UNIVERSITY OF Southampton
School of Physics and Astronomy

HIDDeν
Hunting Invisibles: Dark sectors, Dark matter and Neutrinos

Neutrino Mixing Sum Rules

Steve King,

58th Rencontres de Moriond

La Thuille,

29th March 2024

since 1966 Rencontres de Moriond

Neutrino mass and mixing

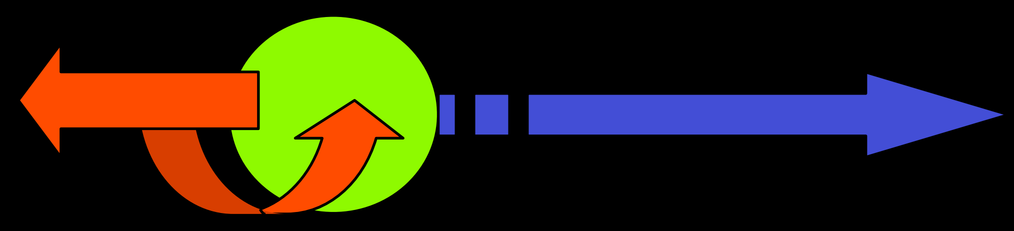
- Neutrinos have tiny masses (much less than electron)
- Neutrinos mix a lot (unlike the quarks)
- Up to 9 new params: 3 masses, 3 angles, 3 phases
- Origin of mass and mixing is unknown

Dirac or Majorana?

