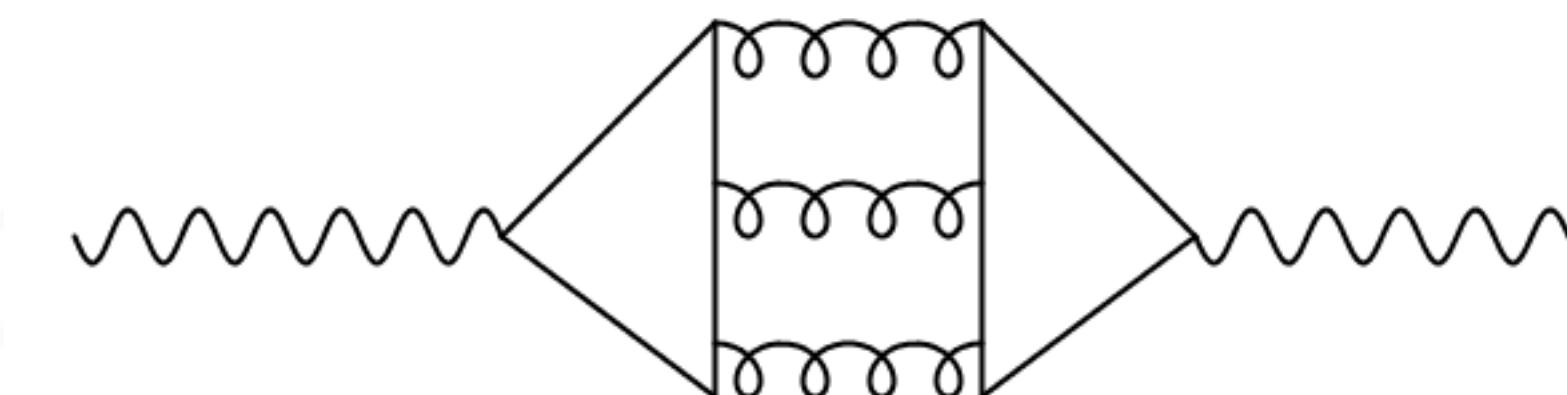
(Bikov et al. arXiv:1206.1284)

$$\beta = \left[\frac{1}{3} \sum_q K_q Q_q^2 + \sigma \left(\sum_q Q_q \right)^2 \right],$$

$$K_q = N_c \left[1 + \frac{\hat{\alpha}_s}{\pi} + \dots \right]$$



$$\sigma = \left(\frac{\hat{\alpha}_s}{\pi} \right)^3 \left(\frac{55}{216} - \frac{5}{9} \zeta_3 \right) + \dots$$

