Precision $b \to s\gamma$

Recontres de Moriond, 2024

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Outline

Introduction

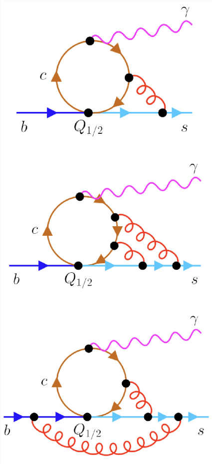
Introduction

$\bar{B} \to X_s\gamma$ is interesting to search (or constraint) new physics in the quark sector:

- $b \to s\gamma$ is forbidden at tree-level in the SM.
- The dominant contributions in the SM come from weak decays.

$\Rightarrow$ The SM rate is small.

$\Rightarrow$ The decay is sensitive to new physics.



Status of $\bar{B} \to X_s\gamma$

Experimental:

$$\mathcal{B}(\bar{B} \to X_s\gamma)^{\text{exp}}_{E_\gamma > 1.6\text{GeV}} = \underbrace{(3.49 \pm 0.19)}_{\pm 5.4\%} \times 10^{-4}$$

- CLEO, BaBar and Belle measurements combined by PDG and HFLAV [arXiv:2206.07501] [1]

[1] Newest BelleII measurement not yet included. [arXiv:2210.10220]

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In the future:

$$\mathcal{B}(\bar{B} \to X_s\gamma)^{\text{exp}}_{E_\gamma > 1.6\text{GeV}} = \underbrace{(3.49 \pm 0.09)}_{\pm 2.6\%} \times 10^{-4}$$

- After Belle II a significant reduction in expected. [arXiv:1808.10567]

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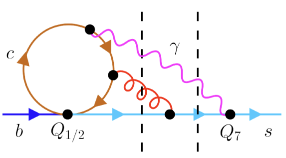
Theoretical: [Misiak, Rehman, Steinhauser '20]

$$\mathcal{B}(\bar{B} \to X_s\gamma)^{\text{exp}}_{E_\gamma > 1.6\text{GeV}} = \underbrace{(3.40 \pm 0.17)}_{\pm 5.0\%} \times 10^{-4}$$

Breakdown of the error: m_c-interpolation

$$\pm 5\% = \sqrt{(\pm 3\%)^2 + (\pm 3\%)^2 + (\pm 2.5\%)^2}$$

higher orders parametric and non-perturbative



$\bar{B} \to X_s\gamma$ in the SM

Determination of $\bar{B} \to X_s\gamma$ in the SM:

$$\mathcal{B}(\bar{B} \to X_s\gamma)_{E_\gamma > E_0} = \mathcal{B}(\bar{B} \to X_c e\bar{\nu})_{\text{exp}} \left|\frac{V_{ts}^* V_{tb}}{V_{cb}}\right|^2 \frac{6\alpha}{\pi C} \left[P(E_0) + N(E_0)\right]$$

- semileptonic phase-space factor: [Alberti, Gambino, Healey, Nandi '14] [2]

$$C = \left|\frac{V_{ub}}{V_{cb}}\right|^2 \frac{\Gamma(\bar{B} \to X_c e\bar{\nu})}{\Gamma(\bar{B} \to X_u e\bar{\nu})}$$

- $P(E_0)$: perturbative contributions

$$P(E_0) \sim \Gamma(b \to X_s^p\gamma) = \Gamma(b \to s\gamma) + \Gamma(b \to sg\gamma) + \Gamma(b \to sq\bar{q}\gamma) + \ldots \approx 96\%$$

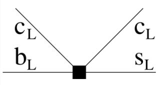
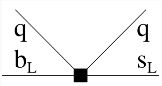
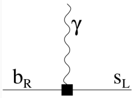
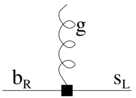
- $N(E_0)$: non perturbtative contributions $\approx 4\%$

[2] A N^{3}LO refinement is possible [Fael, KS, Steinhauser '20; Fael, Usivitsch '23]

