$B_s \to \mu^+\mu^-\gamma$ at large q^2 from lattice QCD

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Motivations

Why $B_s \to \mu^+\mu^-\gamma$ at large q^2 ?

- The $B_s \to \mu^+\mu^-\gamma$ decay allows for a new test of the SM predictions in $b \to s$ FCNC transitions.
- Despite the $\mathcal{O}(\alpha_{\rm em})$-suppression w.r.t. the widely studied $B_s \to \mu^+\mu^-$, removal of helicity-suppression makes the two decay rates comparable in magnitude.
- At very high $\sqrt{q^2}$ = invariant mass of the $\mu^+\mu^-$, the contributions from penguin operators appearing in the weak effective-theory, which are difficult to compute on the lattice, are suppressed [Guadagnoli, Reboud, Zwicky, JHEP '17] ✓.

In this talk I will present the first, ($\simeq$) first-principles lattice QCD calculation of the $B_s \to \mu^+\mu^-\gamma$ decay rate for $q^2 \gtrsim (4.2 \text{ GeV})^2$.

The effective weak-Hamiltonian

The low-energy effective theory describing the $b \to s$ transition, neglecting doubly Cabibbo-suppressed terms, is

$$\mathcal{H}_{\text{eff}}^{b\to s} = 2\sqrt{2} G_F V_{tb} V_{ts}^* \left[\sum_{i=1,2} C_i(\mu) \mathcal{O}_i^c + \sum_{i=3}^{6} C_i(\mu) \mathcal{O}_i + \frac{\alpha_{\text{em}}}{4\pi} \sum_{i=7}^{10} C_i(\mu) \mathcal{O}_i \right]$$

current-current: $\mathcal{O}_1^c = \left(\bar{s}_i \gamma^\mu P_L c_j\right) \left(\bar{c}_j \gamma^\mu P_L b_i\right)$, $\mathcal{O}_2^c = \left(\bar{s} \gamma^\mu P_L c\right) \left(\bar{c} \gamma^\mu P_L b\right)$,

ph./chromo. penguins: $\mathcal{O}_7 = -\frac{m_b}{e} \bar{s} \sigma^{\mu\nu} F_{\mu\nu} P_R b$, $\mathcal{O}_8 = -\frac{g_s m_b}{4\pi\alpha_{\text{em}}} \bar{s} \sigma^{\mu\nu} G_{\mu\nu} P_R b$,

semileptonic: $\mathcal{O}_9 = \left(\bar{s} \gamma^\mu P_L b\right) \left(\bar{\mu} \gamma_\mu \mu\right)$, $\mathcal{O}_{10} = \left(\bar{s} \gamma^\mu P_L b\right) \left(\bar{\mu} \gamma_\mu \gamma^5 \mu\right)$

- The amplitude $\mathcal{A}$ is the sandwich of $\mathcal{H}_{\text{eff}}^{b\to s}$ between initial and final states

$$\mathcal{A}[\bar{B}_s \to \mu^+ \mu^- \gamma] = \langle \gamma(\boldsymbol{k}, \varepsilon) \mu^+(p_1) \mu^-(p_2) | - \mathcal{H}_{\text{eff}}^{b\to s} | \bar{B}_s(\boldsymbol{p}) \rangle_{\text{QCD+QED}} \ ,$$

- To lowest-order in $\mathcal{O}(\alpha_{\text{em}})$ [Beneke et al, EPJC 2011]:

$$\mathcal{A}[\bar{B}_s \to \mu^+ \mu^- \gamma] = -e \frac{\alpha_{\text{em}}}{\sqrt{2}\pi} V_{tb} V_{ts}^* \varepsilon_\mu^* \left[\sum_{i=1}^{9} C_i \overbrace{H_i^{\mu\nu}}^{\text{NP}-\text{QCD}} L_{V\nu} + C_{10} \left(\overbrace{H_{10}^{\mu\nu}}^{\text{NP}-\text{QCD}} L_{A\nu} - \overbrace{\frac{i}{2} f_{B_s} L_A^{\mu\nu} p_\nu}^{\text{PT}-\text{contribution}} \right) \right]$$