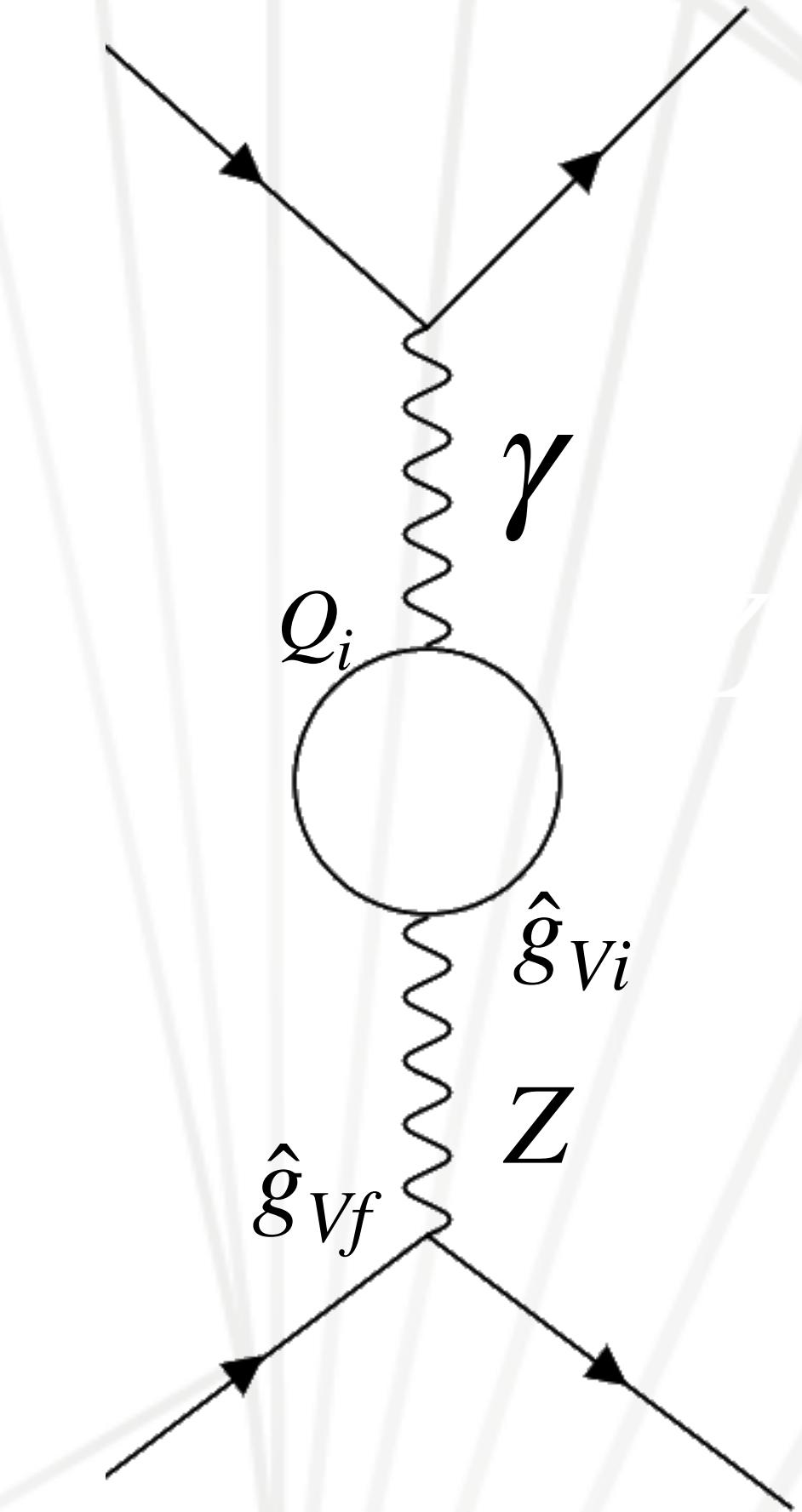