Effective Hamiltonian

- At low energies we want to work in the effective theory to resum large logarithmic contributions: $\left(\alpha_s \ln m_W^2/m_b^2\right)^n$
- For $b \to s\gamma$ (when neglecting NLO EW and CKM suppressed effects) we have:

$$\mathcal{L} = \mathcal{L}_{QCD\times QED}(u,d,s,c,b) + \frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_{i=1}^{8} C_i(\mu) Q_i + \dots$$

$Q_{1,2}$	$(\bar{s}\Gamma_i c)(\bar{c}\Gamma_i' b)$		$\lvert C_i(m_b)\rvert \sim 1$
$Q_{3,4,5,6}$	$(\bar{s}\Gamma_i b)\sum_q (\bar{q}\Gamma_i' q)$		$\lvert C_i(m_b)\rvert < 0.07$
Q_7	$\frac{em_b}{16\pi^2}\bar{s}_L\sigma^{\mu\nu} b_R F_{\mu\nu}$		$\lvert C_7(m_b)\rvert \sim 0.3$
Q_8	$\frac{gm_b}{16\pi^2}\bar{s}_L\sigma^{\mu\nu} T^a b_R G^a_{\mu\nu}$		$\lvert C_8(m_b)\rvert \sim 0.15$

Effective Hamiltonian

$$\Gamma(b \to X_s^p \gamma) = \frac{G_F^2 m_b^5 \alpha}{3} 2\pi^4 \left| V_{ts}^* V_{tb} \right|^2 \underbrace{\sum_{i,j=1}^{8} C_i(\mu_b) C_j(\mu_b) G_{ij}}_{\sim P(E_0)}$$

Three steps for the calculation:

1. Calculate the Wilson coefficients $C_i(\mu_0)$ at the hard scale $\mu_0 = m_W$.
2. Derive the renormalization group equations and anomalous dimensions γ_{ij} in the effective theory to evolve down to the low scale $\mu = m_b$:

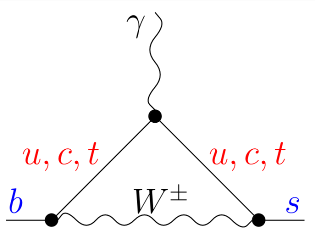
$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} C_i(\mu) = \sum_j \gamma_{ij}(\mu) \cdot C_j(\mu)$$

3. Evaluate the matrix elements $G_{ij}(m_b)$ in the effective theory.

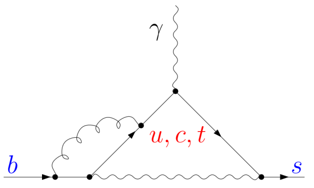
Effective Hamiltonian

Wilson coefficients at hard scale: for expample $C_7(m_W)$

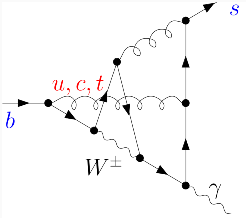
LO [Inami, Lim '81, ...]



NLO [Adel, Yao '93, ...]



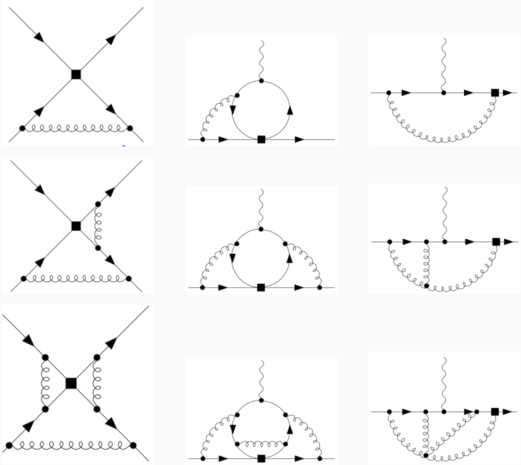
NNLO [Misiak, Steinhauser '04]



Effective Hamiltonian

Anomalous dimensions: $\mu \frac{\mathrm{d}}{\mathrm{d}\mu} C_i(\mu) = \sum_j \gamma_{ij}(\mu) \cdot C_j(\mu)$