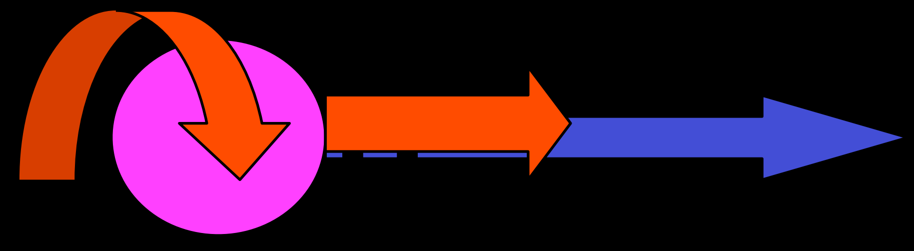
CUORE Irene Nutini

Pasquale Di Bari

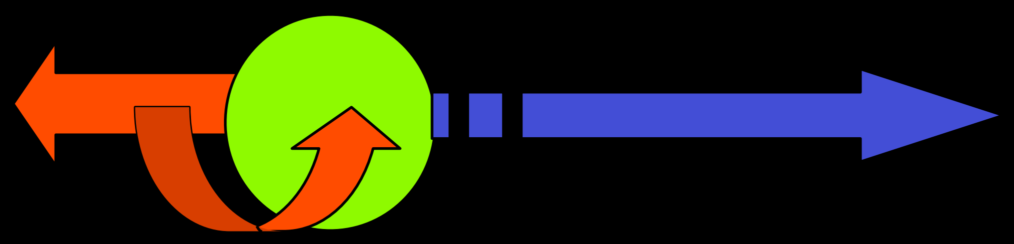
ν_L — **Dirac** — ν_R



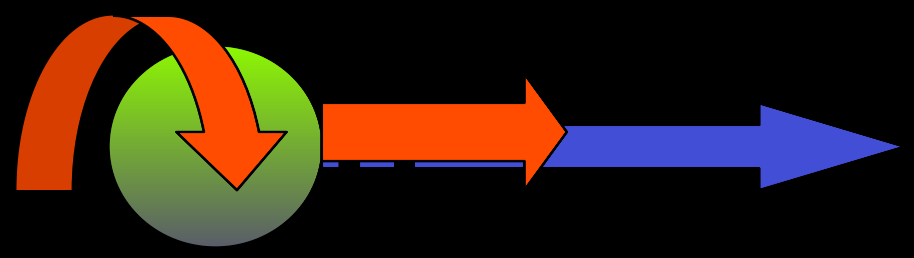
$m_{\rm D}\bar{\nu}_L\nu_R$



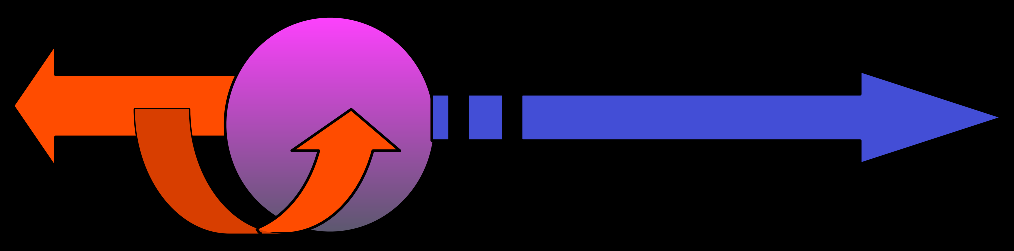
ν_L — **Majorana** — ν_L^c



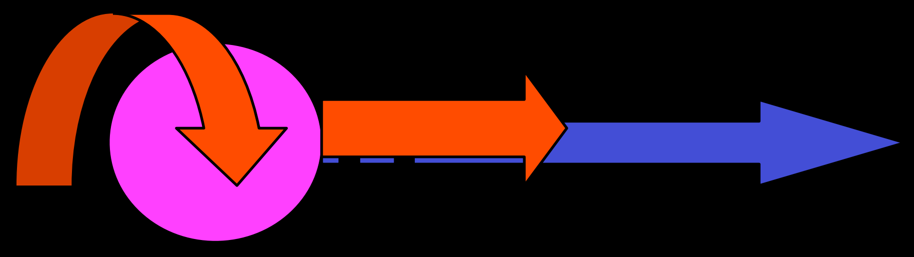
$m_{\rm L}\bar{\nu}_L\nu_L^c$

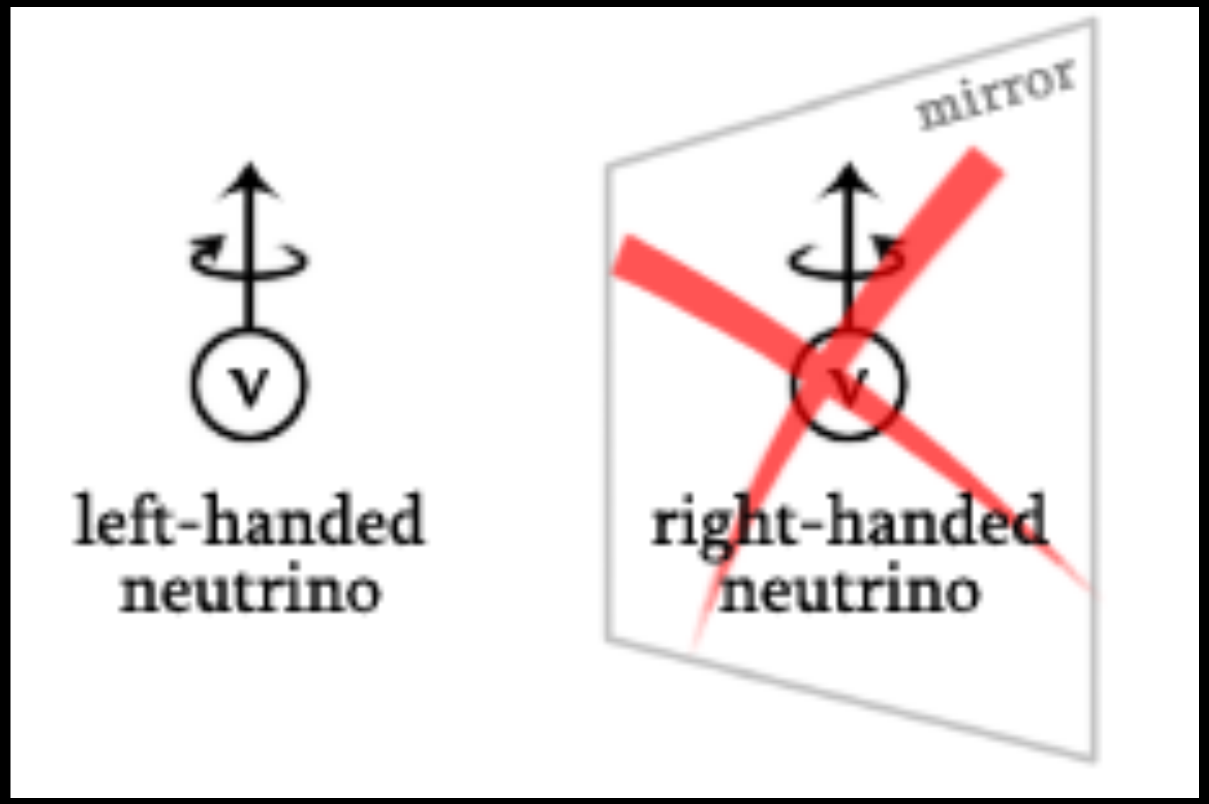


ν_R^c — **Majorana** — ν_R

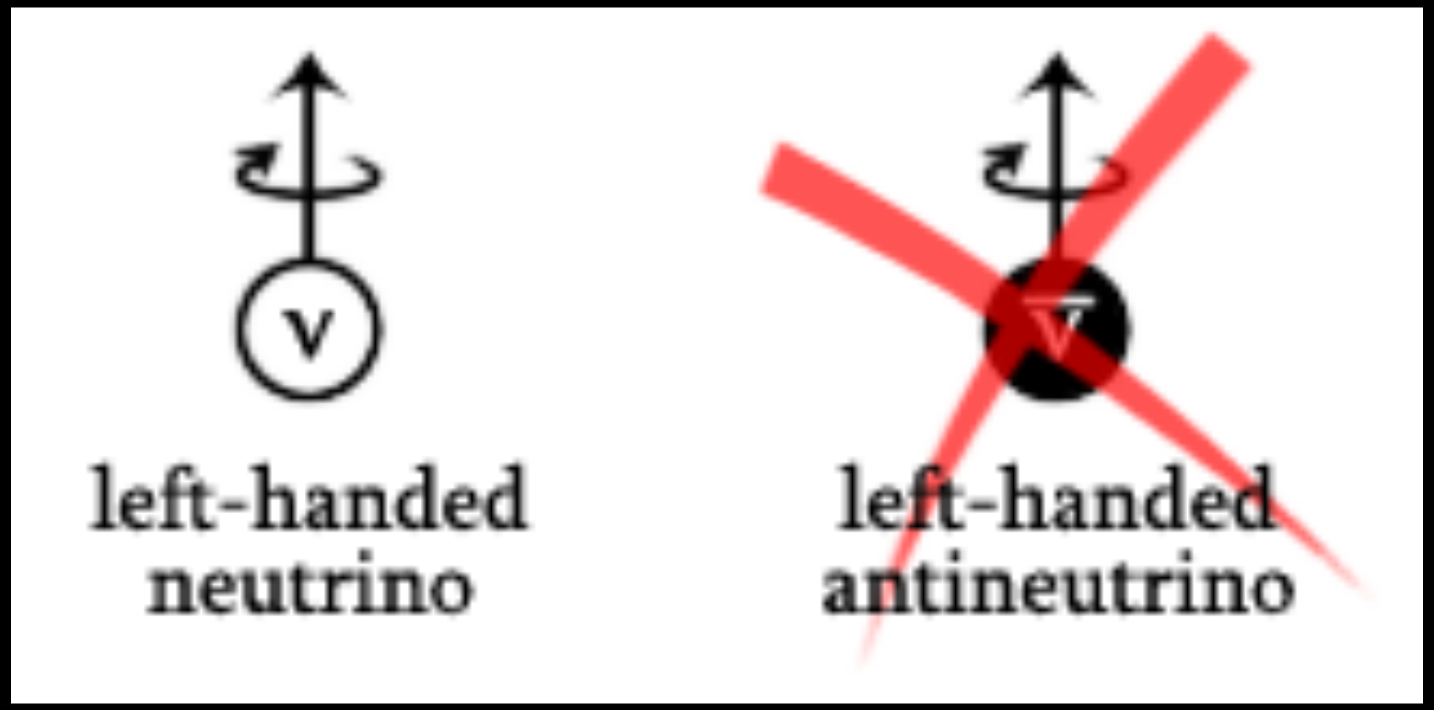


$M_{\rm R}\bar{\nu}_R^c\nu_R$

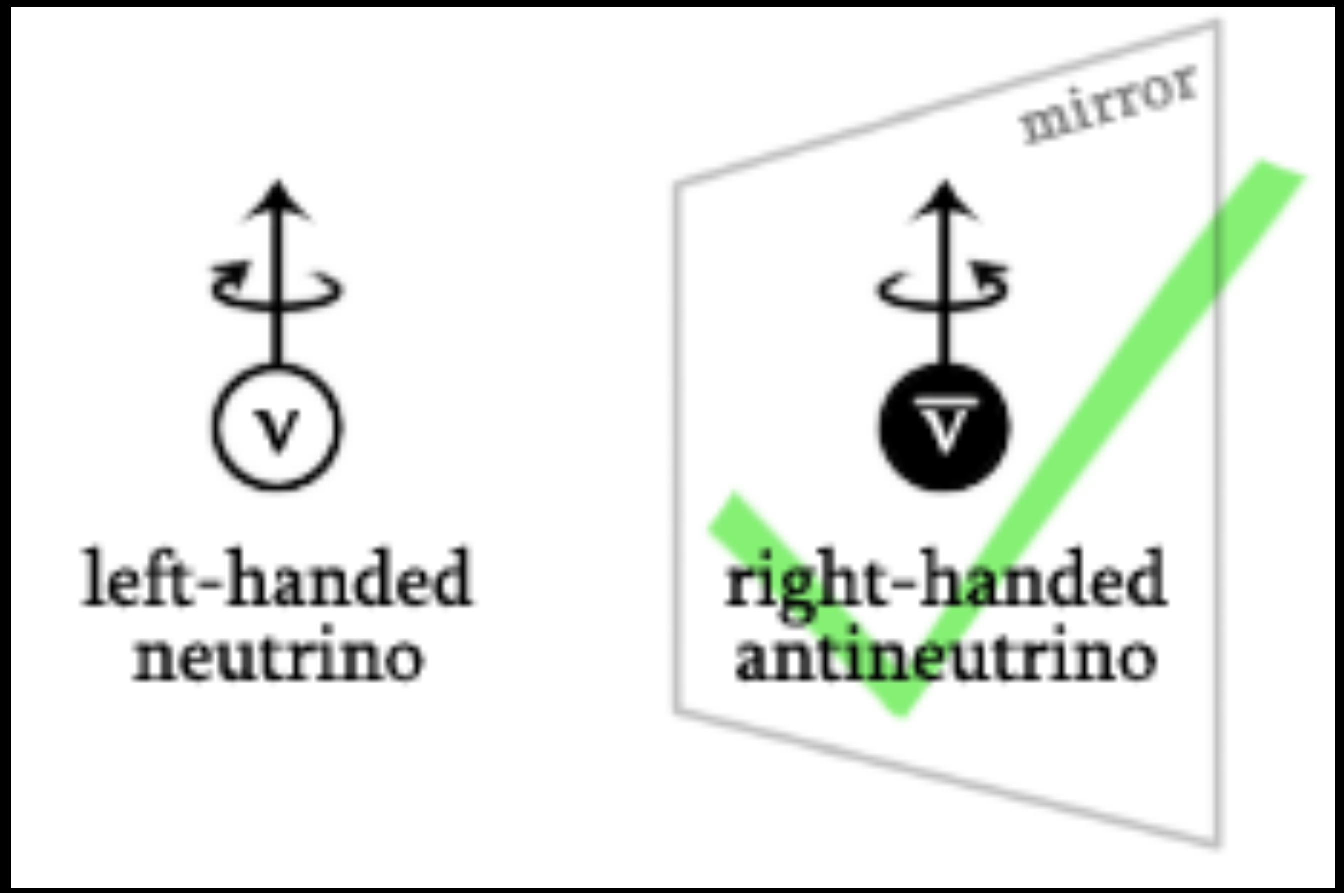




P violated

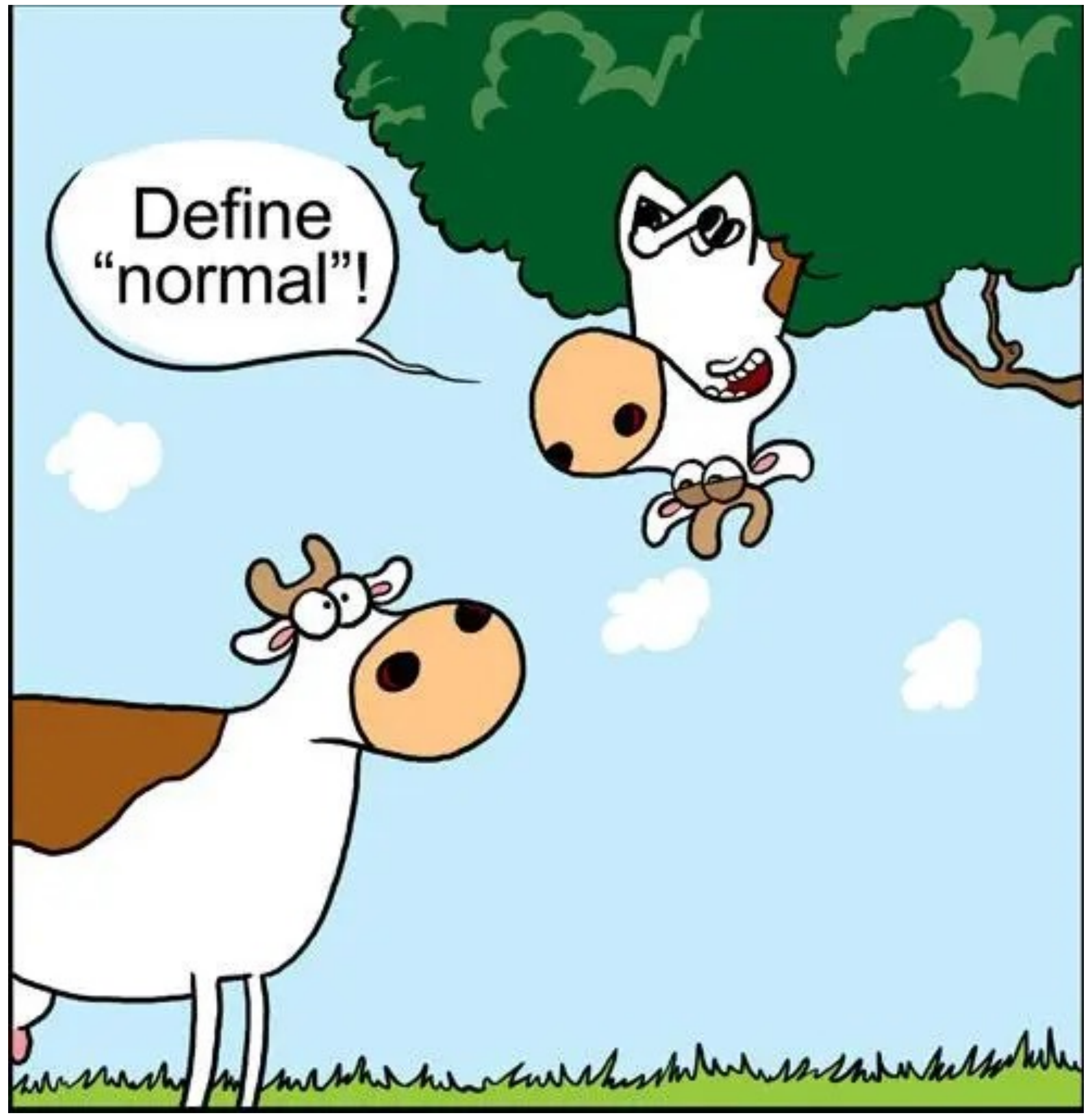


C violated

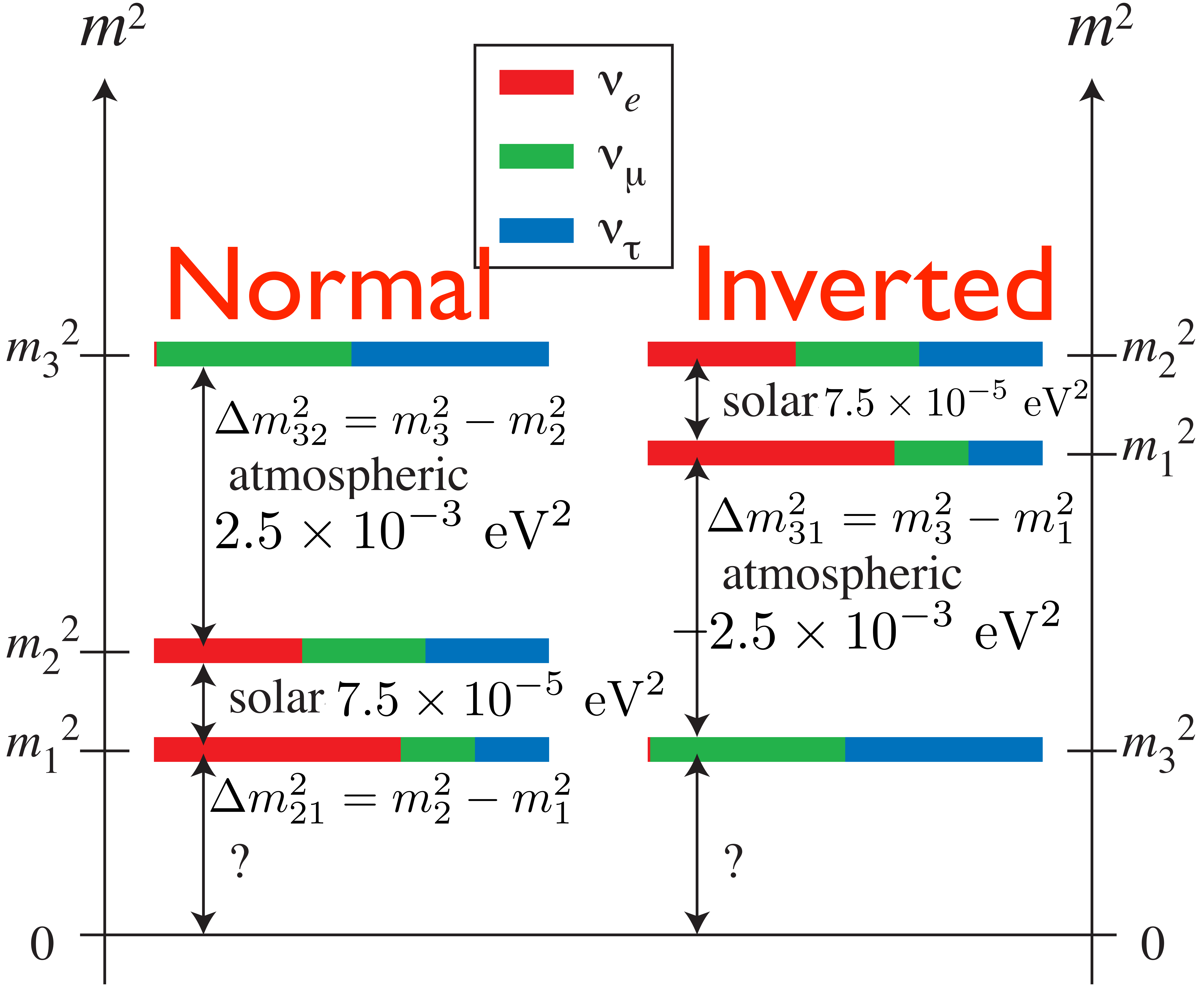


CP conserved

or not?



Normal or Inverted?

