Modular Flavor Symmetries

Michael Ratz

March 26 2024

supported by NSF

Moriond 2024, La Thuile, Italy

Based in part on collaborations with:
Y. Almumin, M.-C. Chen, V. Knapp–Pérez, X. Li, X.-G. Liu, O. Medina, H.P. Nilles, M. Ramos–Hamud, S. Ramos–Sánchez & S. Shukla

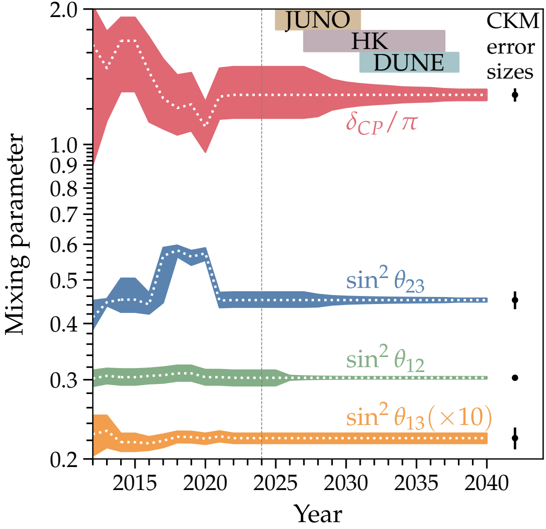
Outline

&

Disclaimers

Current and future precision of neutrino experiments

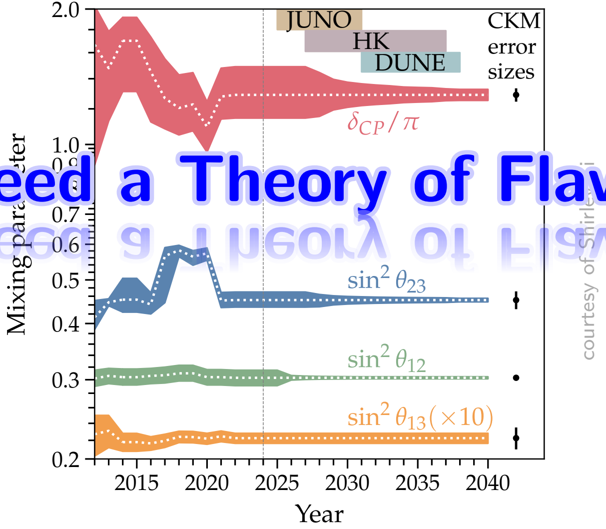
Song, Li, Argüelles, Bustamante & Vincent [2021]



courtesy of Shirley Li

Current and future precision of neutrino experiments

Song, Li, Argüelles, Bustamante & Vincent [2021]



Need a Theory of Flavor

Theories of flavor

- ☞ Froggatt–Nielsen models — Froggatt & Nielsen [1979],. . .
- ☞ Non-Abelian discrete flavor symmetries — Kaplan & Schmaltz [1994],. . .
- ☞ Warped extra dimensions — Randall & Sundrum [1999],. . .

Theories of flavor

- Froggatt–Nielsen models
  Froggatt & Nielsen [1979],...
- Non-Abelian discrete flavor symmetries
  Kaplan & Schmaltz [1994],...
- Warped extra dimensions
  Randall & Sundrum [1999],...
- New game in town:
  Feruglio [2017],...

Modular Flavor Symmetries

for reviews see e.g. Feruglio & Romanino [2021]
Almumin, Chen, Cheng, Knapp-Perez, Li, Mondol, Ramos-Sanchez, MR & Shukla [2023]
Kobayashi & Tanimoto [2023]
Ding & King [2023]
Ding & Valle [2024]

Theories of flavor

- ☞ Froggatt–Nielsen models — Froggatt & Nielsen [1979],...
- ☞ Non-Abelian discrete flavor symmetries — Kaplan & Schmaltz [1994],...
- ☞ Warped extra dimensions — Randall & Sundrum [1999],...
- ☞ New game in town: — Feruglio [2017],...

Modular Flavor Symmetries

Apologies and disclaimers:

- This talk will not have extensive references to model building activities that contribute to this exciting field, sorry!
- I will have to suppress many details which are not essential to get the big picture.

Modular

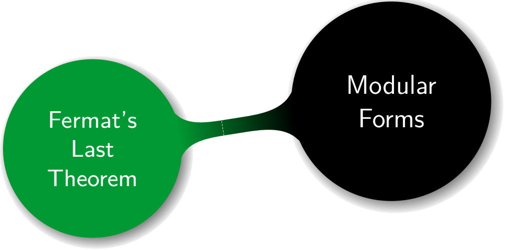
Flavor

Symmetries

Modular forms

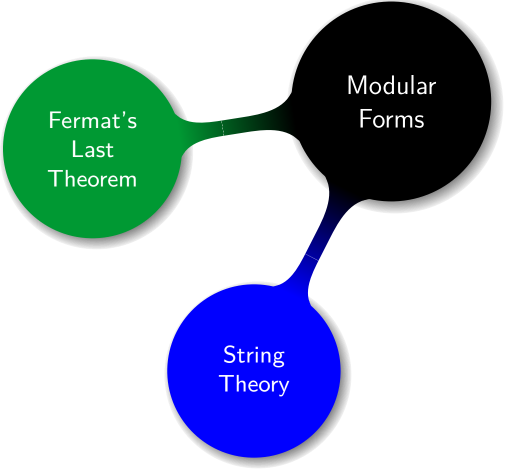
▸ details

☞ Modular forms are well known in mathematics and some areas of physics



Modular forms

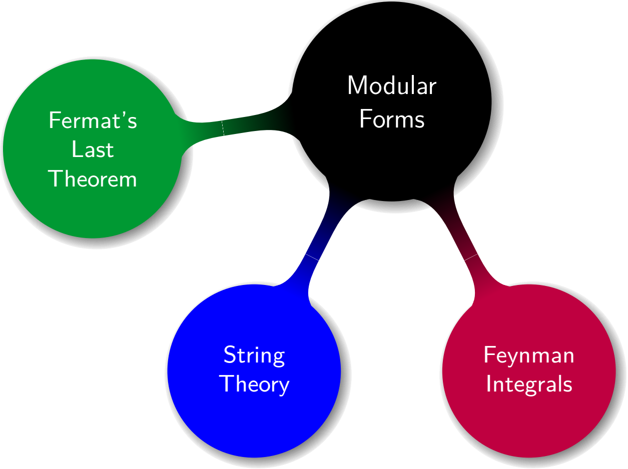
☞ Modular forms are well known in mathematics and some areas of physics



Modular forms & modular flavor symmetries

details

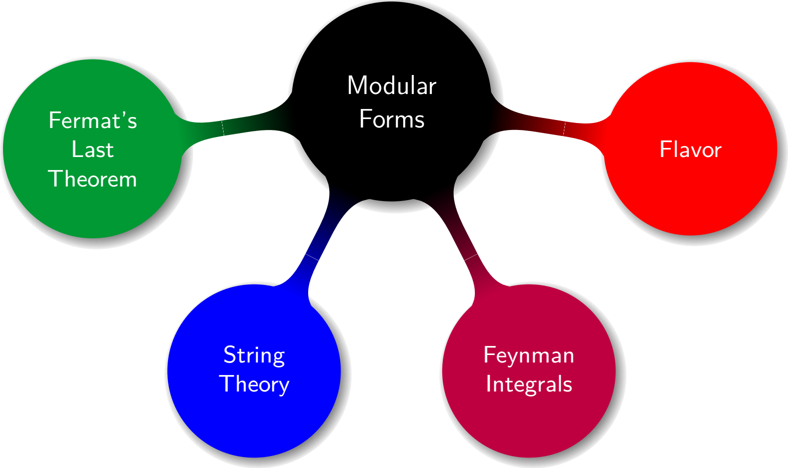
☞ Modular forms are well known in mathematics and some areas of physics