Z_{22} Z_{27} Z_{28}



LO

[Gaillard, Lee '74; Altarelli, Maiani '74] [Grinstein et al '90]

[Shifmann et al '78; Grigjanis et al '88]

NLO

[Altarelli et al '81; Buras, Weisz '90] [Chetryrkin, Misiak, Münz '97]

[Misiak, Münz '95]

NNLO

[Gorbahn, Haisch '04] [Czakon, Haisch, Misiak '06] [Gorbahn, Haisch, Misiak '05]

NNLO corrections give -4% correction to the branching ratio

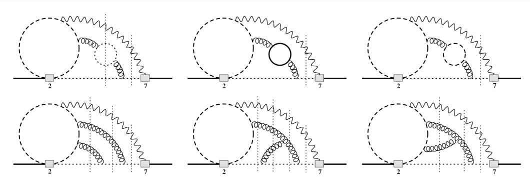
NNLO corrections to $B \to X_s \gamma$

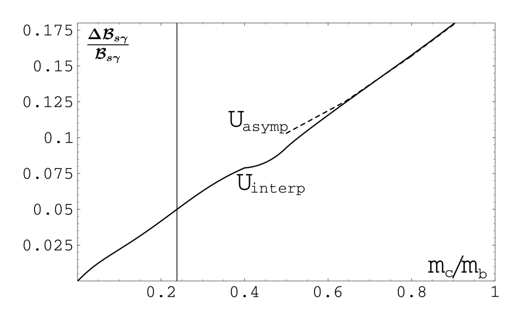
$$\Gamma(b \to X_s^p \gamma) = \frac{G_F^2 m_b^5 \alpha}{3} 2\pi^4 \left| V_{ts}^* V_{tb} \right|^2 \underbrace{\sum_{i,j=1}^{8} C_i(\mu_b) C_j(\mu_b) G_{ij}}_{\sim P(E_0)}$$

Status:

- NLO is known completely. [Greub, Hurth, Wyler '96; Ali, Greub '91-'95; Buras, Czarnecki, Urban, Misiak '02; Pott '95]
- NNLO:
  - G_{77} and G_{78} are known completely. [Blokland et al '05; Melnikov, Mitov '05; Asatrian et al. '06-'10]
  - For numerically small contributions the two body contributions are known, the rest is approximated using BLM.
  - G_{17} and G_{27} interpolated between $m_c = 0$ and $m_c \to \infty$.

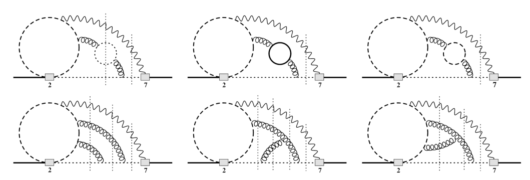
Calculation of G_{27} at NNLO





- Perturbative calculation can be done by considering diagrams with operator insertions and unitarity cuts.
- Calculation for $m_c \to \infty$: [Misiak, Steinhauser '06, '10]
- Calculation for $m_c = 0$: [Czakon, Fiedler, Huber, Misiak, Schutzmeier, Steinhauser '15]
- Calculation of terms proportional to n_f for arbitrary values of m_c: [Misiak, Rehmann, Steinhauser '20]

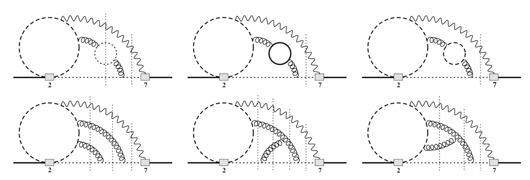
$\Rightarrow$ Interpolation to physical m_c/m_b introduces $\pm 3\%$ error in final result.



General work flow:

1. Generate all diagrams and express the amplitudes in terms of four-loop two-scale scalar integrals with unitarity cuts.