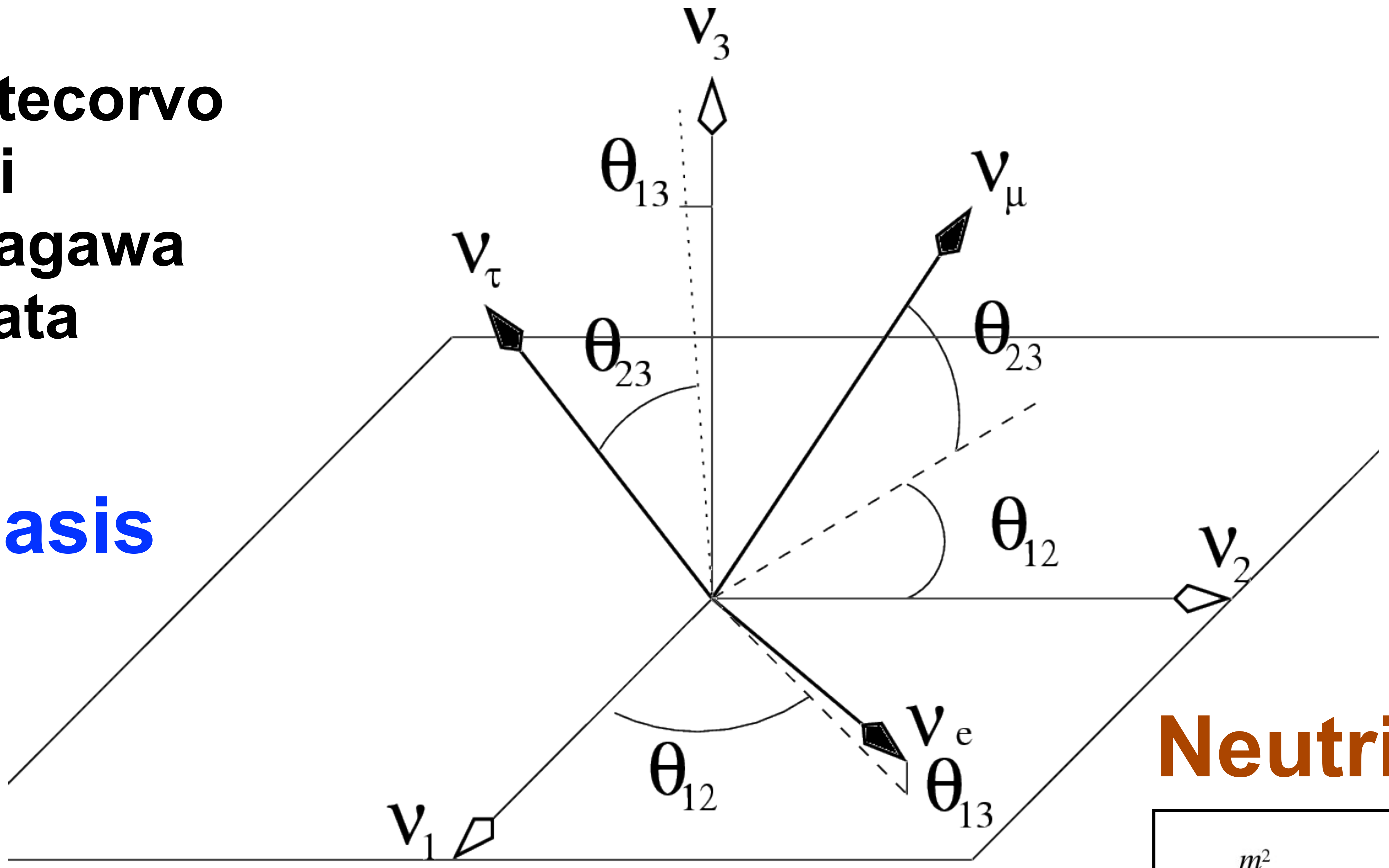


PMNS mixing matrix

B. Pontecorvo

S. Sakata Z. Maki M. Nakagawa

Pontecorvo
Maki
Nakagawa
Sakata

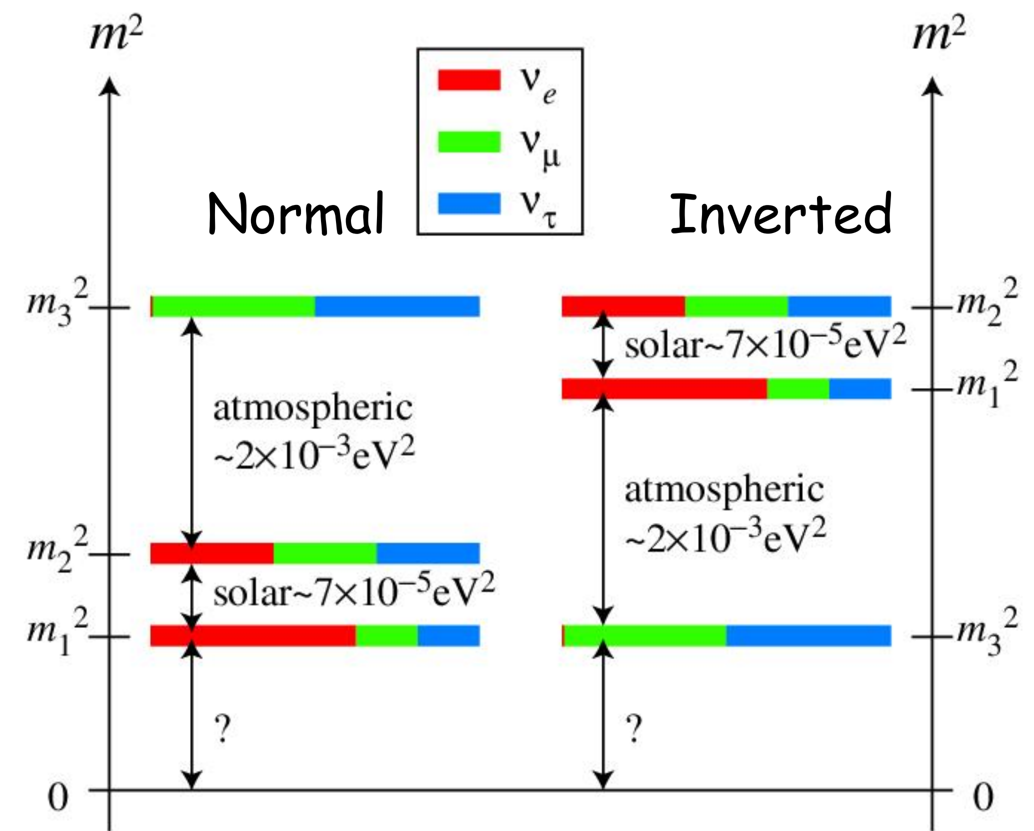


Charged lepton mass basis

$$\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L \quad \begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix}_L \quad \begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix}_L$$

$$\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \begin{pmatrix}U_{e1} & U_{e2} & U_{e3}\\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3}\\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3}\end{pmatrix} \begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix}$$

Neutrino mass basis



PMNS mixing matrix

$$U_{PMNS}=\begin{pmatrix}1 & 0 & 0\\ 0 & c_{23} & s_{23}\\ 0 & -s_{23} & c_{23}\end{pmatrix}\begin{pmatrix}c_{13} & 0 & s_{13}e^{-i\delta}\\ 0 & 1 & 0\\ -s_{13}e^{i\delta} & 0 & c_{13}\end{pmatrix}\begin{pmatrix}c_{12} & s_{12} & 0\\ -s_{12} & c_{12} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & e^{i\frac{\alpha_{21}}{2}} & 0\\ 0 & 0 & e^{i\frac{\alpha_{31}}{2}}\end{pmatrix}$$

Atmospheric **Reactor** **Solar** **Majorana**

CP violating phase

$$=\begin{pmatrix}c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta}\\ -s_{12}c_{23}-c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23}-s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23}\\ s_{12}s_{23}-c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23}-s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23}\end{pmatrix}$$

CP violating Majorana phases

$$\times\,\mathrm{diag}(1, e^{i\alpha_{21}/2}, e^{i\alpha_{31}/2})$$

SuperK Andrew Santos T2K Phil Litchfield NOvA Mayly Sanchez

Muon Neutrino Oscillations

$$P(\nu_\mu \to \nu_\mu; E, L) = 1 - \sin^2(2\theta_{23}) \sin^2\left(\frac{\Delta L}{2}\right) + \mathcal{O}(\epsilon)$$

$$P(\nu_\mu \to \nu_e; E, L) \equiv P_1 + P_{\frac{3}{2}} + \mathcal{O}\left(\epsilon^2\right)$$

Matter effect

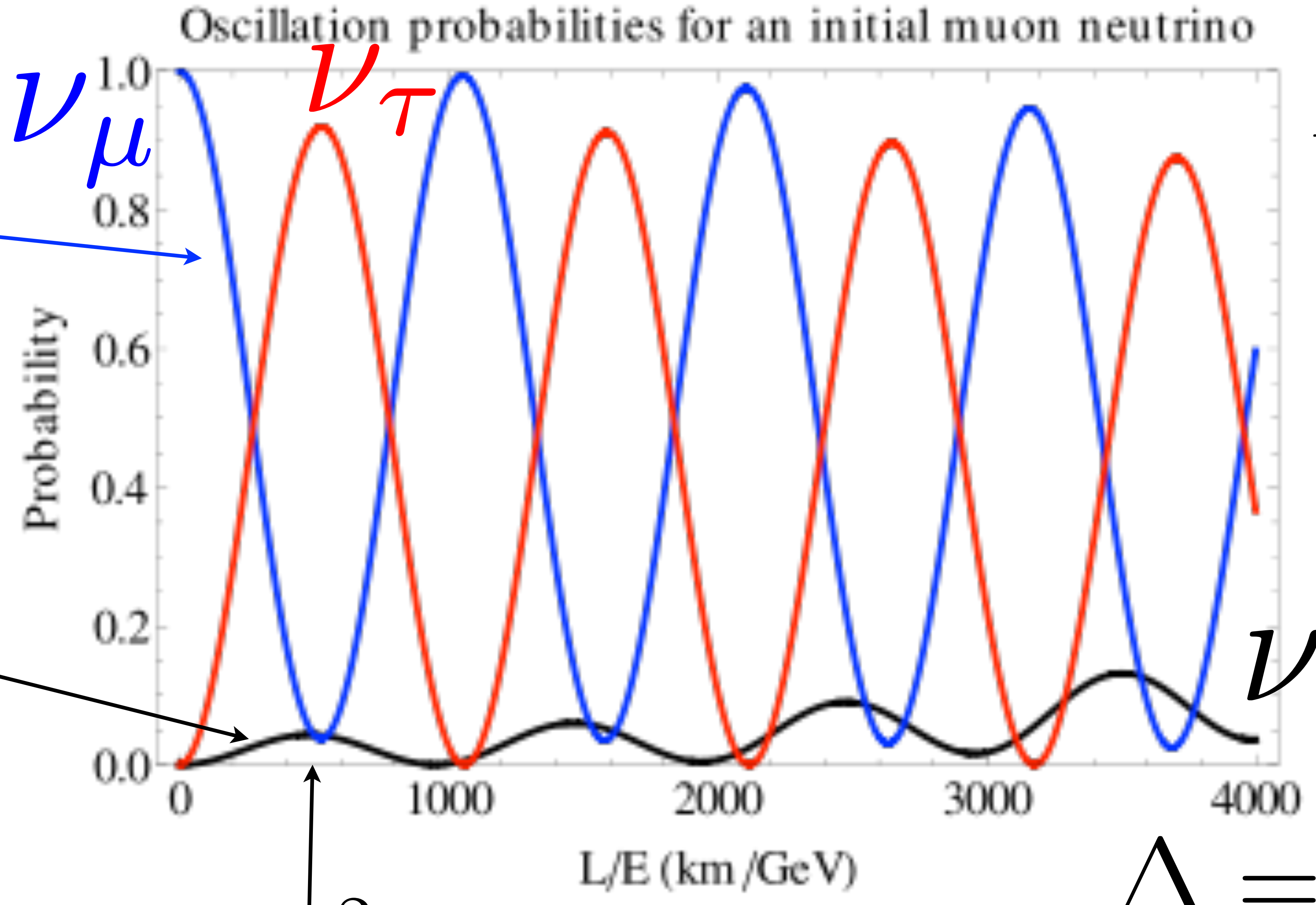
$$P_1 = \frac{4}{(1-r_A)^2} \sin^2\theta_{23} \sin^2\theta_{13} \sin^2\left(\frac{(1-r_A)\Delta L}{2}\right)$$

CP phase

$$P_{\frac{3}{2}} = 8J_r \frac{\epsilon}{r_A(1-r_A)} \cos\left(\delta + \frac{\Delta L}{2}\right) \times \sin\left(\frac{r_A \Delta L}{2}\right) \sin\left(\frac{(1-r_A)\Delta L}{2}\right)$$

Muon disappearance

Electron appearance



$r_A = 2\sqrt{2} G_F N_e E / \Delta m^2_{31}$

r_A, δ change sign for antineutrinos

$J_r = \cos\theta_{12} \sin\theta_{12} \cos\theta_{23} \sin\theta_{23} \sin\theta_{13}$

$$\Delta = \Delta m^2_{31}/2E$$

Accelerator LBL (1st atm max)

$$\frac{\Delta m^2_{31} L}{4E} = \frac{\pi}{2}$$

$$\epsilon \equiv \Delta m^2_{21}/\Delta m^2_{31} \approx 0.03 \sim O(\sin^2\theta_{13})$$

NuFIT 5.2 (2022)

Global Fits (Pre-NOvA/T2K)

3σ ranges

$\theta_{23} = [39.6^\circ, 51.9^\circ]$ **Octant?**

$\sin^2\theta_{23} = \frac{1}{2}$? 45°? **Max Mix?**

$\theta_{12} = [31.31^\circ, 35.74^\circ]$

$\sin^2\theta_{12} = \frac{1}{3}$? 35.26°? **TBM?**

$\delta = [0^\circ, 44^\circ]$ & $[108^\circ, 360^\circ]$

0°? 180°? 270°?

CPC? **Max CPV?**

without SK atmospheric data

	Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 2.3$)	
	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.303^{+0.012}_{-0.011}$	$0.270 \to 0.341$	$0.303^{+0.012}_{-0.011}$	$0.270 \to 0.341$
$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$
$\sin^2\theta_{23}$	$0.572^{+0.018}_{-0.023}$	$0.406 \to 0.620$	$0.578^{+0.016}_{-0.021}$	$0.412 \to 0.623$
$\theta_{23}/^\circ$	$49.1^{+1.0}_{-1.3}$	$39.6 \to 51.9$	$49.5^{+0.9}_{-1.2}$	$39.9 \to 52.1$
$\sin^2\theta_{13}$	$0.02203^{+0.00056}_{-0.00059}$	$0.02029 \to 0.02391$	$0.02219^{+0.00060}_{-0.00057}$	$0.02047 \to 0.02396$
$\theta_{13}/^\circ$	$8.54^{+0.11}_{-0.12}$	$8.19 \to 8.89$	$8.57^{+0.12}_{-0.11}$	$8.23 \to 8.90$
$\delta_{CP}/^\circ$	197^{+42}_{-25}	$108 \to 404$	286^{+27}_{-32}	$192 \to 360$
$\frac{\Delta m^2_{21}}{10^{-5}\ eV^2}$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ eV^2}$	$+2.511^{+0.028}_{-0.027}$	$+2.428 \to +2.597$	$-2.498^{+0.032}_{-0.025}$	$-2.581 \to -2.408$

with SK atmospheric data

	Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 6.4$)	
	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.303^{+0.012}_{-0.012}$	$0.270 \to 0.341$	$0.303^{+0.012}_{-0.011}$	$0.270 \to 0.341$
$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$
$\sin^2\theta_{23}$	$0.451^{+0.019}_{-0.016}$	$0.408 \to 0.603$	$0.569^{+0.016}_{-0.021}$	$0.412 \to 0.613$
$\theta_{23}/^\circ$	$42.2^{+1.1}_{-0.9}$	$39.7 \to 51.0$	$49.0^{+1.0}_{-1.2}$	$39.9 \to 51.5$
$\sin^2\theta_{13}$	$0.02225^{+0.00056}_{-0.00059}$	$0.02052 \to 0.02398$	$0.02223^{+0.00058}_{-0.00058}$	$0.02048 \to 0.02416$
$\theta_{13}/^\circ$	$8.58^{+0.11}_{-0.11}$	$8.23 \to 8.91$	$8.57^{+0.11}_{-0.11}$	$8.23 \to 8.94$
$\delta_{CP}/^\circ$	232^{+36}_{-26}	$144 \to 350$	276^{+22}_{-29}	$194 \to 344$
$\frac{\Delta m^2_{21}}{10^{-5}\ eV^2}$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ eV^2}$	$+2.507^{+0.026}_{-0.027}$	$+2.427 \to +2.590$	$-2.486^{+0.025}_{-0.028}$	$-2.570 \to -2.406$