Modular forms & modular flavor symmetries

▸ details

☞ Modular forms are well known in mathematics and some areas of physics



Modular forms & modular flavor symmetries

details

- Modular forms are well known in mathematics and some areas of physics
- Traditional modular forms

$$f(\gamma\,\tau) = (c\tau + d)^k f(\tau)$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$$

Modular forms & modular flavor symmetries

details

- Modular forms are well known in mathematics and some areas of physics
- Traditional modular forms

  $$f(\gamma\,\tau) = (c\tau + d)^k f(\tau)$$

  $k \in \mathbb{N}$ modular weight

Modular forms & modular flavor symmetries

details

- Modular forms are well known in mathematics and some areas of physics
- Traditional modular forms

$$f(\gamma\,\tau) = (c\tau + d)^k f(\tau)$$

- Vector-valued modular forms

Liu & Ding [2022]

$$f_i(\gamma\,\tau) = (c\tau + d)^k \left[\rho_N(\gamma)\right]_{ij} f_j(\tau)$$

representation matrix of Γ_N
e.g. $\Gamma_3 = A_4$

Modular forms & modular flavor symmetries

details

- Modular forms are well known in mathematics and some areas of physics
- Traditional modular forms

$$f(\gamma\,\tau) = (c\tau + d)^k f(\tau)$$

- Vector-valued modular forms

Liu & Ding [2022]

$$f_i(\gamma\,\tau) = (c\tau + d)^k \left[\rho_N(\gamma)\right]_{ij} f_j(\tau)$$

Feruglio [2017]

Modular flavor symmetries:

What if Yukawa couplings are modular forms?

An explicit example

Feruglio [2017]

☞ Neutrino mass in traditional A_4 models

$$m_\nu = \frac{v_u^2}{\Lambda}\begin{pmatrix} 2a & -c & -b \\ -c & 2b & -a \\ -b & -a & 2c \end{pmatrix} \qquad \text{(traditional)}$$

VEV of the u-type Higgs in the MSSM

An explicit example

Feruglio [2017]

☞ Neutrino mass in traditional A_4 models

$$m_\nu = \frac{v_u^2}{\Lambda}\begin{pmatrix} 2a & -c & -b \\ -c & 2b & -a \\ -b & -a & 2c \end{pmatrix} \qquad \text{(traditional)}$$

☞ Neutrino mass in a "modular" A_4 model

$$m_\nu = \frac{v_u^2}{\Lambda}\begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}$$

known modular functions of τ

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} X_2^2 \\ \sqrt{2}X_1X_2 \\ -X_1^2 \end{pmatrix} \text{ where } \begin{cases} X_1(\tau) := 3\sqrt{2}\dfrac{\eta^3(3\tau)}{\eta(\tau)} \\ X_2(\tau) := -3\dfrac{\eta^3(3\tau)}{\eta(\tau)} - \dfrac{\eta^3(\tau/3)}{\eta(\tau)} \end{cases}$$

An explicit example

Feruglio [2017]

- Neutrino mass in traditional A_4 models

$$m_\nu = \frac{v_u^2}{\Lambda} \begin{pmatrix} 2a & -c & -b \\ -c & 2b & -a \\ -b & -a & 2c \end{pmatrix} \qquad \text{(traditional)}$$

- Neutrino mass in a "modular" A_4 model

$$m_\nu = \frac{v_u^2}{\Lambda} \begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}$$

- Highly predictive:

$$\left.\begin{array}{r} \Lambda \\ \mathrm{Re}\,\tau \\ \mathrm{Im}\,\tau \end{array}\right\} \xrightarrow{\text{predict}} \left\{\begin{array}{l} \text{3 mass eigenvalues } m_i \\ \text{3 mixing angles } \theta_{ij} \\ \text{3 phases (1 Dirac \& 2 Majorana)} \end{array}\right.$$

Predictions of the Feruglio model

Esteban, Gonzalez-Garcia, Maltoni, Schwetz & Zhou [2020]
Song, Li, Argüelles, Bustamante & Vincent [2021]

NuFIT 5.2 (2022)

without SK atmospheric data

	Inverted Ordering ($\Delta\chi^2 = 2.3$)	
	bfp $\pm 1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.303^{+0.012}_{-0.011}$	$0.270 \to 0.341$
$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$
$\sin^2\theta_{23}$	$0.578^{+0.016}_{-0.021}$	$0.412 \to 0.623$
$\theta_{23}/^\circ$	$49.5^{+0.9}_{-1.2}$	$39.9 \to 52.1$
$\sin^2\theta_{13}$	$0.02219^{+0.00060}_{-0.00057}$	$0.02047 \to 0.02396$
$\theta_{13}/^\circ$	$8.57^{+0.12}_{-0.11}$	$8.23 \to 8.90$
$\delta_{\mathrm{CP}}/^\circ$	286^{+27}_{-32}	$192 \to 360$
$\frac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \mathrm{eV}^2}$	$-2.498^{+0.032}_{-0.025}$	$-2.581 \to -2.408$

Feruglio [2017]

☞ Model predictions

$$\sin^2\theta_{12} = 0.295$$

$$\sin^2\theta_{23} = 0.651$$

$$\sin^2\theta_{13} = 0.0447$$

$$\delta_{\mathcal{CP}} = 279^\circ$$

$$\frac{\Delta m^2_{\mathrm{sol}}}{\Delta m^2_{\mathrm{atm}}} = 0.0292$$

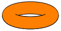
Modular Invariant

Holomorphic

Observables

Predictive power of modular symmetries

☞ Ingredients

 modular invariance

Predictive power of modular symmetries

- Ingredients
- modular invariance
- holomorphy

Predictive power of modular symmetries

- Ingredients
- modular invariance
- holomorphy
- finiteness

Predictive power of modular symmetries

- Ingredients

- modular invariance
- holomorphy
- finiteness

} superpotential essentially $\mathscr{W}$ fixed

Predictive power of modular symmetries

☞ Ingredients

- modular invariance
- holomorphy
- finiteness

} superpotential essentially $\mathscr{W}$ fixed

☞ However, typical observables are not holomorphic

e.g.
$$\left.\begin{aligned} \mathscr{W} &= \frac{\mathcal{M}(\tau)}{2}\,\Phi^2 \\ K &= \frac{1}{(-\mathrm{i}\,\tau+\mathrm{i}\,\overline{\tau})^{k_\Phi}}\,\overline{\Phi}\Phi \end{aligned}\right\} \curvearrowright\quad \begin{aligned} m_{\text{physical}} &= m_{\text{physical}}(\overline{\tau},\tau) \\ &= |\mathcal{M}(\tau)|\,(-\mathrm{i}\,\tau+\mathrm{i}\,\overline{\tau})^{k_\Phi} \end{aligned}$$

not entirely fixed by symmetries

Predictive power of modular symmetries

- Ingredients

- modular invariance
- holomorphy
- finiteness

} superpotential essentially $\mathscr{W}$ fixed

- However, typical observables are not holomorphic

e.g. $$\left.\begin{aligned}\mathscr{W} &= \frac{\mathcal{M}(\tau)}{2}\,\Phi^2\\ K &= \frac{1}{(-\mathrm{i}\,\tau+\mathrm{i}\,\overline{\tau})^{k_\Phi}}\overline{\Phi}\Phi\end{aligned}\right\} \curvearrowright\quad \begin{aligned} m_{\text{physical}} &= m_{\text{physical}}(\overline{\tau},\tau)\\ &= |\mathcal{M}(\tau)|\,(-\mathrm{i}\,\tau+\mathrm{i}\,\overline{\tau})^{k_\Phi}\end{aligned}$$

? Are there observables which fulfill modular invariance, holomorphy & finiteness?