Calculation of G_{27} at NNLO



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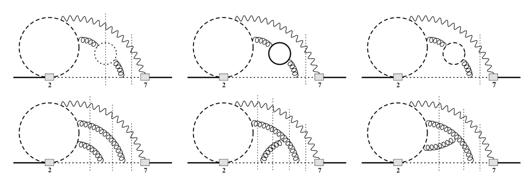

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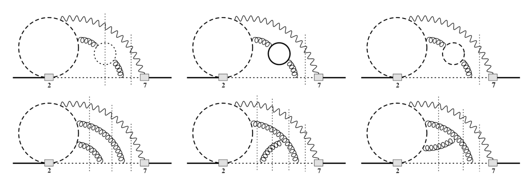
3. Using the IBP reduction we can find a system of differential equations for the masters M_k:

$$\frac{d}{dz} M_k(z = m_c^2/m_b^2, \epsilon) = R_{kl}(z, \epsilon) M_l(z, \epsilon)$$

Calculation of G_{27} at NNLO



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4a. Solve the master integrals numerically with boundary values obtained for $z \to \infty$.

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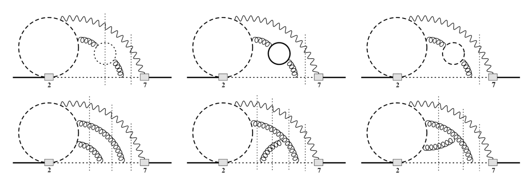

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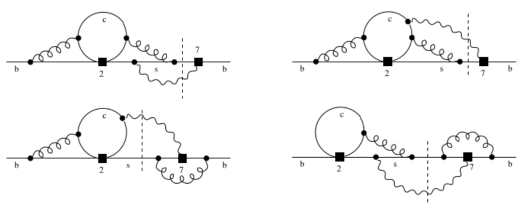
4b. Calculate the master integrals numerically at the physical point with AMFlow [Liu, Ma '22] .

Two Body contributions to G_{27} at NNLO

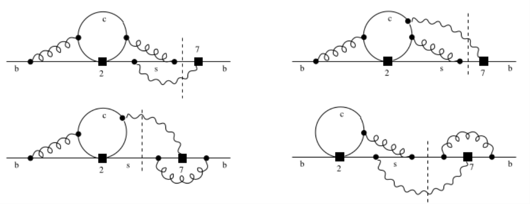
[Czaja, Czakon, Huber, Misiak, Niggetiedt, Rehmann, KS, Steinhauser '23]



- We finished the calculation of the 2-body contributions.
  - We find $\mathcal{O}(500)$ integral familes.
  - The reductions to master integrals are done with `Fire` [Smirnv, Chuharev '19] and `Kira` [Klappert, Lange, Maierhöfer, Usovitsch '20] .
  - For the two body contributions we need to evaluate 447 master integrals.
- The master integrals are evaluated at the physical point with AMFLow.
- We cross-checked the boundary conditions for $z \to \infty$.

Two Body contributions to G_{27} at NNLO

[Czaja, Czakon, Huber, Misiak, Niggetiedt, Rehmann, KS, Steinhauser '23]

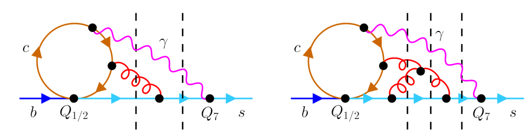


- We finished the calculation of the 2-body contributions.

$$
\begin{aligned}
\Delta_{30}\hat{G}_{27}^{(2)2P}(z=0.04) &\simeq \frac{0.181070}{\epsilon^3} - \frac{6.063805}{\epsilon^2} - \frac{34.087329}{\epsilon} - 127.624515 \\
&+ \left(\frac{0.482853}{\epsilon^2} + \frac{4.093615}{\epsilon} + 10.984004\right) n_b \\
&+ \left(\frac{0.482853}{\epsilon^2} + \frac{4.185427}{\epsilon} + 19.194053\right) n_c \\
&+ \left(\frac{0.482853}{\epsilon^2} + \frac{4.135795}{\epsilon} + 19.647238\right) n_l \,, \\
\Delta_{30}\hat{G}_{17}^{(2)2P}(z=0.04) &\simeq -\frac{1}{6}\Delta_{30}\hat{G}_{27}^{(2)2P}(z=0.04) + \frac{0.987654}{\epsilon^2} + \frac{6.383643}{\epsilon} + 34.077780
\end{aligned}
$$