Is there a pattern in the matrix?

$$|U|_{3\sigma}^{\text{w/o SK-atm}} = \begin{pmatrix} 0.803 \to 0.845 & 0.514 \to 0.578 & 0.142 \to 0.155 \\ 0.233 \to 0.505 & 0.460 \to 0.693 & 0.630 \to 0.779 \\ 0.262 \to 0.525 & 0.473 \to 0.702 & 0.610 \to 0.762 \end{pmatrix}$$

NuFit 5.2

Small

Large

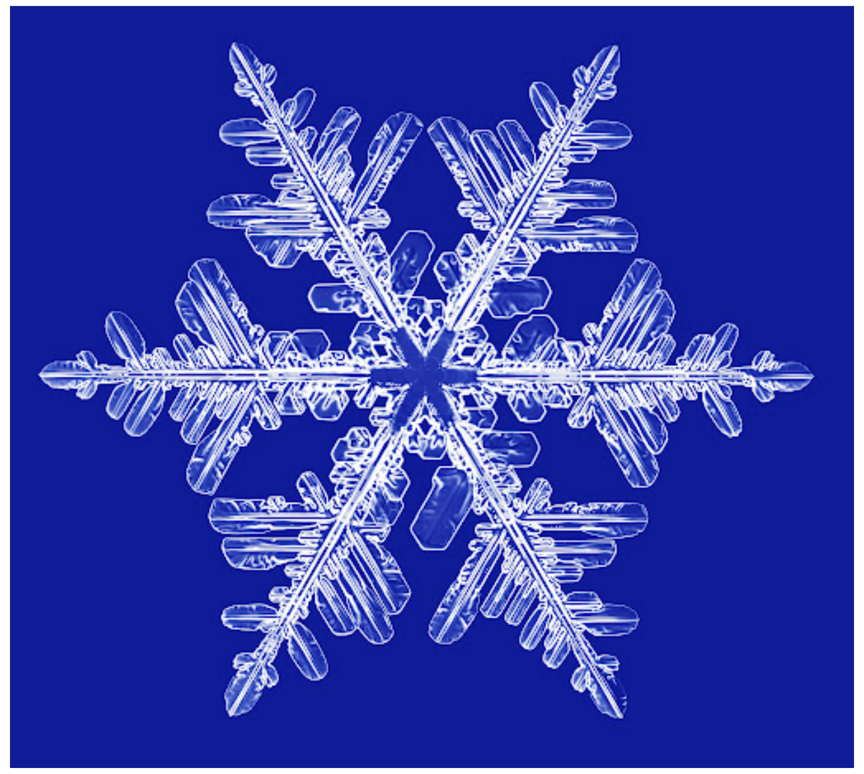
Large

Symmetry can enforce

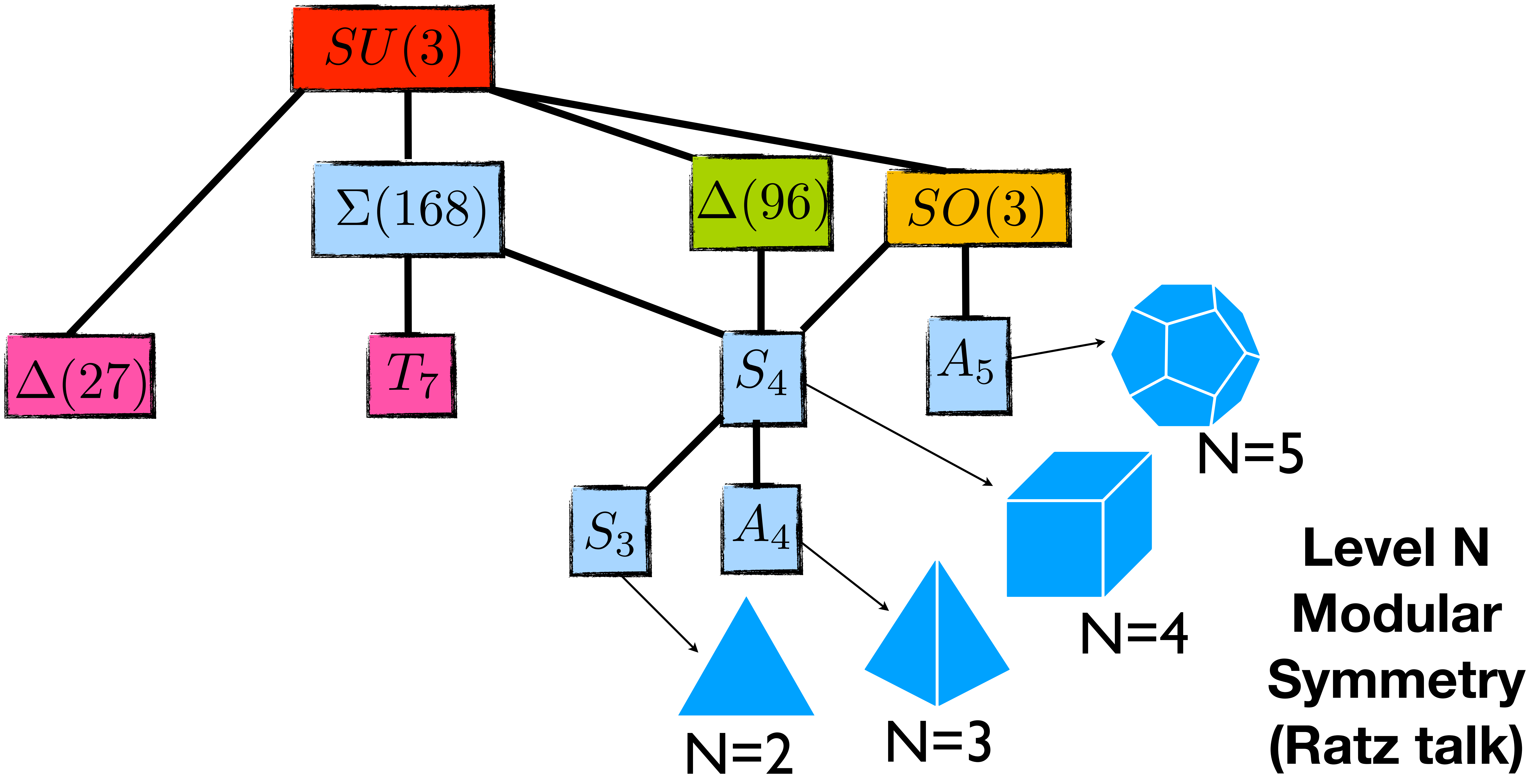
$$\sin\theta_{23} = \frac{1}{\sqrt{2}}$$
$$\sin\theta_{13} = 0$$

$$U_0 = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{s_{12}}{\sqrt{2}} & -\frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Where large $\sin\theta_{12}$ can come from the same symmetry



Non-Abelian Family Symmetry



Some Simple Symmetrical Examples

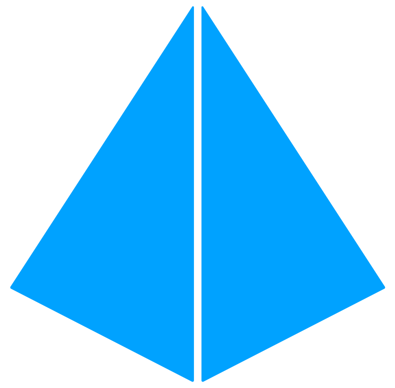
Z_3 Tri, Bi, Z_2

$$\begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Tri-bimaximal

$$\sin\theta_{12} = \frac{1}{\sqrt{3}}$$

A_4



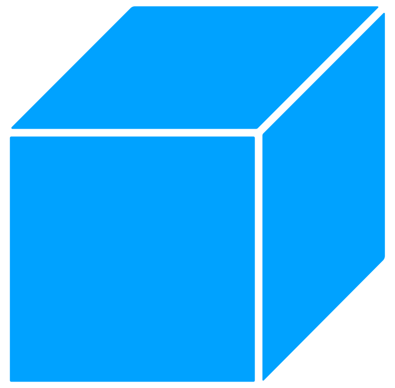
Z_2 Bi, Bi, Z_2

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Bimaximal

$$\sin\theta_{12} = \frac{1}{\sqrt{2}}$$

S_4



Z_5 GR, Bi, Z_2

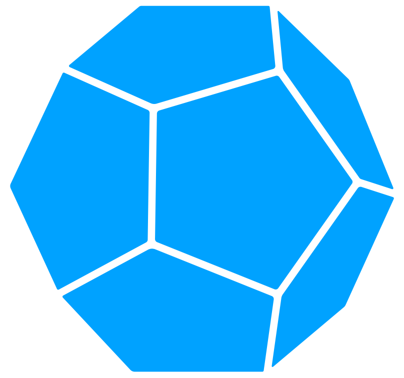
$$\begin{pmatrix} c_{12} & s_{12} & 0 \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{s_{12}}{\sqrt{2}} & -\frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Golden Ratio

a) $\tan\theta_{12} = \frac{2}{1+\sqrt{5}} = \frac{1}{\phi}$

b) $\cos\theta_{12} = \phi/2$

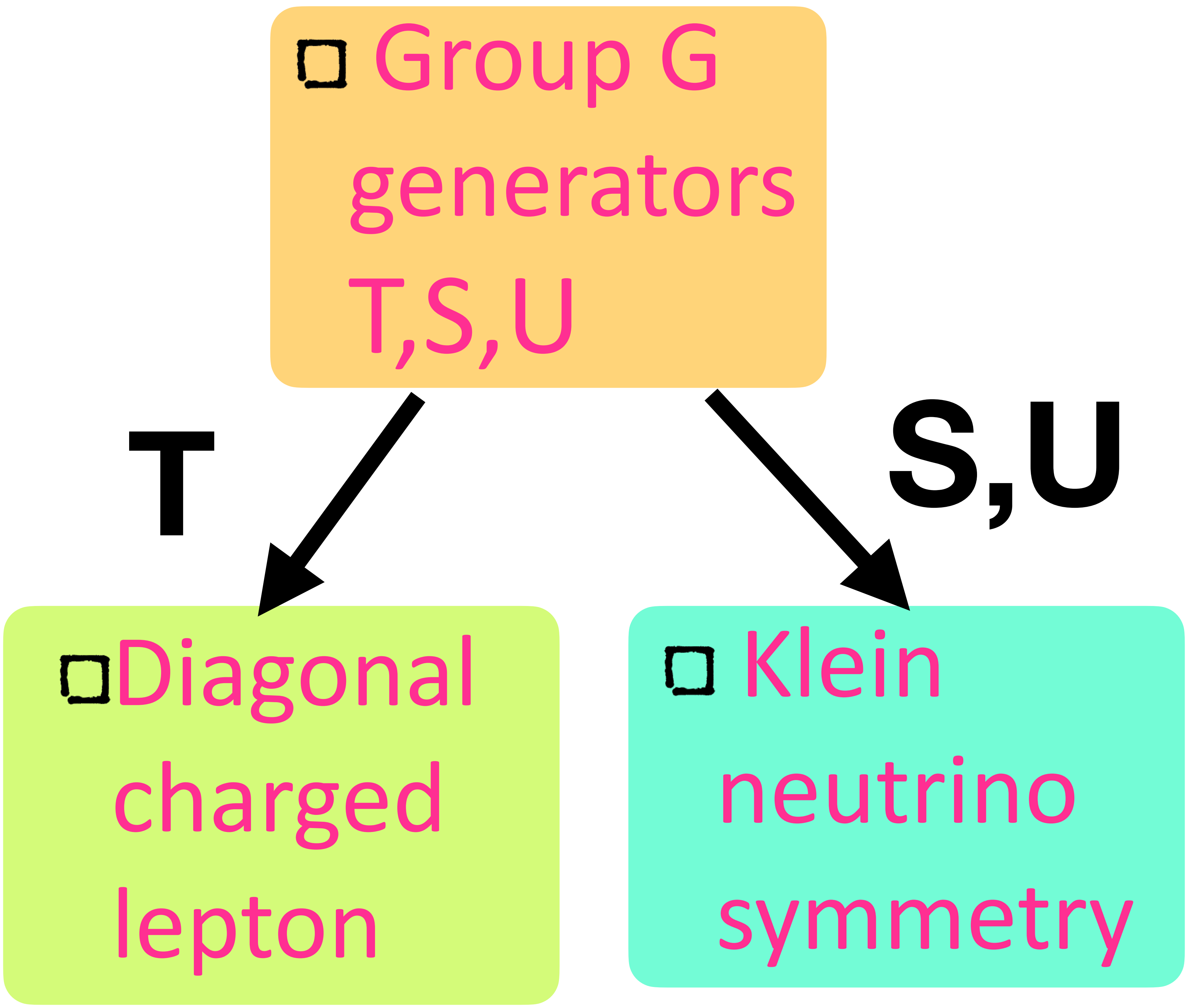
A_5



All these patterns involve $\sin\theta_{13} = 0$ so they need to be corrected

S.F.K. and C.Luhn, 1301.1340

Why is θ_{13} predicted to be zero?