Modular invariant holomorphic observables

- Typical model

$$\mathscr{W}_{\text{lepton}} = Y_e^{ij}\, L_i \cdot H_d\, E_j + \frac{1}{2}\kappa'_{ij}(\tau)\, L_i \cdot H_u\, L_j \cdot H_u$$

Modular invariant holomorphic observables

☞ Typical model

$$\mathscr{W}_{\text{lepton}} = Y_e^i\, L_i \cdot H_d\, E_i + \frac{1}{2}\kappa_{ij}(\tau)\, L_i \cdot H_u\, L_j \cdot H_u$$

diagonal

Modular invariant holomorphic observables

- Typical model

$$\mathscr{W}_{\text{lepton}} = Y_e^i \, L_i \cdot H_d \, E_i + \frac{1}{2} \kappa_{ij}(\tau) \, L_i \cdot H_u \, L_j \cdot H_u$$

- Modular invariant holomorphic observables

Chen, Li, Liu, Medina & MR [2024]

$$I_{ij}(\tau) := \frac{M_{ii}(\tau)\, M_{jj}(\tau)}{\bigl(M_{ij}(\tau)\bigr)^2} = \frac{\kappa_{ii}(\tau)\, \kappa_{jj}(\tau)}{\bigl(\kappa_{ij}(\tau)\bigr)^2} = \frac{m_{ii}(\tau,\overline{\tau})\, m_{jj}(\tau,\overline{\tau})}{\bigl(m_{ij}(\tau,\overline{\tau})\bigr)^2}$$

Modular invariant holomorphic observables

- Typical model

$$\mathscr{W}_{\text{lepton}} = Y_e^i\, L_i \cdot H_d\, E_i + \frac{1}{2}\kappa_{ij}(\tau)\, L_i \cdot H_u\, L_j \cdot H_u$$

- Modular invariant holomorphic observables

Chen, Li, Liu, Medina & MR [2024]

$$I_{ij}(\tau) := \frac{M_{ii}(\tau)\, M_{jj}(\tau)}{\bigl(M_{ij}(\tau)\bigr)^2} = \frac{\kappa_{ii}(\tau)\, \kappa_{jj}(\tau)}{\bigl(\kappa_{ij}(\tau)\bigr)^2} = \frac{m_{ii}(\tau,\overline{\tau})\, m_{jj}(\tau,\overline{\tau})}{\bigl(m_{ij}(\tau,\overline{\tau})\bigr)^2}$$

- I_{ij} are entirely given by masses, mixing angles and phases, e.g.

$$I_{12} = \frac{a_0\left[\widetilde{m}_1\left(\mathrm{e}^{\mathrm{i}\,\delta} c_{23} s_{12} + c_{12} s_{13} s_{23}\right)^2 + \widetilde{m}_2\left(\mathrm{e}^{\mathrm{i}\,\delta} c_{12} c_{23} - s_{12} s_{13} s_{23}\right)^2 + \mathrm{e}^{2\,\mathrm{i}\,\delta} m_3 c_{13}^2 s_{23}^2\right]}{c_{13}^2\left[\widetilde{m}_1 c_{12}\left(\mathrm{e}^{\mathrm{i}\,\delta} c_{23} s_{12} + c_{12} s_{13} s_{23}\right) + \widetilde{m}_2 s_{12}\left(s_{12} s_{13} s_{23} - \mathrm{e}^{\mathrm{i}\,\delta} c_{12} c_{23}\right) - \mathrm{e}^{2\,\mathrm{i}\,\delta} m_3 s_{13} s_{23}\right]^2}$$

Modular invariant holomorphic observables

- Typical model

$$\mathscr{W}_{\text{lepton}} = Y_e^i\, L_i \cdot H_d\, E_i + \frac{1}{2}\kappa_{ij}(\tau)\, L_i \cdot H_u\, L_j \cdot H_u$$

- Modular invariant holomorphic observables

Chen, Li, Liu, Medina & MR [2024]

$$I_{ij}(\tau) := \frac{M_{ii}(\tau)\, M_{jj}(\tau)}{\bigl(M_{ij}(\tau)\bigr)^2} = \frac{\kappa_{ii}(\tau)\, \kappa_{jj}(\tau)}{\bigl(\kappa_{ij}(\tau)\bigr)^2} = \frac{m_{ii}(\tau,\overline{\tau})\, m_{jj}(\tau,\overline{\tau})}{\bigl(m_{ij}(\tau,\overline{\tau})\bigr)^2}$$

- I_{ij} are entirely given by masses, mixing angles and phases, e.g.

$$I_{12} = \frac{a_0\left[\widetilde{m}_1\left(\mathrm{e}^{\mathrm{i}\,\delta} c_{23} s_{12} + c_{12} s_{13} s_{23}\right)^2 + \widetilde{m}_2\left(\mathrm{e}^{\mathrm{i}\,\delta} c_{12} c_{23} - s_{12} s_{13} s_{23}\right)^2 + \mathrm{e}^{2\,\mathrm{i}\,\delta} m_3 c_{13}^2 s_{23}^2\right]}{c_{13}^2\left[\widetilde{m}_1 c_{12}\left(\mathrm{e}^{\mathrm{i}\,\delta} c_{23} s_{12} + c_{12} s_{13} s_{23}\right) + \widetilde{m}_2 s_{12}\left(s_{12} s_{13} s_{23} - \mathrm{e}^{\mathrm{i}\,\delta} c_{12} c_{23}\right) - \mathrm{e}^{2\,\mathrm{i}\,\delta} m_3 s_{13} s_{23}\right]^2}$$

$\widetilde{m}_1 := m_1\, \mathrm{e}^{\mathrm{i}\,\varphi_1}$

$\widetilde{m}_2 := m_2\, \mathrm{e}^{\mathrm{i}\,\varphi_2}$

Modular invariant holomorphic observables

- Typical model

$$\mathscr{W}_{\text{lepton}} = Y_e^i\, L_i \cdot H_d\, E_i + \frac{1}{2}\kappa_{ij}(\tau)\, L_i \cdot H_u\, L_j \cdot H_u$$

- Modular invariant holomorphic observables

Chen, Li, Liu, Medina & MR [2024]

$$I_{ij}(\tau) := \frac{M_{ii}(\tau)\, M_{jj}(\tau)}{\bigl(M_{ij}(\tau)\bigr)^2} = \frac{\kappa_{ii}(\tau)\,\kappa_{jj}(\tau)}{\bigl(\kappa_{ij}(\tau)\bigr)^2} = \frac{m_{ii}(\tau,\overline{\tau})\, m_{jj}(\tau,\overline{\tau})}{\bigl(m_{ij}(\tau,\overline{\tau})\bigr)^2}$$

- I_{ij} are entirely given by masses, mixing angles and phases, e.g.