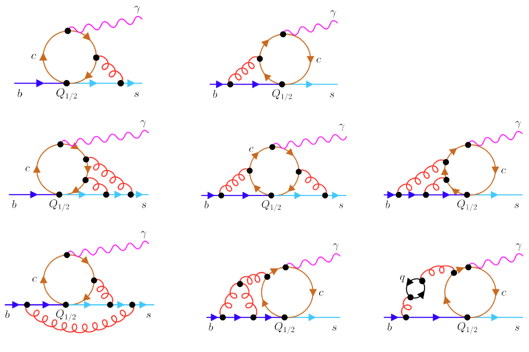
Two Body contributions to G_{27} at NNLO

[Fael, Lange, KS, Steinhauser '23]



Other approach:

- Interpret the cut diagrams as vertex corrections:

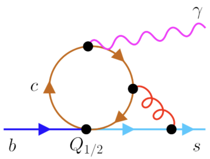


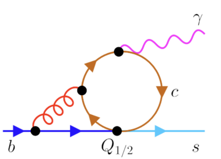
Two Body contributions to G_{27} at NNLO

Calculation of the vertex: $b(p_b) \to s(p_s) + \gamma(q_\gamma)$

$$M^\mu = \bar{u}_s(p_s) P_R \left(t_1 \frac{q_\gamma^\mu}{m_b} + t_2 \frac{p_b^\mu}{m_b} + t_3 \, \gamma^\mu \right) u_b(p_b)$$

- Calculate 30 (591) diagrams at 2-(3-)loop level.
- We find masters 14 (479) master integrals at at 2-(3-)loop level.
- At 2-loop: We are able to solve all master integrals analytically, extending the previously known results. [Misiak, Rehman, Steinhauser '17]





Two Body contributions to G_{27} at NNLO

Master integrals at 3-loop:

- Calculate initial values of the master integrals at $x = m_c/m_b = 1/5$ with AMFlow.
- Construct symbolic expansions around $x = 1/5, 1/10, 0$ by inserting an ansatz into the differential equation and solve a large linear system of equations in terms of a small number of initial conditions.
- Use either the initial boundary value or the previous expansion to fix the initial conditions.

$\Rightarrow$ We obtain a precise semi-analytic result for $0 < m_c/m_b < 1/5$.

We agree with a partial result obtained in [Greub, Asatrian, Saturnino, Wiegand '23] .

Non-perturbative effects

- The matrix elements also receive non-perturbative contributions.
- The most important effects come from photons coupling to light quarks.
- Effects can be described using SCET and non-local soft matrix elements (shape functions). [Benzke, Lee, Neubert, Paz '10]
- Moments of the shape functions can be related to HQET parameters. [Gunawardana, Paz '19]

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For example:

$$\Lambda_{17} = \frac{2}{3}\mathrm{Re}\int_{-\infty}^{+\infty} \frac{\mathrm{d}w_1}{w_1}\left[1 - F\left(\frac{m_c^2}{m_b w_1}\right) + \frac{m_b w_1}{12 m_c^2}\right] h_{17}(\omega_1)\ , \qquad \int_{-\infty}^{\infty} \mathrm{d}\omega_1\, h_{17} = \frac{2}{3}\mu_G^2\ , \ \ldots$$

Non-perturbative effects

- Some non-perturbative effects can be estimated by data driven approaches, e.g. the $Q_7 - Q_8$ interference:

$$\Gamma[B^- \to X_s\gamma] \sim A + B\,Q_u + C\,Q_d + DQ_s, \quad \Gamma[\bar{B}^0 \to X_s\gamma] \sim A + B\,Q_d + C\,Q_u + DQ_s$$

- Isospin averaged: $\Gamma \sim A + \frac{1}{2}(B+C)(Q_u + Q_d) + DQ_s = A + \delta\Gamma_{78}$
- Isospin asymmetry: $\Delta_{0-} \sim \frac{C-B}{2\Gamma}(Q_u - Q_d)$

$$\frac{\delta\Gamma_{78}}{\Gamma} \sim \frac{Q_u + Q_d}{Q_d - Q_u}\left[1 + \underbrace{\pm 0.3}_{SU_F(3)\ \text{breaking}}\right]\Delta_{0-}$$

- Belle [arXiv:1807.04236] : $\Gamma_{0-} = \left(-0.48 \pm 1.49 \pm 0.97 \pm 1.15\right)\%$

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- Belle II expects a factor of 4 improvement.