S.F.K. and C.Luhn, 1301.1340

Why is θ_{13} predicted to be zero?

- Diagonal charged lepton

$$T^\dagger (M_e M_e^\dagger) T = M_e M_e^\dagger$$

$$T = \mathrm{diag}(1, \omega^2, \omega)$$

$$\omega = e^{i2\pi/N}$$

- Group G generators T,S,U

T → Diagonal charged lepton

S,U → Klein neutrino symmetry

- Klein neutrino symmetry

$$M^\nu = S^\dagger M^\nu S^* \quad M^\nu = U^\dagger M^\nu U^*$$

$$S = U_{\mathrm{PMNS}}\ \mathrm{diag}(-1,+1,-1)\ U_{\mathrm{PMNS}}^\dagger$$

$$U = U_{\mathrm{PMNS}}\ \mathrm{diag}(-1,-1,+1)\ U_{\mathrm{PMNS}}^\dagger$$

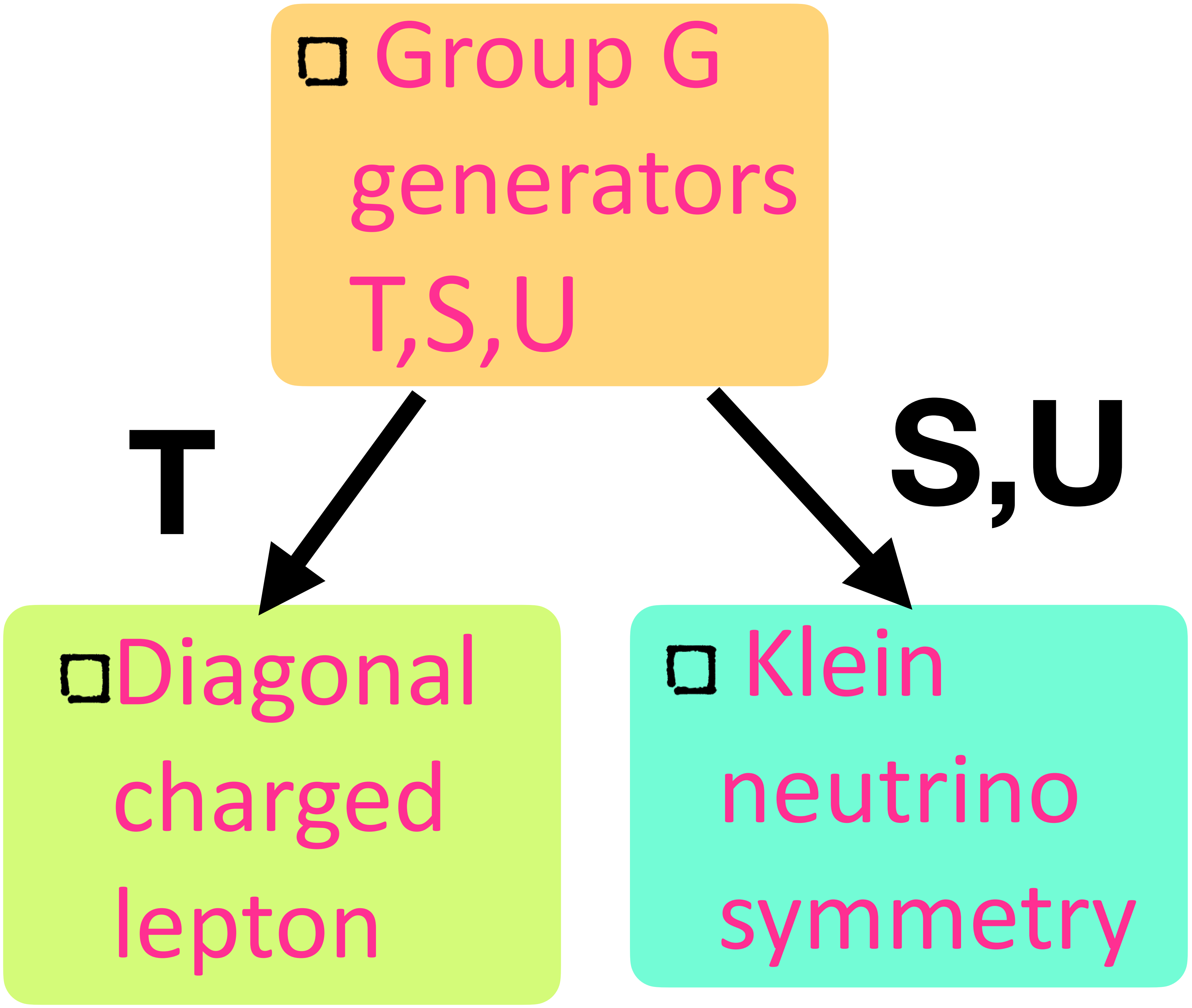
$$\begin{pmatrix} c_{12} & s_{12} & 0 \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{s_{12}}{\sqrt{2}} & -\frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$\sin\theta_{12}$

Fixed by symmetry

S.F.K. and C.Luhn, 1301.1340

How to switch on θ_{13}?



S.F.K. and C.Luhn, 1301.1340

How to switch on θ_{13}?

1. Break T

Charged Lepton Corrections

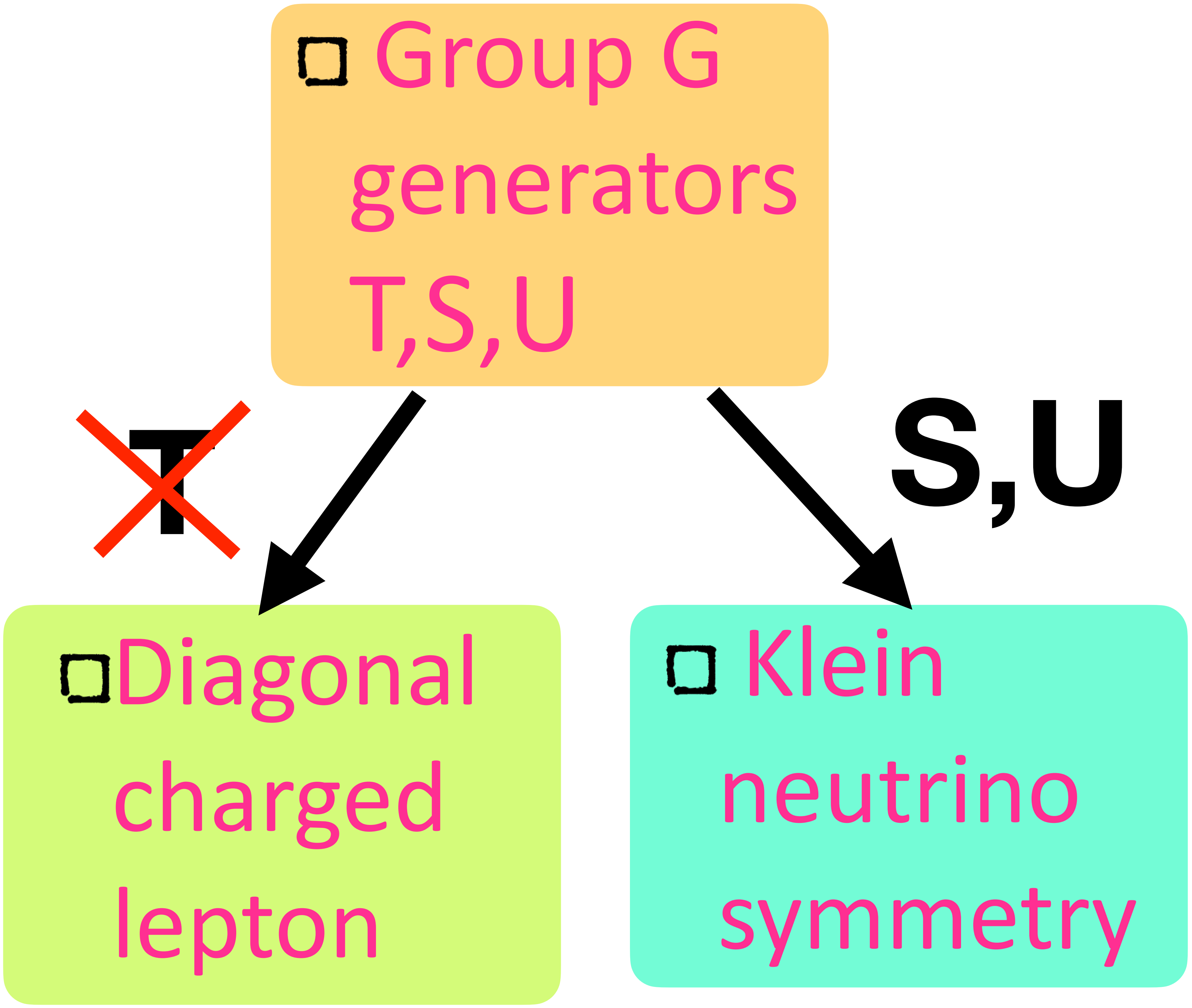
$\theta_{12}^e \neq 0$

Assume

$\theta_{23}^e \neq 0$ $\quad \theta_{13}^e = 0$

$$U_{\text{PMNS}} = U_e U_\nu$$

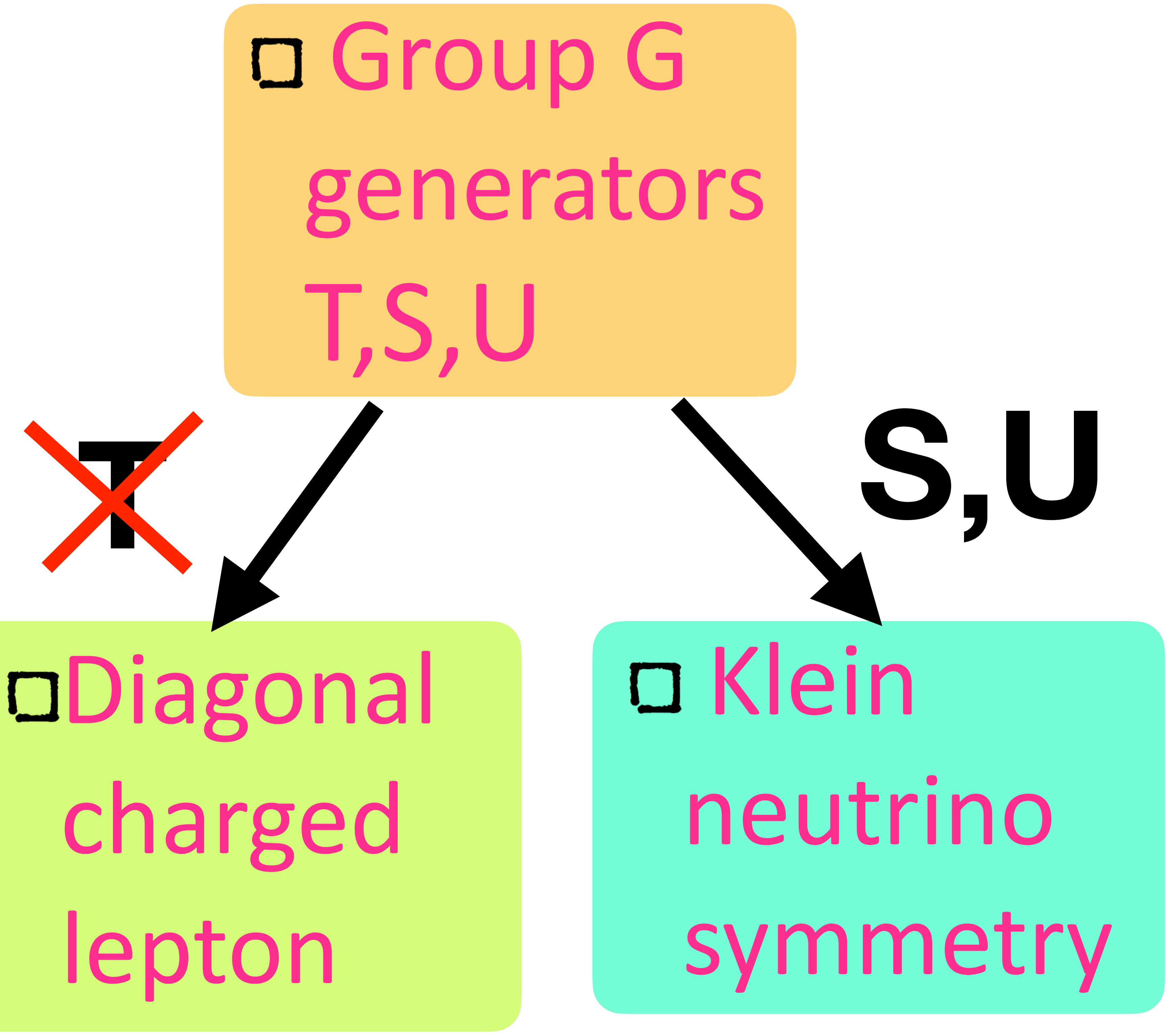
$$s_{13} = \frac{s_{12}^e}{\sqrt{2}}$$



S.F.K. and C.Luhn, 1301.1340

How to switch on θ_{13}?