$$I_{12} = \frac{a_0\left[\widetilde{m}_1\left(\mathrm{e}^{\mathrm{i}\,\delta} c_{23} s_{12} + c_{12} s_{13} s_{23}\right)^2 + \widetilde{m}_2\left(\mathrm{e}^{\mathrm{i}\,\delta} c_{12} c_{23} - s_{12} s_{13} s_{23}\right)^2 + \mathrm{e}^{2\,\mathrm{i}\,\delta} m_3 c_{13}^2 s_{23}^2\right]}{c_{13}^2\left[\widetilde{m}_1 c_{12}\left(\mathrm{e}^{\mathrm{i}\,\delta} c_{23} s_{12} + c_{12} s_{13} s_{23}\right) + \widetilde{m}_2 s_{12}\left(s_{12} s_{13} s_{23} - \mathrm{e}^{\mathrm{i}\,\delta} c_{12} c_{23}\right) - \mathrm{e}^{2\,\mathrm{i}\,\delta} m_3 s_{13} s_{23}\right]^2}$$

$$c_{ij} := \cos\theta_{ij}$$

Modular invariant holomorphic observables

- Typical model

$$\mathscr{W}_{\text{lepton}} = Y_e^i\, L_i \cdot H_d\, E_i + \frac{1}{2}\kappa_{ij}(\tau)\, L_i \cdot H_u\, L_j \cdot H_u$$

- Modular invariant holomorphic observables

Chen, Li, Liu, Medina & MR [2024]

$$I_{ij}(\tau) := \frac{M_{ii}(\tau)\, M_{jj}(\tau)}{\bigl(M_{ij}(\tau)\bigr)^2} = \frac{\kappa_{ii}(\tau)\, \kappa_{jj}(\tau)}{\bigl(\kappa_{ij}(\tau)\bigr)^2} = \frac{m_{ii}(\tau,\overline{\tau})\, m_{jj}(\tau,\overline{\tau})}{\bigl(m_{ij}(\tau,\overline{\tau})\bigr)^2}$$

- I_{ij} are entirely given by masses, mixing angles and phases, e.g.

$$I_{12} = \frac{a_0\left[\widetilde{m}_1\left(\mathrm{e}^{\mathrm{i}\,\delta} c_{23} s_{12} + c_{12} s_{13} s_{23}\right)^2 + \widetilde{m}_2\left(\mathrm{e}^{\mathrm{i}\,\delta} c_{12} c_{23} - s_{12} s_{13} s_{23}\right)^2 + \mathrm{e}^{2\,\mathrm{i}\,\delta} m_3 c_{13}^2 s_{23}^2\right]}{c_{13}^2\left[\widetilde{m}_1 c_{12}\left(\mathrm{e}^{\mathrm{i}\,\delta} c_{23} s_{12} + c_{12} s_{13} s_{23}\right) + \widetilde{m}_2 s_{12}\left(s_{12} s_{13} s_{23} - \mathrm{e}^{\mathrm{i}\,\delta} c_{12} c_{23}\right) - \mathrm{e}^{2\,\mathrm{i}\,\delta} m_3 s_{13} s_{23}\right]^2}$$

$$s_{ij} := \sin\theta_{ij}$$

Modular invariant holomorphic observables

- Typical model

$$\mathscr{W}_{\text{lepton}} = Y_e^i\, L_i \cdot H_d\, E_i + \frac{1}{2}\kappa_{ij}(\tau)\, L_i \cdot H_u\, L_j \cdot H_u$$

- Modular invariant holomorphic observables

Chen, Li, Liu, Medina & MR [2024]

$$I_{ij}(\tau) := \frac{M_{ii}(\tau)\, M_{jj}(\tau)}{\bigl(M_{ij}(\tau)\bigr)^2} = \frac{\kappa_{ii}(\tau)\, \kappa_{jj}(\tau)}{\bigl(\kappa_{ij}(\tau)\bigr)^2} = \frac{m_{ii}(\tau,\overline{\tau})\, m_{jj}(\tau,\overline{\tau})}{\bigl(m_{ij}(\tau,\overline{\tau})\bigr)^2}$$

- I_{ij} are entirely given by masses, mixing angles and phases, e.g.

$$I_{12} = \frac{a_0\left[\widetilde{m}_1\left(\mathrm{e}^{\mathrm{i}\,\delta}c_{23}s_{12}+c_{12}s_{13}s_{23}\right)^2+\widetilde{m}_2\left(\mathrm{e}^{\mathrm{i}\,\delta}c_{12}c_{23}-s_{12}s_{13}s_{23}\right)^2+\mathrm{e}^{2\,\mathrm{i}\,\delta}m_3c_{13}^2s_{23}^2\right]}{c_{13}^2\left[\widetilde{m}_1c_{12}\left(\mathrm{e}^{\mathrm{i}\,\delta}c_{23}s_{12}+c_{12}s_{13}s_{23}\right)+\widetilde{m}_2s_{12}\left(s_{12}s_{13}s_{23}-\mathrm{e}^{\mathrm{i}\,\delta}c_{12}c_{23}\right)-\mathrm{e}^{2\,\mathrm{i}\,\delta}m_3s_{13}s_{23}\right]^2}$$

$$a_0 := \left(\widetilde{m}_1c_{12}^2+\widetilde{m}_2s_{12}^2\right)c_{13}^2+\mathrm{e}^{2\,\mathrm{i}\,\delta}m_3s_{13}^2$$

Modular invariant holomorphic observables

- Typical model

$$\mathscr{W}_{\text{lepton}} = Y_e^i\, L_i \cdot H_d\, E_i + \frac{1}{2}\kappa_{ij}(\tau)\, L_i \cdot H_u\, L_j \cdot H_u$$

- Modular invariant holomorphic observables

Chen, Li, Liu, Medina & MR [2024]

$$I_{ij}(\tau) := \frac{M_{ii}(\tau)\, M_{jj}(\tau)}{\bigl(M_{ij}(\tau)\bigr)^2} = \frac{\kappa_{ii}(\tau)\, \kappa_{jj}(\tau)}{\bigl(\kappa_{ij}(\tau)\bigr)^2} = \frac{m_{ii}(\tau,\overline{\tau})\, m_{jj}(\tau,\overline{\tau})}{\bigl(m_{ij}(\tau,\overline{\tau})\bigr)^2}$$

- I_{ij} are entirely given by masses, mixing angles and phases, e.g.

$$I_{12} = \frac{a_0\left[\widetilde{m}_1\left(\mathrm{e}^{\mathrm{i}\,\delta}c_{23}s_{12}+c_{12}s_{13}s_{23}\right)^2+\widetilde{m}_2\left(\mathrm{e}^{\mathrm{i}\,\delta}c_{12}c_{23}-s_{12}s_{13}s_{23}\right)^2+\mathrm{e}^{2\,\mathrm{i}\,\delta}m_3c_{13}^2s_{23}^2\right]}{c_{13}^2\left[\widetilde{m}_1c_{12}\left(\mathrm{e}^{\mathrm{i}\,\delta}c_{23}s_{12}+c_{12}s_{13}s_{23}\right)+\widetilde{m}_2s_{12}\left(s_{12}s_{13}s_{23}-\mathrm{e}^{\mathrm{i}\,\delta}c_{12}c_{23}\right)-\mathrm{e}^{2\,\mathrm{i}\,\delta}m_3s_{13}s_{23}\right]^2}$$

- I_{ij} are invariant under the renormalization group

Chang & Kuo [2002]

Invariants in the Feruglio model

- Invariants

$$I_{12}(\tau) = -2$$

Invariants in the Feruglio model

☞ Invariants

$$\left.\begin{aligned} I_{12}(\tau) &= -2 \\ I_{13}(\tau) &= -2\left(1+\frac{1}{3}j_3(\tau)\right)^3 \\ I_{23}(\tau) &= -\frac{32}{I_{13}} = \frac{16}{\left(1+\frac{1}{3}j_3(\tau)\right)^3} \end{aligned}\right\} \curvearrowright \; I_{13}\,I_{23} = -32$$