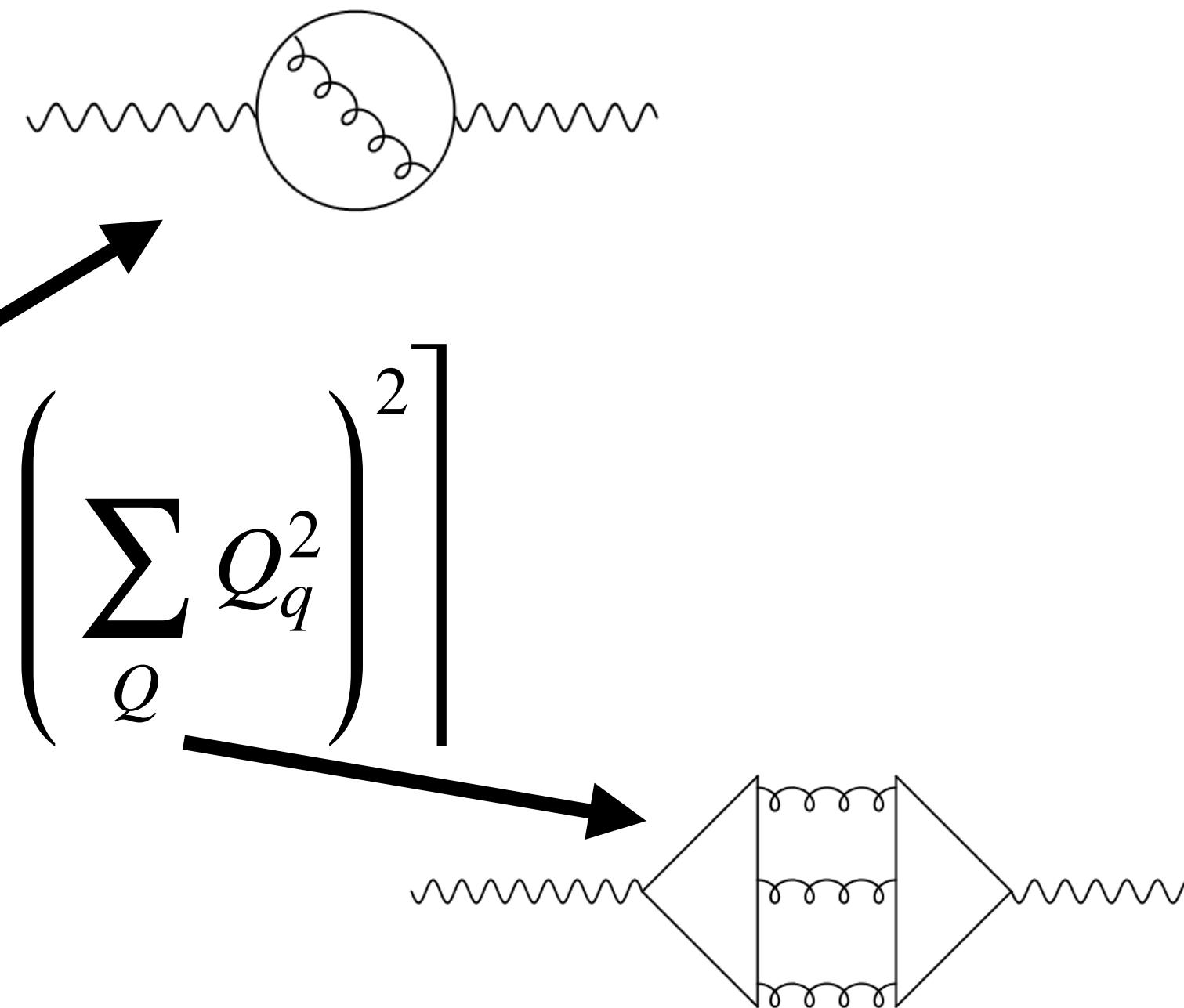