B → Vγ [Paul, Straub '16]

$$\mathcal{B}(B_q \to V\gamma) = \tau_{B_q} \frac{G_F^2 e^2 \left|V_{tb}^* V_{tq}\right|^2 m_{B_q}^3 m_b^2}{128\pi^4} \left(|C_7|^2 + |C_7'|^2\right) T_1(0) \,,$$

$$A_{CP}(B_q(t) \to V\gamma) = \frac{\Gamma(\bar{B}_q(t) \to \bar{V}\gamma) - \Gamma(B_q(t) \to V\gamma)}{\Gamma(\bar{B}_q(t) \to \bar{V}\gamma) + \Gamma(B_q(t) \to V\gamma)}$$

$T_1(0)$: non-perturbative form factor, $C_7' = m_s/m_b C_7$ in the SM

- The main source of uncertainty in the branching ratio comes from $T_1(0)$.
- Other hadronic contributions are estimated using QCD factorization and light-cone-sum-rules.

$\mathbf{B \to V}\gamma$ [Paul, Straub '16, Straub (flavio) '18]

	$\mathcal{B}_{\text{exp}} \cdot 10^5$		$\mathcal{B}_{\text{SM}} \cdot 10^5$
$B^0 \to K^{*0}\gamma$	$4.47 \pm 0.10 \pm 0.16$	BaBar	4.18 ± 0.84
$B^0 \to K^{*0}\gamma$	$3.96 \pm 0.07 \pm 0.14$	Belle	
$B^+ \to K^{*+}\gamma$	$4.22 \pm 0.14 \pm 0.16$	BaBar	4.25 ± 0.88
$B^+ \to K^{*+}\gamma$	$3.76 \pm 0.10 \pm 0.12$	Belle	
$B_s^0 \to \phi\gamma$	$3.6 \pm 0.5 \pm 0.3 \pm 0.6$	Belle	4.02 ± 0.52
$B_s^0 \to \phi\gamma$	$3.38 \pm 0.34 \pm 0.20$	LHCb	

- The SM predicition for the branching ratios has large uncertainties due to non-perturbative effects.

Exclusive decays

$B \to V\gamma$

	experimental		SM
$A_{CP}(B^0 \to K^*\gamma)$	$(-0.4 \pm 1.4 \pm 0.3)\%$	Belle	$(0.3 \pm 0.1)\%$
	$(-0.3 \pm 1.7 \pm 0.7)\%$	BaBar	
	$(+0.8 \pm 1.7 \pm 0.9)\%$	LHCb	
	$(-0.4 \pm 0.21)\%$	Belle II	

- Specific angular observables and asymmetries are theoretically clean.
- In these cases the theoretical precision often exceeds the experimental one.

$\Rightarrow$ Global fits to inclusive and exclusive decay modes can severely constrain NP.

Summary and Outlook

Experimental:

$$\mathcal{B}(\bar{B} \to X_s\gamma)^{\text{exp}}_{E_\gamma > 1.6\text{GeV}} = \underbrace{(3.49 \pm 0.19)}_{\pm 5.4\%} \times 10^{-4}$$

Theoretical: [Misiak, Rehman, Steinhauser '20]

$$\mathcal{B}(\bar{B} \to X_s\gamma)^{\text{exp}}_{E_\gamma > 1.6\text{GeV}} = \underbrace{(3.40 \pm 0.17)}_{\pm 5.0\%} \times 10^{-4}$$