Hauptmodul of $\Gamma(3)$
$j_3(\tau) := \eta(\tau/3)^3/\eta(3\tau)^3$

Invariants in the Feruglio model

☞ Invariants

$$\left.\begin{aligned} I_{12}(\tau) &= -2 \\ I_{13}(\tau) &= -2\left(1+\frac{1}{3}j_3(\tau)\right)^3 \\ I_{23}(\tau) &= -\frac{32}{I_{13}} = \frac{16}{\left(1+\frac{1}{3}j_3(\tau)\right)^3} \end{aligned}\right\} \curvearrowright \quad I_{13}\, I_{23} = -32$$

☞ A_4 sum rule

Chuliá Centelles, Cepedello & Medina [2022]; Centelles Chuliá, Kumar, Popov & Srivastava [2024]

$$m_3 = \begin{cases} m_2 + m_1 & \text{for normal ordering (NO)} \\ m_2 - m_1 & \text{for inverted ordering (IO)} \end{cases}$$

see talk by S. King for more sum rules

Invariants in the Feruglio model

☞ Invariants

$$\left.\begin{aligned} I_{12}(\tau) &= -2 \\ I_{13}(\tau) &= -2\left(1+\frac{1}{3}j_3(\tau)\right)^3 \\ I_{23}(\tau) &= -\frac{32}{I_{13}} = \frac{16}{\left(1+\frac{1}{3}j_3(\tau)\right)^3} \end{aligned}\right\} \curvearrowright \; I_{13}\, I_{23} = -32$$

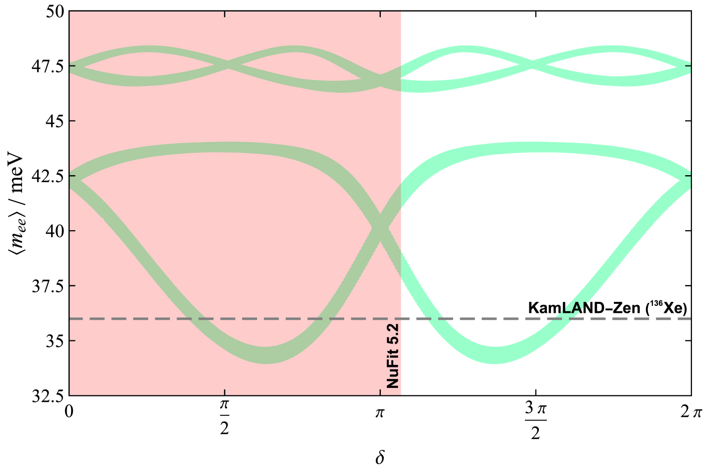
☞ A_4 sum rule

Chuliá Centelles, Cepedello & Medina [2022]; Centelles Chuliá, Kumar, Popov & Srivastava [2024]

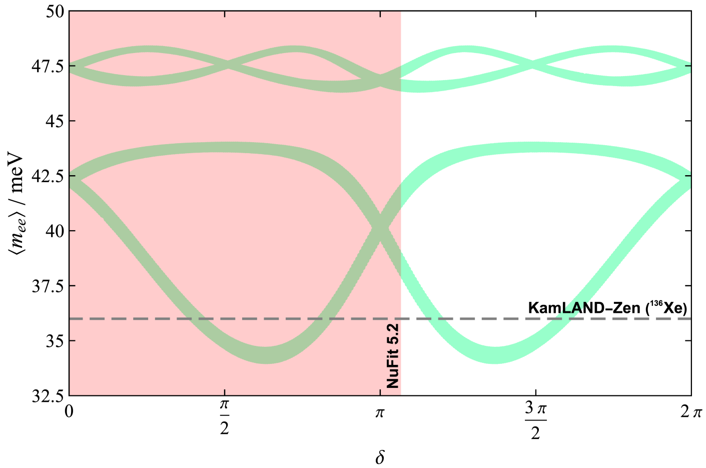
$$m_3 = \begin{cases} m_2 + m_1 & \text{for normal ordering (NO)} \\ m_2 - m_1 & \text{for inverted ordering (IO)} \end{cases}$$

➥ Imposing $I_{12} = -2$ and experimental constraints leaves one free parameter, e.g. δ

An application: m_{ee} as a function of δ



An application: m_{ee} as a function of δ



☞ Imposing $I_{13}\,I_{23} = -32$ rules out the model for all τ

Summary

&

Outlook

Take-home messages

☞ New game in town: modular flavor symmetries

Take-home messages

- New game in town: modular flavor symmetries
- Superpotential constrained by modular invariance (infinite group)

UV completion of modular symmetries

☞ Bottom-up constructions may face problems in the UV

https://workshops.ift.uam-csic.es/uploads/poster/poster_congreso_299.pdf

UV completion of modular symmetries

- ☞ Bottom-up constructions may face problems in the UV
- ☞ Couplings *are* modular forms in a large class of string compactifications

e.g. Chun, Mas, Lauer & Nilles [1989]; Erler, Jungnickel, Spaliński & Stieberger [1993]; Quevedo [1996]

This is nothing but one of the $SL(2,\mathbf{Z})_{T,U}$ transformation for toroidal orbifold compactifications ($a = b = d = 1, c = 0$ in eq. (10)). Therefore the only conditions these symmetries impose on W is that it should transform as a modular form of a given weight ($W \to (cT+d)^{-3}W$ for the simplest toroidal orbifolds with T the overall size of the compactification space)[36]. In fact, explicit calculations for specific orbifold models show that

$$W_{tree}(T, Q^I) = \chi_{IJK}(T)\, Q^I Q^J Q^K + \cdots \qquad (19)$$

with $\chi(T)$ a particular modular form of $SL(2,\mathbf{Z})$ or any other duality group and the ellipsis represent higher powers of Q, exponentially suppressed. The identification of $\chi(T)$ with modular forms was a highly nontrivial check of the explicit orbifold calculations which were preformed in refs. [37] without any relation (nor knowledge) of the underlying duality symmetry $SL(2,\mathbf{Z})$. This kind of symmetry puts also strong constraints to the higher order, nonrenormalizable, corrections to W, since each matter field Q transforms in a particular way under that symmetry ($Q \to (cT+d)^n Q$ with n the modular weight of Q). There are also other discrete symmetries, as those defined by the point group $\mathcal{P}$ and space group $\mathcal{S}$ of an orbifold which have to be respected by the superpotential W. These 'selection rules' are very important to find vanishing

UV completion of modular symmetries

- Bottom-up constructions may face problems in the UV
- Couplings *are* modular forms in a large class of string compactifications

e.g. Chun, Mas, Lauer & Nilles [1989]; Erler, Jungnickel, Spaliński & Stieberger [1993]; Quevedo [1996]

- Eclectic flavor symmetries

Baur, Nilles, Trautner & Vaudrevange [2019]
Nilles, Ramos–Sánchez & Vaudrevange [2021]; Baur, Kade, Nilles, Ramos-Sanchez & Vaudrevange [2021]

UV completion of modular symmetries

- Bottom-up constructions may face problems in the UV
- Couplings *are* modular forms in a large class of string compactifications

  e.g. Chun, Mas, Lauer & Nilles [1989]; Erler, Jungnickel, Spaliński & Stieberger [1993]; Quevedo [1996]

- Eclectic flavor symmetries

  Baur, Nilles, Trautner & Vaudrevange [2019]
  Nilles, Ramos–Sánchez & Vaudrevange [2021]; Baur, Kade, Nilles, Ramos-Sanchez & Vaudrevange [2021]