Renormalization group equation

$$\frac{d\hat{\alpha}}{d \ln \mu^2} = \frac{\hat{\alpha}^2}{\pi} \left[\frac{1}{24} \sum_i K_i \gamma_i Q_i^2 + \sigma \left(\sum_Q Q_q^2 \right)^2 \right]$$

$$\hat{g}_{Vf} = T_f - 2Q_f^2 \sin^2 \hat{\theta}$$

$$\frac{d\hat{g}_{Vf}}{d \ln \mu^2} = \frac{\hat{\alpha}^2}{\pi} \left[\frac{1}{24} \sum_i K_i \gamma_i \hat{g}_{Vi} Q_i + 12\sigma \left(\sum_Q Q_q \right) \left(\sum_Q \hat{g}_{Vq} \right) \right]$$



RGE method

Steps:

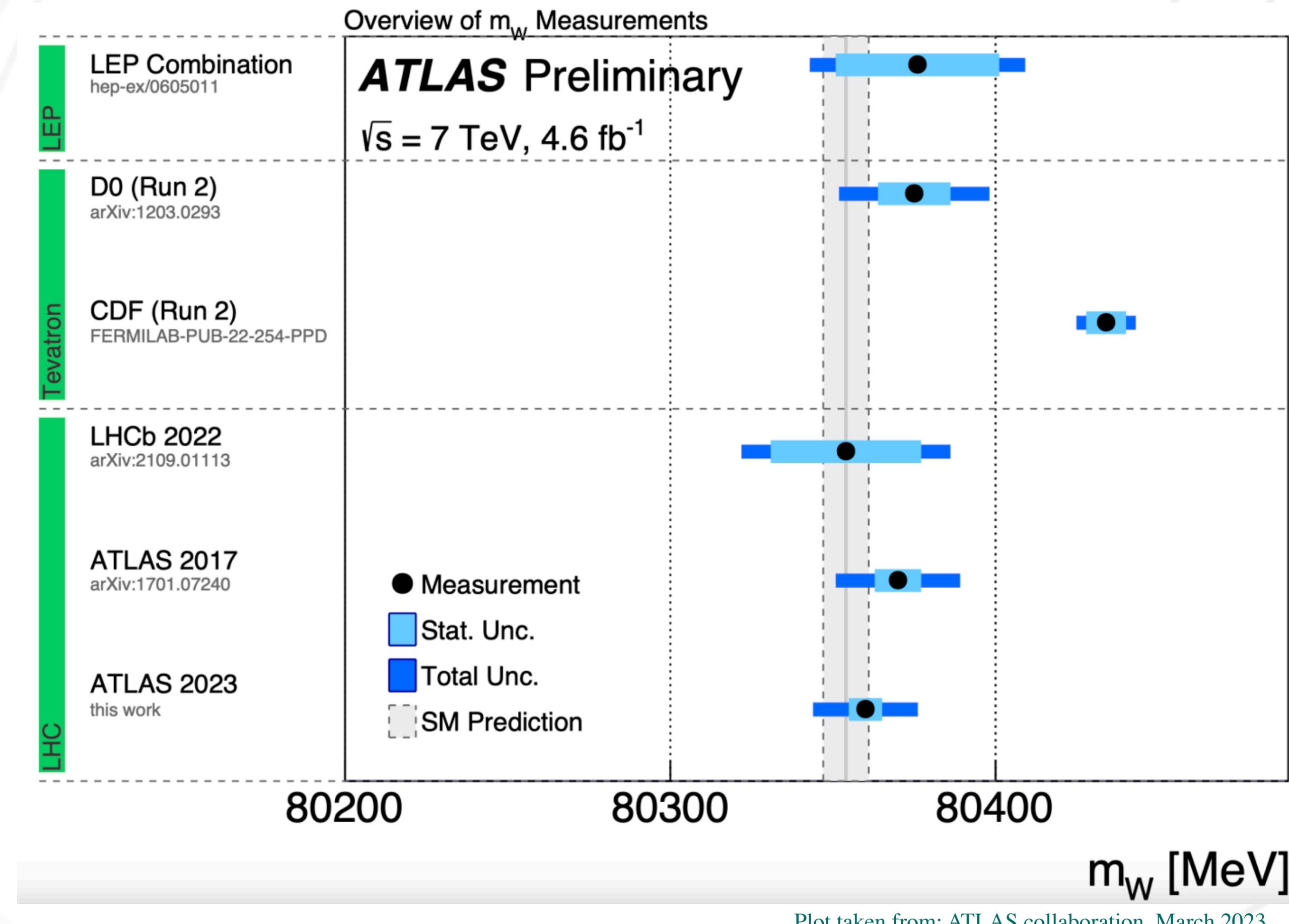
1. Determine light quark contributions to $\hat{\Pi}(0, \mu^2)$ at a low energy scale.
2. Match the charm quark contribution.
3. Run to bottom quark, match it and run.
4. Convert back to the on-shell scheme (effective coupling).

Use again

$$\Delta\alpha^{(5)}(M_Z^2) = \Delta\hat{\alpha}^{(5)}(\mu^2 = M_Z^2) - 4\pi\alpha\text{Re} \left[\hat{\Pi}^{(5)}(M_Z^2, \mu^2 = M_Z^2) \right]$$

From the running Scheme conversion

Some anomalies in the SM



More back ups

$$\text{Atomic spectroscopy } R_\infty = \frac{\alpha^2 m_e c}{4\pi\hbar}$$

1) Extract R_∞ from the data $\nu_{ij} = \varepsilon_j - \varepsilon_i$

$$\varepsilon_i = -\frac{R_\infty c}{n_i^2} (1 + \delta_i)$$

2) Determine $\frac{\hbar}{m_e}$

$$\frac{\hbar}{m_e} = \frac{u}{m_e} \frac{M_X}{u} \frac{\hbar}{M_X}$$

From the bound g factor of the electron

Recoil velocity

↑
cyclotron frequency of an ion in the constant magnetic field