1. Break T



Charged Lepton Corrections

$\theta^e_{12} \neq 0$

$\theta^e_{23} \neq 0$

Assume $\theta^e_{13} = 0$

$$U_{\rm PMNS} = U_e U_\nu$$

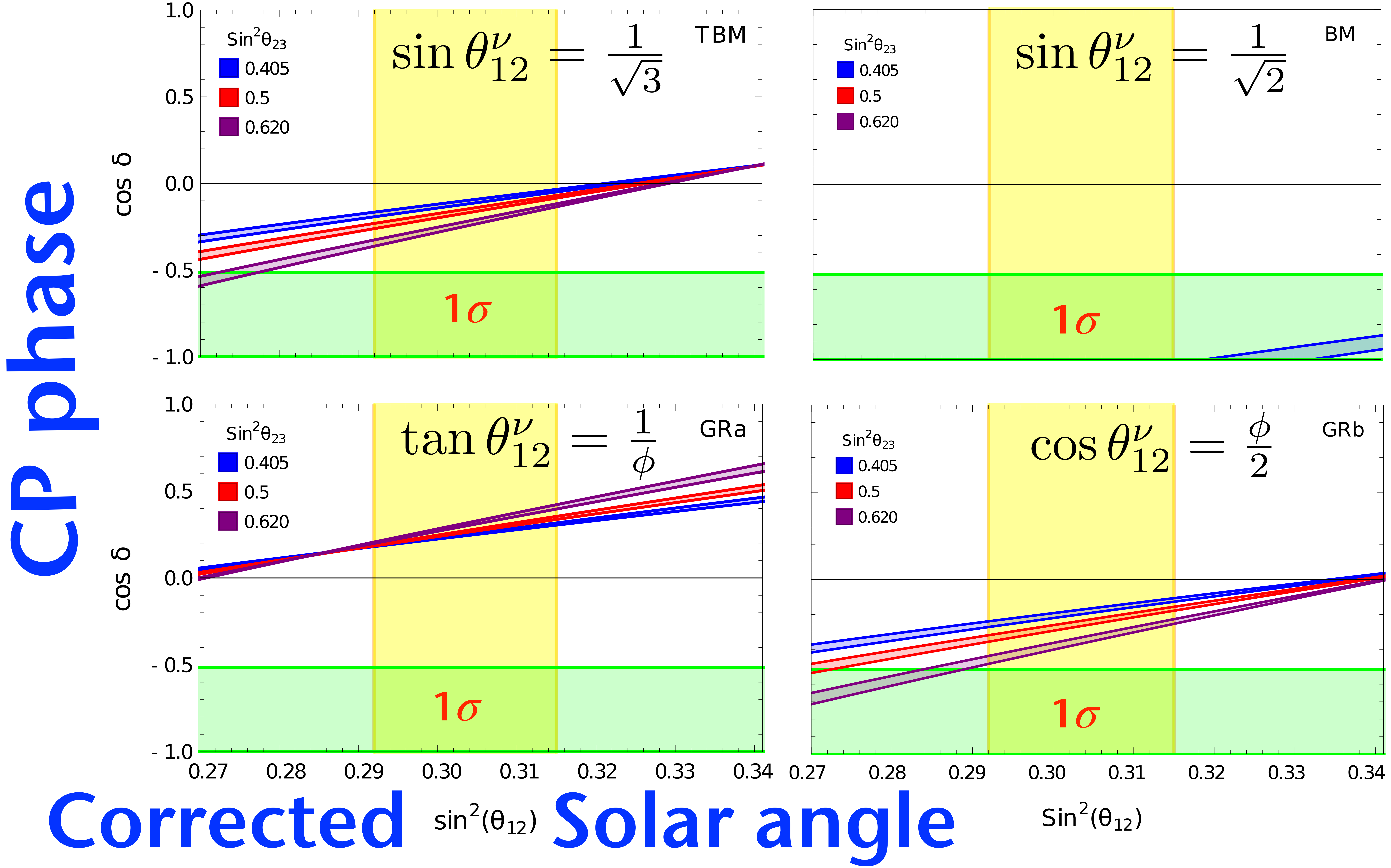
$$s_{13} = \frac{s^e_{12}}{\sqrt{2}}$$

Solar Sum Rule

$$\cos\delta = \frac{\tan\theta_{23}\sin\theta_{12}^2 + \sin\theta_{13}^2\cos\theta_{12}^2/\tan\theta_{23} - (\sin\theta^\nu_{12})^2\left(\tan\theta_{23} + \sin\theta_{13}^2/\tan\theta_{23}\right)}{\sin 2\theta_{12}\sin\theta_{13}}$$

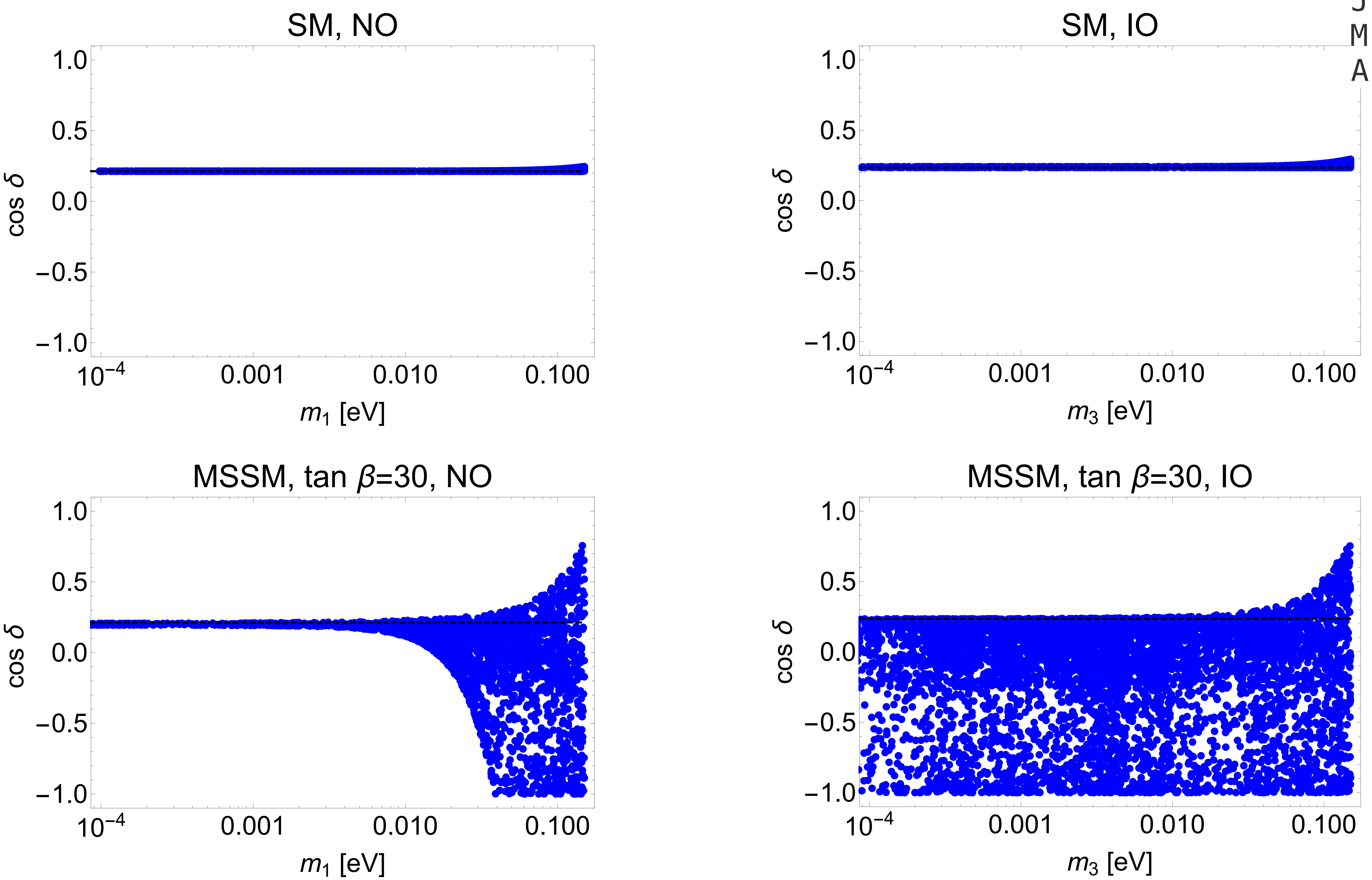
F.Costa and S.F.K., 2307.13895

Solar Sum Rule Predictions



RG corrections to GRa solar sum rule

J.Gehrlein, S.T.Petcov, M.Spinrath and A.V.Titov, 1608.08409



$$10^{13}\ \text{GeV} \to M_Z$$

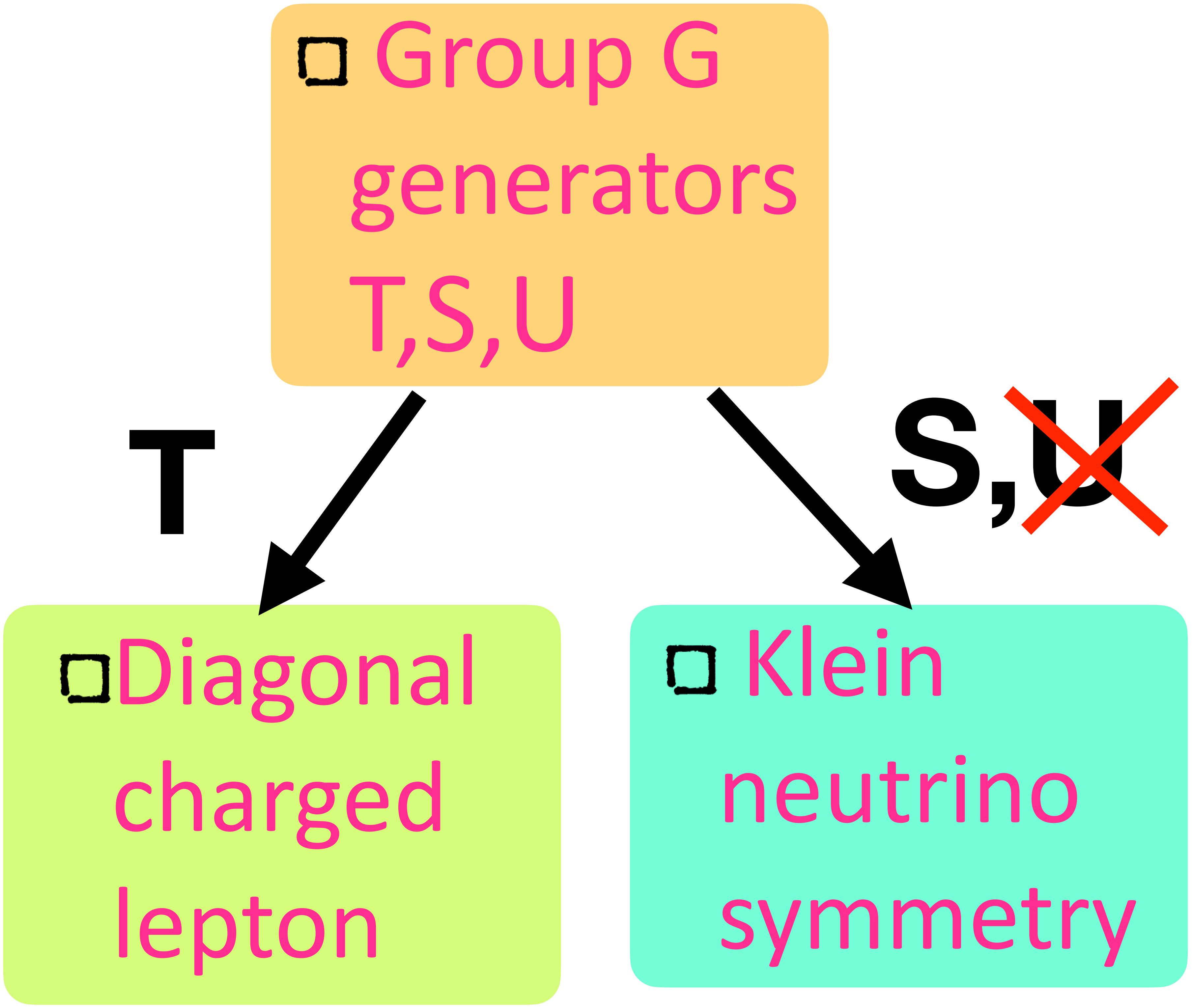
$$\theta^e_{12} \neq 0$$

$$\theta^e_{23} \neq 0$$

$$\theta^e_{13} = 0$$

S.F.K. and C.Luhn, 1301.1340

How to switch on θ_{13}?