Breakdown of the error: m_c-interpolation

$$\pm 5\% = \sqrt{(\pm 3\%)^2 + (\pm 3\%)^2 + (\pm 2.5\%)^2}$$

higher orders parametric and non-perturbative

Status of $B \to X_s\gamma$

In the future:

$$\mathcal{B}(\bar{B} \to X_s\gamma)^{\text{exp}}_{E_\gamma > 1.6\text{GeV}} = \underbrace{(3.49 \pm 0.09)}_{\pm 2.6\%} \times 10^{-4}$$

Theoretical: [Misiak, Rehman, Steinhauser '20]

$$\mathcal{B}(\bar{B} \to X_s\gamma)^{\text{exp}}_{E_\gamma > 1.6\text{GeV}} = \underbrace{(3.40 \pm 0.13)}_{\pm 3.9\%} \times 10^{-4}$$

Breakdown of the error: ~~m_c-interpolation~~

$$\pm 3.9\% = \sqrt{(\pm 3\%)^2 + \cancel{(\pm 3\%)^2} + (\pm 2.5\%)^2}$$

higher orders parametric and non-perturbative

Backup

Two Body contributions to G_{27} at NLO

New analytic results at 2-loop:

$$G_{27}^{(1),2P} = -\frac{92}{81\epsilon} + f_0(z) + \epsilon f_1(z) + \mathcal{O}(\epsilon^2),$$

$$\begin{aligned}
f_0 = C_F\Bigg[& -\frac{971 + 1916w + 1602w^2 + 1916w^3 + 971w^4}{162(1+w)^4} + \frac{2wH_0(w)^3}{3(1+w)^2} \\
& + \frac{8w(27 + 57w + 26w^2 + 7w^3 + 5w^4)H_0(w)}{27(1+w)^5} - \frac{16w(2 + 3w + 2w^2)H_0(x)^3}{9(1+w)^4} \\
& - \frac{2w(-1 - 2w + 4w^2 + 6w^3 + 3w^4)H_0(w)^2}{3(1+w)^6} - \frac{8w(1+w^2)H_{0,0,-1}(w)}{3(1+w)^4} \\
& - \frac{8(5 + 29w + 54w^2 + 29w^3 + 5w^4)H_{-1}(w)}{27(1+w)^4} + \frac{16w^2(3 + 13w + 15w^2 + 6w^3)H_{0,-1}(w)}{9(1+w)^6} \\
& - \frac{16w(1 - \sqrt{w} + w)}{9(1+w)^6}(3 + 8w + 8w^2 + 3w^3 + 2\sqrt{w} + 3w^{3/2} + 2w^{5/2})H_{1,0}(x) \\
& + \frac{16w(1 + \sqrt{w} + w)}{9(1+w)^6}(3 + 8w + 8w^2 + 3w^3 - 2\sqrt{w} - 3w^{3/2} - 2w^{5/2})H_{-1,0}(x) \\
& + \frac{16w(3 + 9w + 13w^2 + 9w^3 + 3w^4)H_{-1,0}(w)}{9(1+w)^6} - \frac{8w\zeta_3}{(1+w)^2} \\
& - \frac{32w(3 + 9w + 13w^2 + 9w^3 + 3w^4)H_{-1,-1}(w)}{9(1+w)^6} - \frac{16w(2 + 3w + 2w^2)H_{0,1,0}(x)}{3(1+w)^4} \\
& + \frac{16w(2 + 3w + 2w^2)H_{0,-1,0}(x)}{3(1+w)^4} - \frac{16w(1 + 3w + w^2)H_{-1,0,0}(w)}{3(1+w)^4} \\
& + \pi^2\Bigg(-\frac{2w}{27(1+w)^6}(15 + 60w + 94w^2 + 84w^3 + 27w^4 - 12\sqrt{w} - 36w^{3/2} \\
& - 36w^{5/2} - 12w^{7/2}) - \frac{2w(3 + 8w + 3w^2)H_0(w)}{3(1+w)^4} + \frac{8w(5 + 12w + 5w^2)H_{-1}(w)}{9(1+w)^4}\Bigg)\Bigg]
\end{aligned}$$

with

$$x = m_c/m_b,$$

$$w = (1 - \sqrt{1 - 4x^2})/(1 + \sqrt{1 - 4x^2})$$