The local form factors and penguin operators

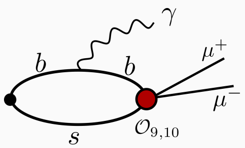
The non-perturbative, structure-dependent, information is encoded in the hadronic tensors $H_i^{\mu\nu}$, which can be grouped in three categories:

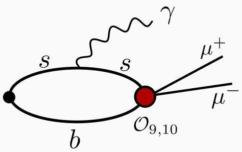
Contributions from semileptonic operators:

$$H_9^{\mu\nu}(p,k) = H_{10}^{\mu\nu}(p,k) = i \int d^4y \; e^{iky} \; \hat{\mathrm{T}}\langle 0| \left[\bar{s}\gamma^\nu P_L b\right](0) J_{\mathrm{em}}^\mu(y) |\bar{B}_s(\boldsymbol{p})\rangle$$

$$= -i\left[g^{\mu\nu}(k\cdot q) - q^\mu k^\nu\right]\frac{F_A}{2m_{B_s}} + \varepsilon^{\mu\nu\rho\sigma}k_\rho q_\sigma \frac{F_V}{2m_{B_s}}$$

- Parametrized by vector and axial form factors $F_V(x_\gamma)$ and $F_A(x_\gamma)$ $[x_\gamma \equiv 2E_\gamma/m_{B_s}]$. E_γ is the photon energy in the rest-frame of the $\bar{B}_s$.





- It can be computed using standard lattice techniques.

The local form factors and penguin operators

The non-perturbative, structure-dependent, information is encoded in the hadronic tensors $H_i^{\mu\nu}$, which can be grouped in three categories:

Contributions from photon-penguin operator (A-type):

$$H_{7A}^{\mu\nu}(p,k) = i\frac{2m_b}{q^2}\int d^4y\ e^{iky}\ \hat{\mathrm{T}}\langle 0|\left[-i\bar{s}\sigma^{\nu\rho}q_\rho P_R b\right](0)J^\mu_{\mathrm{em}}(y)|\bar{B}_s(\boldsymbol{p})\rangle$$

$$= -i\left[g^{\mu\nu}(k\cdot q) - q^\mu k^\nu\right]\frac{F_{TA}m_b}{q^2} + \varepsilon^{\mu\nu\rho\sigma}k_\rho q_\sigma\frac{F_{TV}m_b}{q^2}$$

- Parametrized by tensor and axial-tensor form factors $F_{TV}(x_\gamma)$ and $F_{TA}(x_\gamma)$.

- It can be computed using standard lattice techniques.

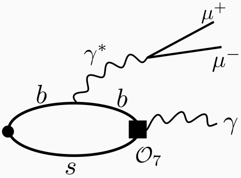
The local form factors and penguin operators

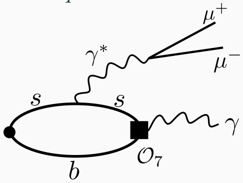
The non-perturbative, structure-dependent, information is encoded in the hadronic tensors $H_i^{\mu\nu}$, which can be grouped in three categories:

Contributions from photon-penguin operator (B-type):

$$H_{7B}^{\mu\nu}(p,k) = i\frac{2m_b}{q^2}\int d^4y\ e^{iqy}\ \hat{\mathrm{T}}\langle 0|\left[-i\bar{s}\sigma^{\mu\rho}k_\rho P_R b\right](0)J_{\mathrm{em}}^{\nu}(y)|\bar{B}_s(\boldsymbol{p})\rangle$$

$$= -i\left[g^{\mu\nu}(k\cdot q) - q^\mu k^\nu\right]\frac{\bar{F}_T m_b}{q^2} + \varepsilon^{\mu\nu\rho\sigma}k_\rho q_\sigma\frac{\bar{F}_T m_b}{q^2}$$





- Computing $\bar{F}_T$ on the lattice is challenging due to lack of analytic continuation to Euclidean spacetime of the correlation functions of interest. We evaluate $\bar{F}_T$ using the spectral density technique developed in [Frezzotti et al, PRD 108 '23] (Backup). Its contribution to the branching is negligible within current accuracy.