- Modular flavor symmetries arise from magnetized tori

  Cremades, Ibáñez & Marchesano [2004]
  Kikuchi, Kobayashi & Uchida [2021]; Almumin, Chen, Knapp-Pérez, Ramos-Sánchez, MR & Shukla [2021]

UV completion of modular symmetries

- Bottom-up constructions may face problems in the UV
- Couplings *are* modular forms in a large class of string compactifications

  e.g. Chun, Mas, Lauer & Nilles [1989]; Erler, Jungnickel, Spaliński & Stieberger [1993]; Quevedo [1996]

- Eclectic flavor symmetries

  Baur, Nilles, Trautner & Vaudrevange [2019]
  Nilles, Ramos–Sánchez & Vaudrevange [2021]; Baur, Kade, Nilles, Ramos-Sanchez & Vaudrevange [2021]

- Modular flavor symmetries arise from magnetized tori

  Cremades, Ibáñez & Marchesano [2004]
  Kikuchi, Kobayashi & Uchida [2021]; Almumin, Chen, Knapp-Pérez, Ramos-Sánchez, MR & Shukla [2021]

- Top-down mechanisms to stabilize τ are in qualitative agreement with τ values required by phenomenological fits

  Knapp-Perez, Liu, Nilles, Ramos-Sanchez & MR [2023]

Open questions

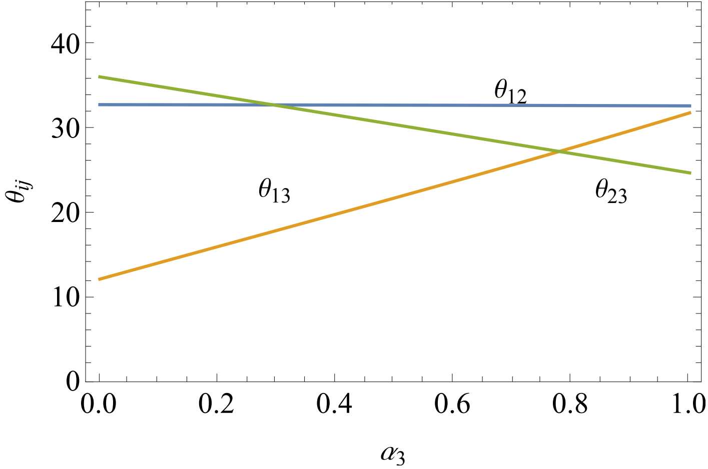
☞ Can one replicate the features without SUSY?

speculations were made in Cremades, Ibáñez & Marchesano [2004]
Almumin, Chen, Knapp-Pérez, Ramos-Sánchez, MR & Shukla [2021]

Open questions

- ☞ Can one replicate the features without SUSY?
- ☞ Extra parameters in the Kähler potential

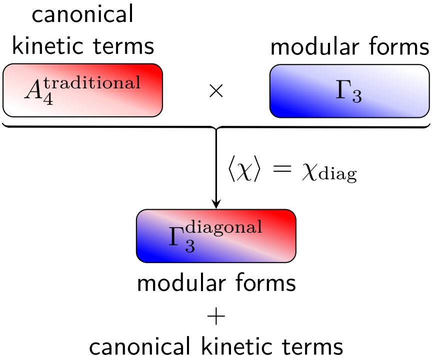
Chen, Ramos-Sánchez & MR [2020]



Open questions

- Can one replicate the features without SUSY?
- Extra parameters in the Kähler potential — Chen, Ramos-Sánchez & MR [2020]
- Proof-of-principle solutions exist but are admittedly not too elegant

Chen, Knapp–Pérez, Ramos-Hamud, Ramos–Sánchez, MR & Shukla [2022]



Executive Summary & Outlook

Modular flavor symmetries

- ☞ work surprisingly well phenomenologically
- ☞ host modular invariant holomorphic observables
- ☞ have a clear top-down motivation and can emerge from explicit string models
- ☞ are not fully explored and some key questions (importance of supersymmetry, corrections to kinetic terms, etc.) are awaiting answers

FLASY
2024
@ UCI

Mille grazie!

Merci beaucoup!

Vielen Dank!

Thank you very much!

Backup slides

Modular

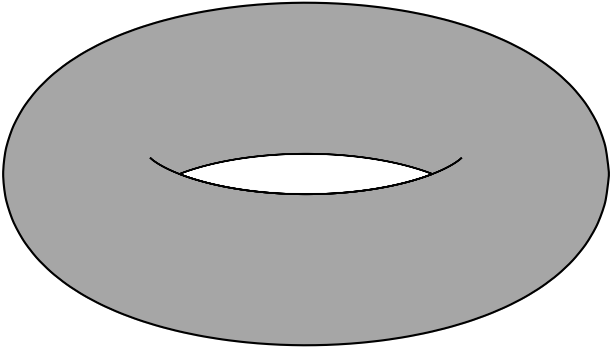
Flavor

Symmetries

(Some details)

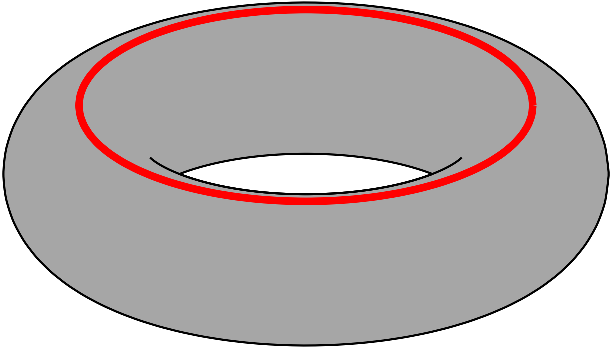
Tori

- Torus = donut



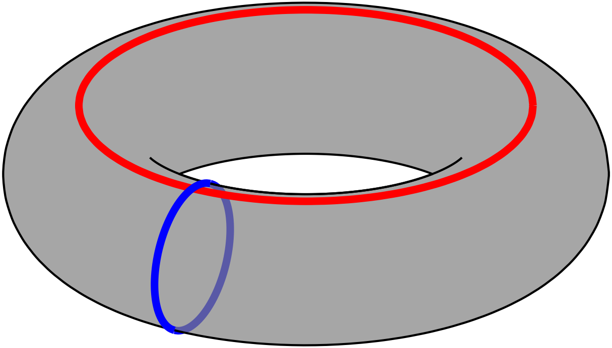
Tori

- Torus = donut
- Two cycles



Tori

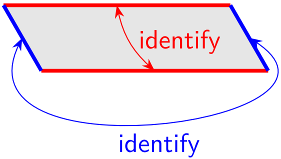
- Torus = donut
- Two cycles



Tori

☞ Torus can be thought of as a parallelogram (which emerges by cutting the torus open along the red and blue cycles)

Tori

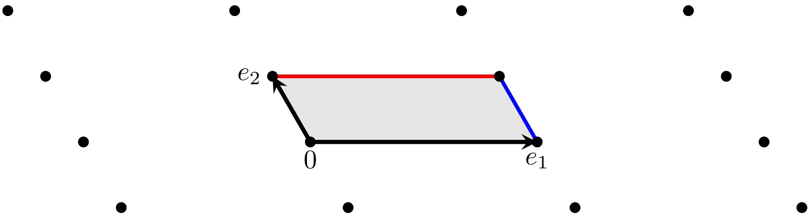


☞ Opposite edges get identified

Tori

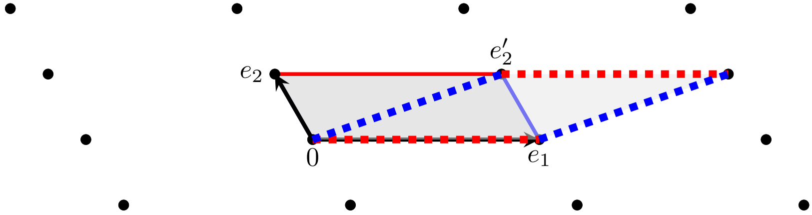
☞ Edges define basis vectors of a lattice