2. Break U

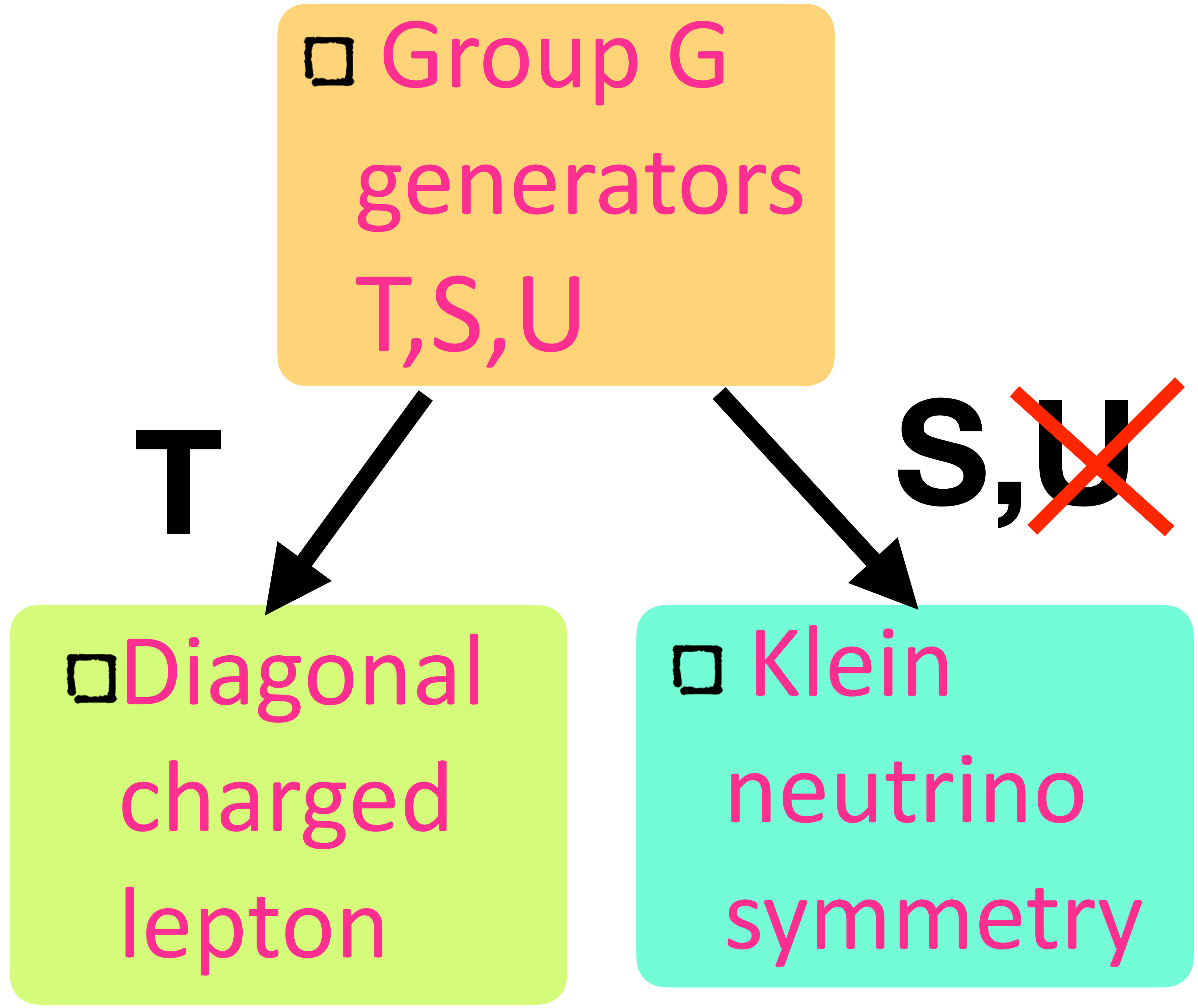
First or second PMNS column preserved

$$\begin{pmatrix} \sqrt{\frac{2}{3}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \\ \frac{1}{\sqrt{6}} & - & - \end{pmatrix} \quad \begin{pmatrix} - & \sqrt{\frac{1}{3}} & - \\ - & \sqrt{\frac{1}{3}} & - \\ - & -\sqrt{\frac{1}{3}} & - \end{pmatrix}$$

s_{13} **free parameter**

S.F.K. and C.Luhn, 1301.1340

How to switch on θ_{13}?



2. Break U

First or second PMNS column preserved

$$\begin{pmatrix} \sqrt{\frac{2}{3}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \\ \frac{1}{\sqrt{6}} & - & - \end{pmatrix} \quad \begin{pmatrix} - & \sqrt{\frac{1}{3}} & - \\ - & \sqrt{\frac{1}{3}} & - \\ - & -\sqrt{\frac{1}{3}} & - \end{pmatrix}$$

Atmospheric Sum Rules

s_{13} **free parameter**

$$\begin{pmatrix} \sqrt{\frac{2}{3}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \\ \frac{1}{\sqrt{6}} & - & - \end{pmatrix}$$

$$s_{12}^2 = \frac{(1-3s_{13}^2)}{3\,(1-s_{13}^2)} \qquad \cos\delta = -\frac{\cot 2\theta_{23}\,(1-5s_{13}^2)}{2\sqrt{2}s_{13}\sqrt{1-3s_{13}^2}}$$

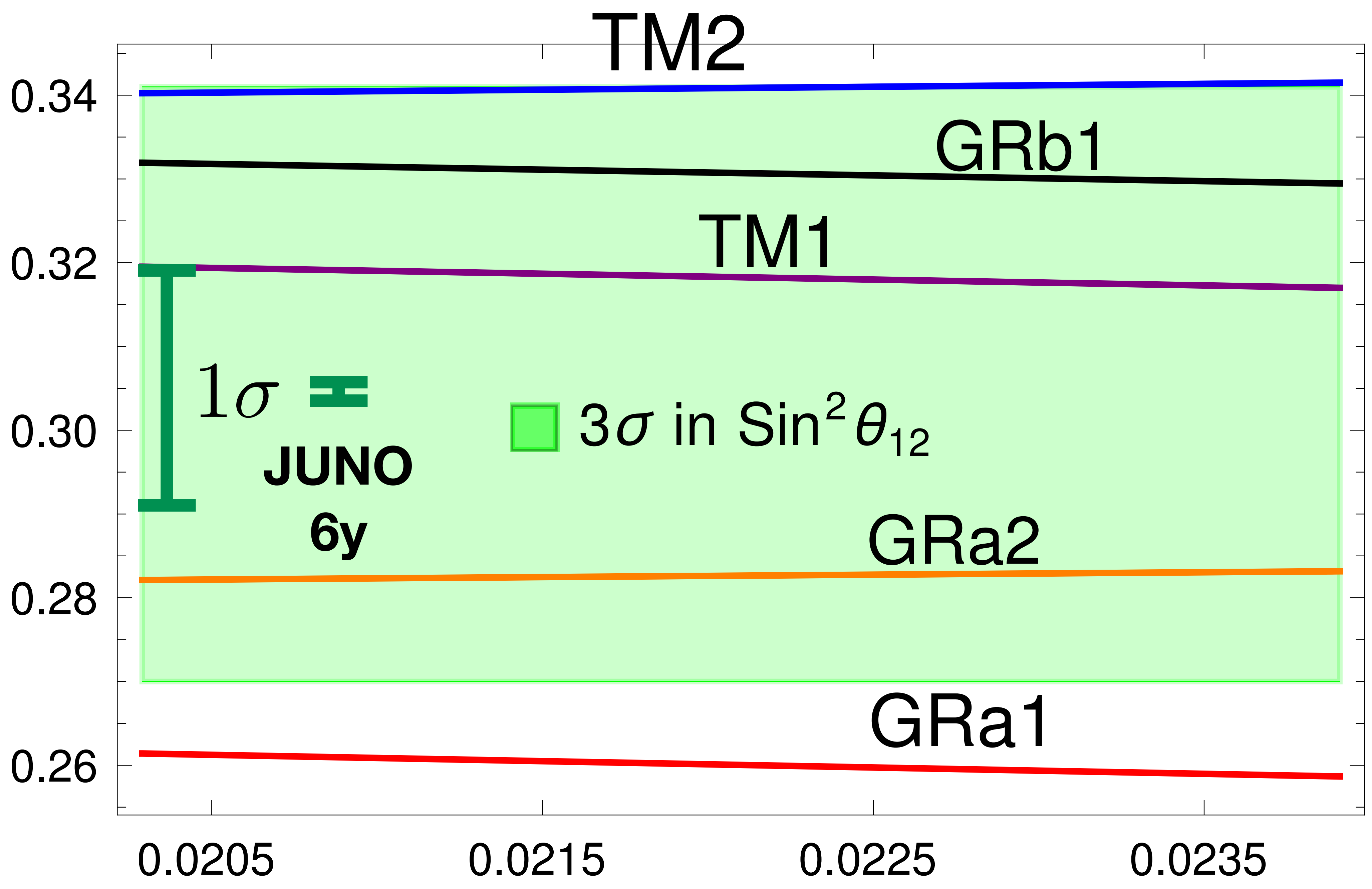
$$s_{12}^2 = \frac{1}{3\,(1-s_{13}^2)} \qquad \cos\delta = \frac{2c_{13}\cot 2\theta_{23}\cot 2\theta_{13}}{\sqrt{2-3s_{13}^2}}$$

$$\begin{pmatrix} - & \sqrt{\frac{1}{3}} & - \\ - & \sqrt{\frac{1}{3}} & - \\ - & -\sqrt{\frac{1}{3}} & - \end{pmatrix}$$

F.Costa and S.F.K., 2307.13895

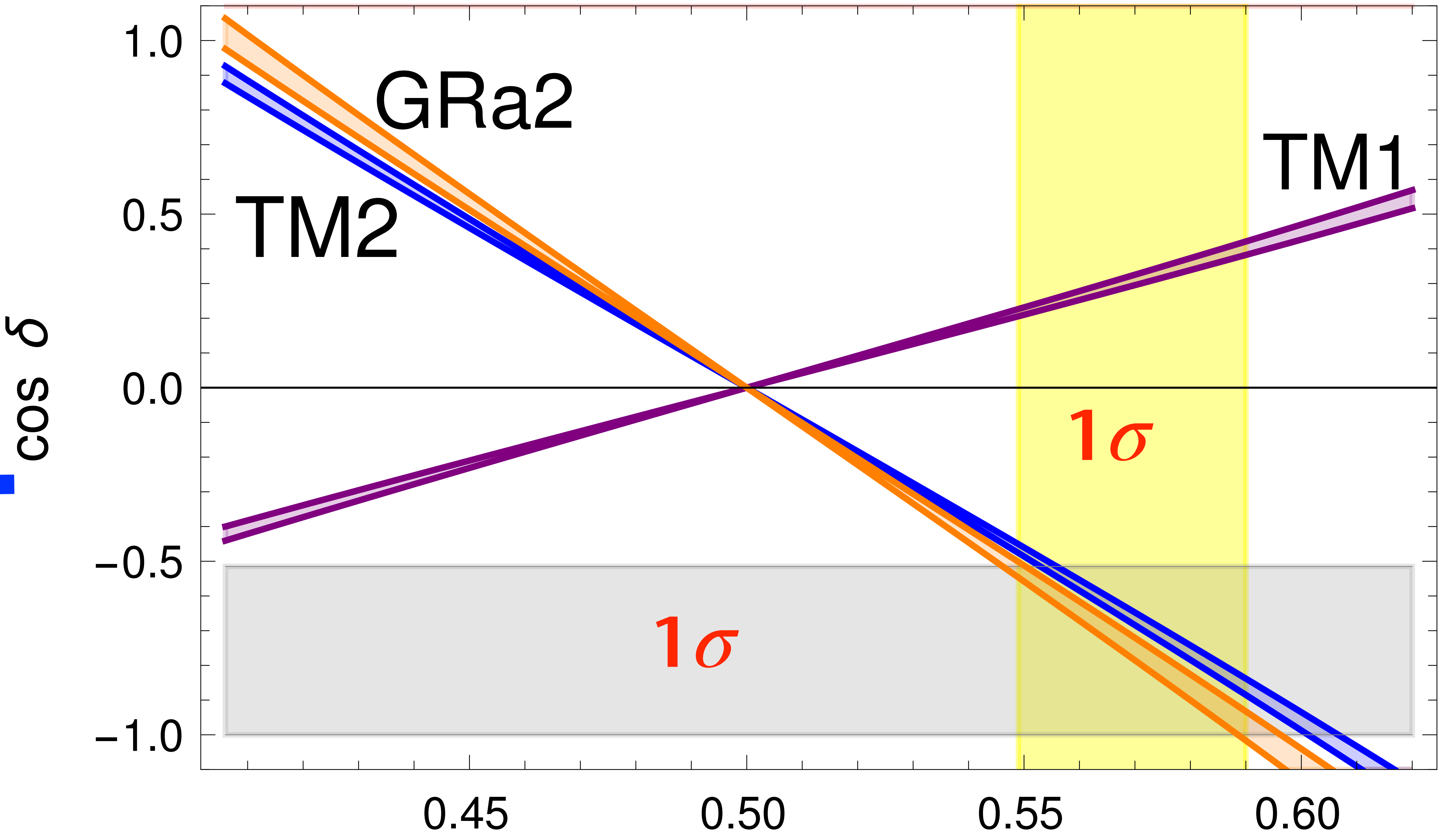
Atmospheric Sum Rule Predictions

Solar angle



Reactor angle $\sin^2(\theta_{13})$

CP phase



Atmospheric angle $\sin^2(\theta_{23})$

Only viable patterns

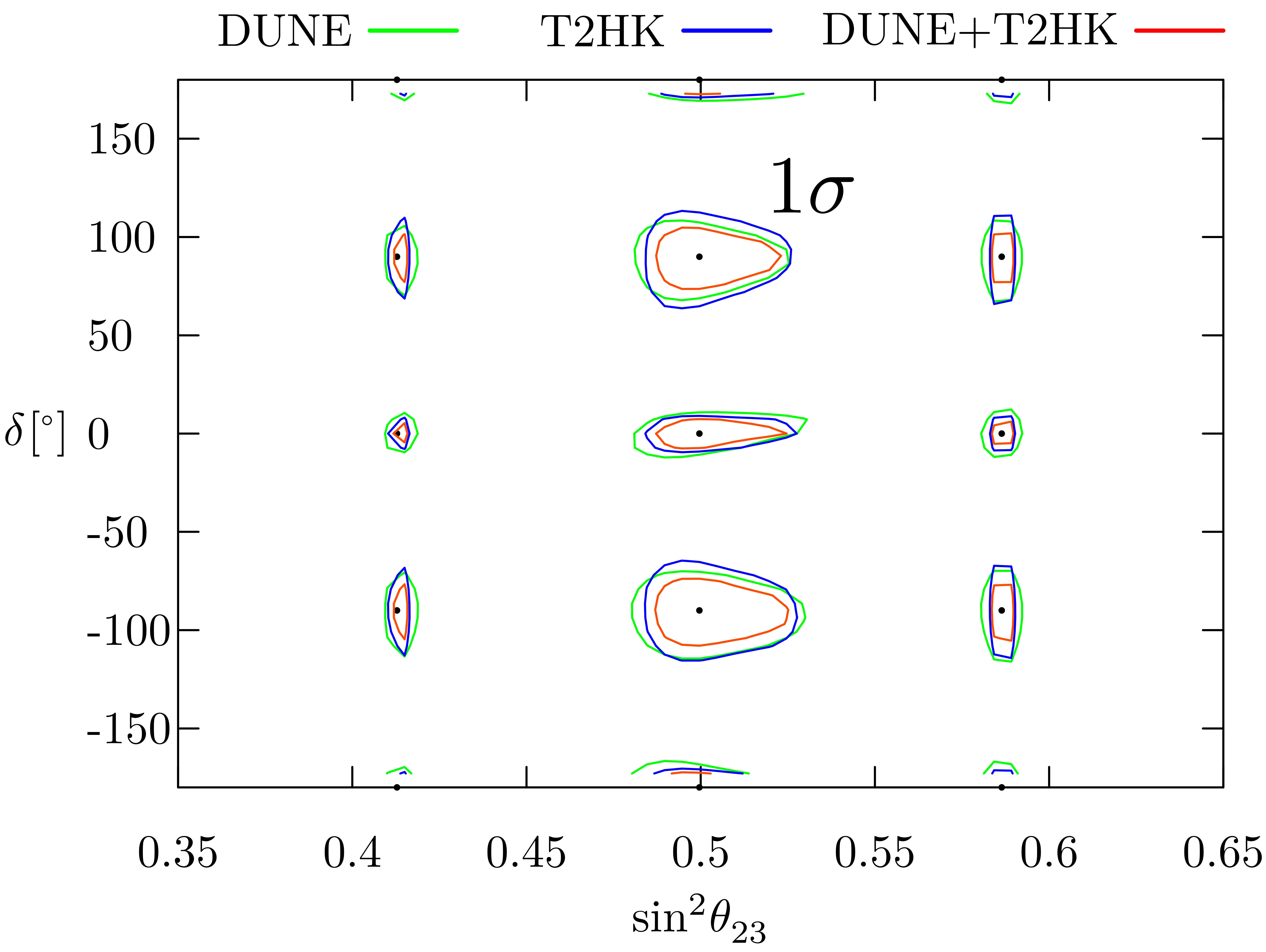
$$\text{TM1}\begin{pmatrix} \sqrt{\frac{2}{3}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \\ \frac{1}{\sqrt{6}} & - & - \end{pmatrix} \quad \text{TM2}\begin{pmatrix} - & \sqrt{\frac{1}{3}} & - \\ - & \sqrt{\frac{1}{3}} & - \\ - & -\sqrt{\frac{1}{3}} & - \end{pmatrix} \quad \text{GRa2}\begin{pmatrix} - & s^\nu_{12} & - \\ - & \frac{c^\nu_{12}}{\sqrt{2}} & - \\ - & -\frac{c^\nu_{12}}{\sqrt{2}} & - \end{pmatrix}$$

disfavoured

$$\tan\theta^\nu_{12} = \frac{1}{\phi}$$

Future Prospects

P.Ballett, S.F.K., S.Pascoli, N.W.Prouse and T.Wang,1612.07275



Will put sum rules to the test!

Conclusions

- Mixing sum rules are relics of simple PMNS matrices enforced by remnant symmetry which **allows** non-zero $\sin\theta_{13}$ and **predicts** $\cos\delta$ (not δ)

- Solar sum rules from charged lepton corrections to simple PMNS matrices

- Atmospheric sum rules from first/second column of simple PMNS matrix

- RG corrections can be small (NH) or large (2HDM, large $\tan\beta$)

- Future precision expts will test sum rules and the symmetry approach

Bibliography

TB

- TB P. F. Harrison, D. H. Perkins, and W. G. Scott, "Tri-bimaximal mixing and the neutrino oscillation data," Phys. Lett. B 530 (2002) 167, arXiv:hep-ph/0202074.
- Z.-z. Xing, "Nearly tri bimaximal neutrino mixing and CP violation," Phys. Lett. B 533 (2002) 85–93, arXiv:hep-ph/0204049.

BM

- V. D. Barger, S. Pakvasa, T. J. Weiler, and K. Whisnant, "Bimaximal mixing of three neutrinos," Phys. Lett. B 437 (1998) 107–116, arXiv:hep-ph/9806387.
- S. Davidson and S. F. King, "Bimaximal neutrino mixing in the MSSM with a single right-handed neutrino," Phys. Lett. B 445 (1998) 191–198, arXiv:hep-ph/9808296.

GR

- A. Datta, F.-S. Ling, and P. Ramond, "Correlated hierarchy, Dirac masses and large mixing angles," Nucl. Phys. B 671 (2003) 383–400, arXiv:hep-ph/0306002.
- Y. Kajiyama, M. Raidal, and A. Strumia, "The Golden ratio prediction for the solar neutrino mixing," Phys. Rev. D 76 (2007) 117301, arXiv:0705.4559 [hep-ph].
- L. L. Everett and A. J. Stuart, "Icosahedral (A(5)) Family Symmetry and the Golden Ratio Prediction for Solar Neutrino Mixing," Phys. Rev. D 79 (2009) 085005, arXiv:0812.1057 [hep-ph].
- W. Rodejohann, "Unified Parametrization for Quark and Lepton Mixing Angles," Phys. Lett. B 671 (2009) 267–271, arXiv:0810.5239 [hep-ph].

Klein neutrino symmetry

- S. F. King and C. Luhn, "On the origin of neutrino flavour symmetry," JHEP 10 (2009) 093, arXiv:0908.1897 [hep-ph].
- S. F. King and C. Luhn, "Neutrino Mass and Mixing with Discrete Symmetry," Rept. Prog. Phys. 76 (2013) 056201, arXiv:1301.1340 [hep-ph].

Experiment

- Super-Kamiokande Collaboration, Y. Fukuda et al., "Evidence for oscillation of atmospheric neutrinos," Phys. Rev. Lett. 81 (1998) 1562–1567, arXiv:hep-ex/9807003.
- SNO Collaboration, Q. R. Ahmad et al., "Measurement of the rate of $\nu_e + d \to p + p + e^-$ interactions produced by ^{8}B solar neutrinos at the Sudbury Neutrino Observatory," Phys. Rev. Lett. 87 (2001) 071301, arXiv:nucl-ex/0106015.
- Daya Bay Collaboration, F. P. An et al., "Observation of electron-antineutrino disappearance at Daya Bay," Phys. Rev. Lett. 108 (2012) 171803, arXiv:1203.1669 [hep-ex].
- RENO Collaboration, J. K. Ahn et al., "Observation of Reactor Electron Antineutrino Disappearance in the RENO Experiment," Phys. Rev. Lett. 108 (2012) 191802, arXiv:1204.0626 [hep-ex].

Discrete Symmetries

- H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada, and M. Tanimoto, "Non-Abelian Discrete Symmetries in Particle Physics," Prog. Theor. Phys. Suppl. 183 (2010) 1–163, arXiv:1003.3552 [hep-th].
- S. F. King, A. Merle, S. Morisi, Y. Shimizu, and M. Tanimoto, "Neutrino Mass and Mixing: from Theory to Experiment," New J. Phys. 16 (2014) 045018, arXiv:1402.4271 [hep-ph].

Atmospheric sum rules

- C. H. Albright and W. Rodejohann, Eur. Phys. J. C 62 (2009), 599-608 doi:10.1140/epjc/s10052-009-1074-3 [arXiv:0812.0436 [hep-ph]].
- C. Luhn, Nucl. Phys. B 875 (2013), 80-100 doi:10.1016/j.nuclphysb.2013.07.003 [arXiv:1306.2358 [hep-ph]].
- S. F. King and C. Luhn, JHEP 09 (2011), 042 doi:10.1007/JHEP09(2011)042 [arXiv:1107.5332 [hep-ph]].
- P. Ballett, S. F. King, C. Luhn, S. Pascoli and M. A. Schmidt, Phys. Rev. D 89 (2014) no.1, 016016 doi:10.1103/PhysRevD.89.016016 [arXiv:1308.4314 [hep-ph]].
- F. Costa and S. F. King, "Neutrino mixing sum rules and the Littlest Seesaw," arXiv:2307.13895 [hep-ph].

Solar sum rule

- S. F. King, JHEP 08 (2005), 105 doi:10.1088/1126-6708/2005/08/105 [arXiv:hep-ph/0506297 [hep-ph]].
- I. Masina, Phys. Lett. B 633 (2006), 134-140 doi:10.1016/j.physletb.2005.10.097 [arXiv:hep-ph/0508031 [hep-ph]].
- S. Antusch and S. F. King, Phys. Lett. B 631 (2005), 42-47 doi:10.1016/j.physletb.2005.09.075 [arXiv:hep-ph/0508044 [hep-ph]].
- S. Antusch, P. Huber, S. F. King and T. Schwetz, JHEP 04 (2007), 060 doi:10.1088/1126-6708/2007/04/060 [arXiv:hep-ph/0702286 [hep-ph]].
- P. Ballett, S. F. King, C. Luhn, S. Pascoli and M. A. Schmidt, JHEP 12 (2014), 122 doi:10.1007/JHEP12(2014)122 [arXiv:1410.7573 [hep-ph]]
- D. Marzocca, S. T. Petcov, A. Romanino, M. C. Sevilla, JHEP 1305 (2013) 073 [arXiv:1302.0423];
- I. Girardi, S. T. Petcov, A. V. Titov, Nucl. Phys. B 894 (2015) 733 [arXiv:1410.8056 [hep-ph]].
- F. Costa and S. F. King, "Neutrino mixing sum rules and the Littlest Seesaw," arXiv:2307.13895 [hep-ph].

Simple Patterns

$$U_\nu^{\rm TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$U_\nu^{\rm BM} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$U_\nu^{\rm GR} = \begin{pmatrix} c_{12}^\nu & s_{12}^\nu & 0 \\ -\frac{s_{12}^\nu}{\sqrt{2}} & \frac{c_{12}^\nu}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{s_{12}^\nu}{\sqrt{2}} & -\frac{c_{12}^\nu}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$a. \quad \tan\theta_{12}^\nu = \frac{2}{1+\sqrt{5}} = \frac{1}{\phi}$$

$$b. \quad \cos\theta_{12}^\nu = \frac{1+\sqrt{5}}{4} = \frac{\phi}{2}$$

Atmospheric sum rules

$$|U_{e1}| = \sqrt{\frac{2}{3}}, \quad |U_{\mu 1}| = |U_{\tau 1}| = \frac{1}{\sqrt{6}}$$

$$U_{\rm TM1} \approx \begin{pmatrix} \sqrt{\frac{2}{3}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \\ \frac{1}{\sqrt{6}} & - & - \end{pmatrix}$$

$$s_{12}^2 = \frac{(1-3s_{13}^2)}{3\,(1-s_{13}^2)} \qquad \cos\delta = -\frac{\cot 2\theta_{23}(1-5s_{13}^2)}{2\sqrt{2}s_{13}\sqrt{1-3s_{13}^2}}$$

$$|U_{e2}| = |U_{\mu 2}| = |U_{\tau 2}| = \frac{1}{\sqrt{3}}$$

$$U_{\rm TM2} \approx \begin{pmatrix} - & \sqrt{\frac{1}{3}} & - \\ - & \sqrt{\frac{1}{3}} & - \\ - & -\sqrt{\frac{1}{3}} & - \end{pmatrix}$$

$$s_{12}^2 = \frac{1}{3\,(1-s_{13}^2)} \qquad \cos\delta = \frac{2c_{13}\cot 2\theta_{23}\cot 2\theta_{13}}{\sqrt{2-3s_{13}^2}}$$

s_{13}

Charged lepton corrections (also s_{23}^e but not s_{13}^e)

$$U_{\rm PMNS} = \begin{pmatrix} c_{12}^e & s_{12}^e e^{-i\delta_{12}^e} & 0 \\ -s_{12}^e e^{i\delta_{12}^e} & c_{12}^e & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cdots & \cdots & \frac{s_{12}^e}{\sqrt{2}} e^{-i\delta_{12}^e} \\ \cdots & \cdots & \frac{c_{12}^e}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\frac{|U_{\tau 1}|}{|U_{\tau 2}|} = \frac{|s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta}|}{|-c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta}|} = \frac{1}{\sqrt{2}}$$

Solar sum rule

$$\cos\delta = \frac{\tan\theta_{23}\sin\theta_{12}^2 + \sin\theta_{13}^2\cos\theta_{12}^2/\tan\theta_{23} - (\sin\theta_{12}^\nu)^2\left(\tan\theta_{23} + \sin\theta_{13}^2/\tan\theta_{23}\right)}{\sin 2\theta_{12}\sin\theta_{13}}$$