Tori



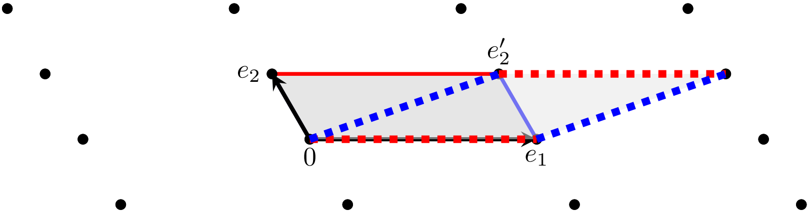
☞ Torus is $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$: two points in the plane get identified if they differ by a lattice translation

Tori



☞ Fundamental domain is not unique

Tori

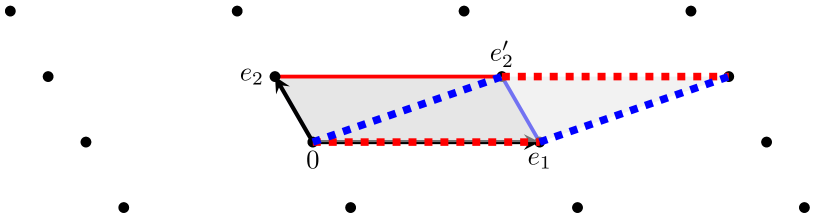


- ☞ Fundamental domain is not unique
- ☞ We can build linear combinations of the basis vectors

$$\begin{pmatrix} e_2 \\ e_1 \end{pmatrix} \xmapsto{\gamma} \begin{pmatrix} e_2' \\ e_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_2 \\ e_1 \end{pmatrix} =: \gamma \begin{pmatrix} e_2 \\ e_1 \end{pmatrix}$$

$a, b, c, d \in \mathbb{Z}$

Tori



- Fundamental domain is not unique
- We can build linear combinations of the basis vectors

$$\begin{pmatrix} e_2 \\ e_1 \end{pmatrix} \xrightarrow{\gamma} \begin{pmatrix} e_2' \\ e_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_2 \\ e_1 \end{pmatrix} =: \gamma \begin{pmatrix} e_2 \\ e_1 \end{pmatrix}$$

- Volume of fundamental domain stays the same $\Leftrightarrow \det\gamma = 1 \curvearrowright$ $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ (there is a superfluous sign, so $\gamma \in \Gamma = \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$)

$\mathrm{SL}(2,\mathbb{Z})$

☞ Two basic transformations

$$T \ : e_2 \xrightarrow{T} e_2' = e_2 + e_1 \qquad \curvearrowright \gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} =: T$$

$$S \ : e_1 \xrightarrow{S} e_1' = e_2 \quad \& \quad e_2 \mapsto e_2' = -e_1 \qquad \curvearrowright \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} =: S$$

$\mathrm{SL}(2,\mathbb{Z})$

☞ Two basic transformations

$$T \;:\; e_2 \xrightarrow{T} e_2' = e_2 + e_1 \qquad \curvearrowright \gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} =: T$$

$$S \;:\; e_1 \xrightarrow{S} e_1' = e_2 \quad \& \quad e_2 \mapsto e_2' = -e_1 \qquad \curvearrowright \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} =: S$$

☞ Complex structure modulus $\tau = e_2/e_1$ w/ $\operatorname{Im}\tau > 0$

$\tau \xrightarrow{T} \tau + 1$ and $\tau \xrightarrow{S} \dfrac{-1}{\tau}$

$\mathrm{SL}(2,\mathbb{Z})$ and modular flavor symmetries

- Two basic transformations

$$T \;:\; e_2 \xmapsto{T} e_2' = e_2 + e_1 \qquad \curvearrowright \gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} =: T$$

$$S \;:\; e_1 \xmapsto{S} e_1' = e_2 \quad \& \quad e_2 \mapsto e_2' = -e_1 \qquad \curvearrowright \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} =: S$$

- Complex structure modulus $\tau = e_2/e_1$ w/ $\operatorname{Im}\tau > 0$

$$\tau \xmapsto{T} \tau + 1 \quad \text{and} \quad \tau \xmapsto{S} \frac{-1}{\tau}$$

- S and T generate $\mathrm{SL}(2,\mathbb{Z})$ and $S^2 = -\mathbb{1}$ & $(S\,T)^3 = \mathbb{1}$

$\mathrm{SL}(2,\mathbb{Z})$ and modular flavor symmetries

- Two basic transformations

$$T\ :\ e_2 \xmapsto{T} e_2' = e_2 + e_1 \qquad \curvearrowright \gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} =: T$$

$$S\ :\ e_1 \xmapsto{S} e_1' = e_2 \quad \& \quad e_2 \mapsto e_2' = -e_1 \qquad \curvearrowright \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} =: S$$

- Complex structure modulus $\tau = e_2/e_1$ w/ $\operatorname{Im}\tau > 0$

$$\tau \xmapsto{T} \tau + 1 \quad \text{and} \quad \tau \xmapsto{S} \frac{-1}{\tau}$$

- S and T generate $\mathrm{SL}(2,\mathbb{Z})$ and $S^2 = -\mathbb{1}$ & $(S\,T)^3 = \mathbb{1}$

Modular flavor symmetries:

Identify finite groups with generators satisfying $\Gamma = \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$ relations

$$S^2 = (S\,T)^3 = \mathbb{1} \quad \& \quad \text{additional relations}$$

$\mathrm{SL}(2,\mathbb{Z})$ and modular flavor symmetries

- Two basic transformations

$$T \ : e_2 \xrightarrow{T} e_2' = e_2 + e_1 \qquad \curvearrowright \gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} =: T$$

$$S \ : e_1 \xrightarrow{S} e_1' = e_2 \quad \& \quad e_2 \mapsto e_2' = -e_1 \qquad \curvearrowright \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} =: S$$

- Complex structure modulus $\tau = e_2/e_1$ w/ $\mathrm{Im}\,\tau > 0$

$$\tau \xrightarrow{T} \tau + 1 \quad \text{and} \quad \tau \xrightarrow{S} \frac{-1}{\tau}$$

- S and T generate $\mathrm{SL}(2,\mathbb{Z})$ and $S^2 = -\mathbb{1}$ & $(S\,T)^3 = \mathbb{1}$

Modular flavor symmetries:

Identify finite groups with generators satisfying $\Gamma = \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$ relations

$$S^2 = (S\,T)^3 = \mathbb{1} \quad \& \quad \text{additional relations}$$

Modular flavor symmetries

back

☞ Finite subgroups $\Gamma_N := \Gamma/\Gamma(N)$ where

Feruglio [2017]

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;;\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

level

$\Gamma = \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$

Modular flavor symmetries

back

☞ Finite subgroups $\Gamma_N := \Gamma/\Gamma(N)$ where

Feruglio [2017]

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \; ; \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

☞ E.g. $\Gamma_3 \simeq A_4$ (symmetry of tetrahedron)

Modular flavor symmetries

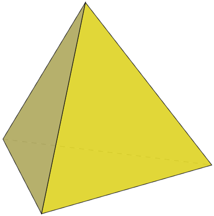
back

☞ Finite subgroups $\Gamma_N := \Gamma/\Gamma(N)$ where

Feruglio [2017]

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\ ;\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

☞ E.g. $\Gamma_3 \simeq A_4$ (symmetry of tetrahedron)



Modular flavor symmetries

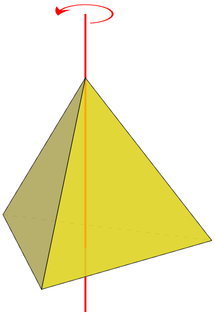
back

- Finite subgroups $\Gamma_N := \Gamma/\Gamma(N)$ where

Feruglio [2017]

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \; ; \; \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

- E.g. $\Gamma_3 \simeq A_4$ (symmetry of tetrahedron)



Modular flavor symmetries

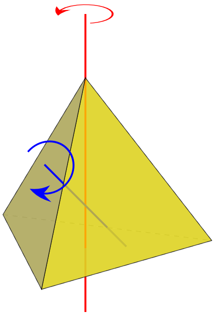
back

- Finite subgroups $\Gamma_N := \Gamma/\Gamma(N)$ where

Feruglio [2017]

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \; ; \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

- E.g. $\Gamma_3 \simeq A_4$ (symmetry of tetrahedron)



References I

Yahya Almumin, Mu-Chun Chen, Víctor Knapp-Pérez, Saúl Ramos-Sánchez, Michael Ratz, and Shreya Shukla. Metaplectic Flavor Symmetries from Magnetized Tori. *JHEP*, 05:078, 2021. doi: 10.1007/JHEP05(2021)078.

Yahya Almumin, Mu-Chun Chen, Murong Cheng, Victor Knapp-Perez, Yulun Li, Adreja Mondol, Saul Ramos-Sanchez, Michael Ratz, and Shreya Shukla. Neutrino Flavor Model Building and the Origins of Flavor and Violation. *Universe*, 9(12):512, 2023. doi: 10.3390/universe9120512.

Alexander Baur, Hans Peter Nilles, Andreas Trautner, and Patrick K. S. Vaudrevange. A String Theory of Flavor and $\mathscr{C}\mathscr{P}$. *Nucl. Phys. B*, 947: 114737, 2019. doi: 10.1016/j.nuclphysb.2019.114737.

Alexander Baur, Moritz Kade, Hans Peter Nilles, Saul Ramos-Sanchez, and Patrick K. S. Vaudrevange. The eclectic flavor symmetry of the $\mathbb{Z}_2$ orbifold. *JHEP*, 02:018, 2021. doi: 10.1007/JHEP02(2021)018.

Salvador Centelles Chuliá, Ranjeet Kumar, Oleg Popov, and Rahul Srivastava. Neutrino mass sum rules from modular A4 symmetry. *Phys. Rev. D*, 109(3): 035016, 2024. doi: 10.1103/PhysRevD.109.035016.

References II

Sanghyeon Chang and Tzee-Ke Kuo. Renormalization invariants of the neutrino mass matrix. *Phys. Rev. D*, 66:111302, 2002. doi: 10.1103/PhysRevD.66.111302.

Mu-Chun Chen, Saúl Ramos-Sánchez, and Michael Ratz. A note on the predictions of models with modular flavor symmetries. *Phys. Lett. B*, 801: 135153, 2020. doi: 10.1016/j.physletb.2019.135153.

Mu-Chun Chen, Víctor Knapp–Pérez, Mario Ramos-Hamud, Saúl Ramos–Sánchez, Michael Ratz, and Shreya Shukla. Quasi–eclectic modular flavor symmetries. *Phys. Lett. B*, 824:136843, 2022. doi: 10.1016/j.physletb.2021.136843.

Mu-Chun Chen, Xueqi Li, Xiang-Gan Liu, Omar Medina, and Michael Ratz. Modular invariant holomorphic observables. 1 2024.

Salvador Chuliá Centelles, Ricardo Cepedello, and Omar Medina. Absolute neutrino mass scale and dark matter stability from flavour symmetry. *JHEP*, 10:080, 2022. doi: 10.1007/JHEP10(2022)080.

References III

E. J. Chun, J. Mas, J. Lauer, and Hans Peter Nilles. Duality and Landau–Ginzburg Models. *Phys. Lett. B*, 233:141–146, 1989. doi: 10.1016/0370-2693(89)90630-8.

D. Cremades, L. E. Ibáñez, and F. Marchesano. Computing Yukawa couplings from magnetized extra dimensions. *JHEP*, 05:079, 2004. doi: 10.1088/1126-6708/2004/05/079.

Gui-Jun Ding and Stephen F. King. Neutrino Mass and Mixing with Modular Symmetry. 11 2023.

Gui-Jun Ding and Jose W. F. Valle. The symmetry approach to quark and lepton masses and mixing. 2 2024.

J. Erler, D. Jungnickel, M. Spaliński, and S. Stieberger. Higher twisted sector couplings of z(n) orbifolds. *Nucl. Phys.*, B397:379–416, 1993.

Ivan Esteban, M. C. Gonzalez-Garcia, Michele Maltoni, Thomas Schwetz, and Albert Zhou. The fate of hints: updated global analysis of three-flavor neutrino oscillations. *JHEP*, 09:178, 2020. doi: 10.1007/JHEP09(2020)178.

Ferruccio Feruglio. Are neutrino masses modular forms? 2017.

References IV

Ferruccio Feruglio and Andrea Romanino. Lepton flavor symmetries. *Rev. Mod. Phys.*, 93(1):015007, 2021. doi: 10.1103/RevModPhys.93.015007.

C. D. Froggatt and Holger Bech Nielsen. Hierarchy of quark masses, cabibbo angles and cp violation. *Nucl. Phys.*, B147:277, 1979.

David B. Kaplan and Martin Schmaltz. Flavor unification and discrete nonAbelian symmetries. *Phys. Rev. D*, 49:3741–3750, 1994. doi: 10.1103/PhysRevD.49.3741.

Shota Kikuchi, Tatsuo Kobayashi, and Hikaru Uchida. Modular flavor symmetries of three-generation modes on magnetized toroidal orbifolds. 1 2021.

Victor Knapp-Perez, Xiang-Gan Liu, Hans Peter Nilles, Saul Ramos-Sanchez, and Michael Ratz. Matter matters in moduli fixing and modular flavor symmetries. *Phys. Lett. B*, 844:138106, 2023. doi: 10.1016/j.physletb.2023.138106.

Tatsuo Kobayashi and Morimitsu Tanimoto. Modular flavor symmetric models. 7 2023.

References V

Xiang-Gan Liu and Gui-Jun Ding. Modular flavor symmetry and vector-valued modular forms. *JHEP*, 03:123, 2022. doi: 10.1007/JHEP03(2022)123.

Hans Peter Nilles, Saúl Ramos–Sánchez, and Patrick K. S. Vaudrevange. Eclectic flavor scheme from ten-dimensional string theory - II detailed technical analysis. *Nucl. Phys. B*, 966:115367, 2021. doi: 10.1016/j.nuclphysb.2021.115367.

Fernando Quevedo. Lectures on superstring phenomenology. 1996. doi: 10.1063/1.49735. [AIP Conf. Proc.359,202(1996)].

Lisa Randall and Raman Sundrum. A Large mass hierarchy from a small extra dimension. *Phys. Rev. Lett.*, 83:3370–3373, 1999. doi: 10.1103/PhysRevLett.83.3370.

Ningqiang Song, Shirley Weishi Li, Carlos A. Argüelles, Mauricio Bustamante, and Aaron C. Vincent. The Future of High-Energy Astrophysical Neutrino Flavor Measurements. *JCAP*, 04:054, 2021. doi: 10.1088/1475-7516/2021/04/054.