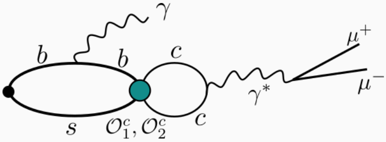
The local form factors and penguin operators

The non-perturbative, structure-dependent, information is encoded in the hadronic tensors $H_i^{\mu\nu}$, which can be grouped in three categories:

Contributions from four-quark and chromomagnetic operators:

$$H^{\mu\nu}_{i=1-6,8}(p,k) = \frac{(4\pi)^2}{q^2} \int d^4y \; d^4x \; e^{iky} e^{iqx} \; \hat{\mathrm{T}} \langle 0 | J^{\mu}_{\text{em}}(y) J^{\nu}_{\text{em}}(x) \mathcal{O}_i(0) | \bar{B}_s(\boldsymbol{p}) \rangle$$

- In the high-q^2 region, they are formally of higher-order in the $1/m_b$ expansion [Guadagnoli, Reboud, Zwicky, JHEP '17].
- We did not compute them, but have future plans to do so.
- In the evaluation of the branching fractions we only included a phenomenological description of the allegedly dominant contribution from the following charming-penguin diagram:



This contribution is dominated by vector $c\bar{c}$ resonances. Some of them overlap with the q^2 region we consider. A description of our parameterization will come later.

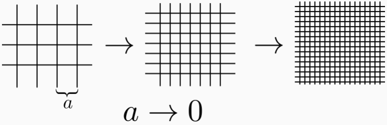
The local form factors on the lattice (I)

We computed on the lattice the local form factors F_V, F_A, F_{TV}, F_{TA} and $\bar{F}_T$ for

$$x_\gamma \in [0.1 : 0.4] \implies 4.16 \text{ GeV} < \sqrt{q^2} < 5.1 \text{ GeV}$$

Two main sources of systematics on the lattice, which must be controlled:

- Continuum-limit extrapolation $(a \to 0)$...



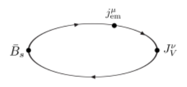
- which we handle by simulating at four values of the lattice spacing $a \in [0.057 : 0.09]$ fm using configurations produced by the ETM Collaboration.
- Extrapolation to the physical B_s meson mass, which we handle by simulating at five different values of the heavy-strange meson mass $m_{H_s} \in [m_{D_s} : 2m_{D_s}]$...
- and then performing the extrapolation $m_{H_s} \to m_{B_s}$ via pole-like+HQET scaling relations. On current lattices in fact we cannot simulate directly the B_s meson, which is too heavy.

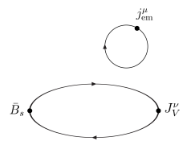
The local form factors on the lattice (II)

We evaluate on the lattice (e.g. in the case of vector FF, F_V):

$$H_V^{\mu\nu}(x_\gamma) = \int \mathrm{d}t_y\, \mathrm{d}^3y\, e^{E_\gamma t_y}\, e^{-i\boldsymbol{k}\boldsymbol{y}}\, \hat{\mathrm{T}}\langle 0|\underbrace{J_V^\nu}_{\bar{s}\gamma^\nu b}(0) J_{\mathrm{em}}^\mu(t_y,\boldsymbol{y})|\bar{B}_s\rangle$$

in the so-called electroquenched approximation





i.e., we neglect the quark disconnected diagram, which vanishes in the $\mathrm{SU}(3)$-symmetric limit and for $m_c \to \infty$.

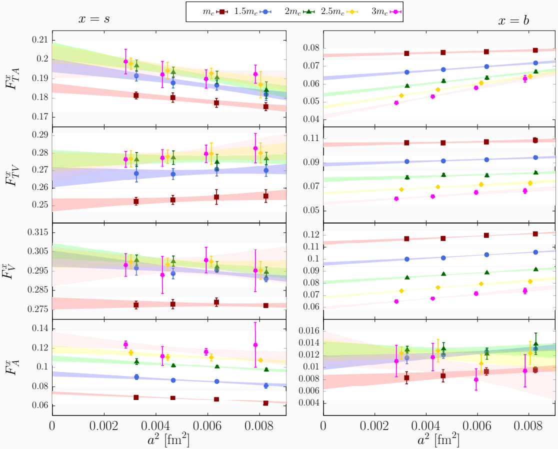
- From $H_V^{\mu\nu}$ it is straightforward to isolate the form factor F_V (similarly for F_A, F_{TA}, F_{TV}).
- We evaluated $H_W^{\mu\nu}$ for $W = \{V, A, TV, TA\}$, for all four lattice spacings, and for all simulated heavy-strange quark masses $m_{H_s} \in [m_{D_s} : 2m_{D_s}]$.
- When simulating at a given m_{H_s} we perform the kinematical rescaling:

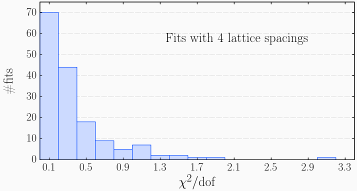
$$E_\gamma \propto m_{H_s}$$

i.e., we always keep $x_\gamma = 2E_\gamma/m_{H_s}$ fixed.

Continuum limit extrapolation

We perform the continuum-limit extrapolation at fixed m_{H_s} and x_γ





We performed a total of 160 continuum-limit extrapolations.

$\Longleftarrow$ Example for $x_\gamma = 0.4$.

Systematic errors evaluated performing fits using only the three finest lattice spacings.

Results obtained using three or four lattice spacings combined using AIC.

Extrapolation to the physical B_s meson mass (I)

In the limit of large E_γ and m_{H_s} the heavy-mass/large-energy EFT predicts up to radiative corrections [Beneke et al, EPJC 2011, JHEP 2020]

$$\frac{F_W(x_\gamma, m_{H_s})}{f_{H_s}} \propto \frac{|q_s|}{x_\gamma} \frac{1}{\lambda_{B_s}} + \mathcal{O}(\frac{1}{E_\gamma}, \frac{1}{m_{H_s}}) \qquad W = \{V, A, TV, TA\} \qquad [1]$$

In the high-q^2 region we consider ($x_\gamma \in [0.1 : 0.4]$) sizable corrections to [1] due to resonance contributions are to be expected. Relying on VMD one has

$$\frac{F_W(x_\gamma, m_{H_s})}{f_{H_s}} \propto \frac{1}{r_W + \frac{x_\gamma}{2} - 1} + \mathcal{O}(\frac{1}{E_\gamma}, \frac{1}{m_{H_s}})$$

$$r_V = r_{TV} = \frac{m_{H_s^*}}{m_{H_s}} \simeq \underbrace{1 + \frac{\lambda_2}{m_{H_s}^2}}_{\text{LO-HQET}} \; , \qquad r_A = r_{TA} = \frac{m_{H_{s1}}}{m_{H_s}} \simeq \underbrace{1 + \frac{\Lambda_1}{m_{H_s}}}_{\text{LO-HQET}}$$

- $\sqrt{\lambda_2} \sim \Lambda_1 \simeq 0.5$ GeV. H_s^* and H_{s1} are respectively the ground state $J^P = 1^-$ and $J^P = 1^+$ mesons, made of an heavy quark and a strange anti-quark.
- If $x_\gamma \ll 2\lambda_2/m_{H_s}^2$ ($x_\gamma \ll 2\Lambda_1/m_{H_s}$), the presence of a quasi-pole generates an enhancement of $F_{V/TV}$ ($F_{A/TA}$) of order $\mathcal{O}(m_{H_s}^2)$ ($\mathcal{O}(m_{H_s})$), w.r.t. to [1].

Extrapolation to the physical B_s meson mass (II)

To extrapolate to the physical B_s we build a phenomenological fit Ansatz which combines the scaling laws valid for very hard photons, with the quasi-pole correction due to resonance contributions.

$$\frac{F_V(x_\gamma, z)}{f_{H_s}} = \frac{|q_s|}{x_\gamma} \frac{1}{1 + C_V \frac{2z^2}{x_\gamma}} \; [K + \text{NLO+NNLO}]$$

$$\frac{F_A(x_\gamma, z)}{f_{H_s}} = \frac{|q_s|}{x_\gamma} \frac{1}{1 + C_A \frac{2z}{x_\gamma}} \; [K + \text{NLO+NNLO}]$$

$$\frac{F_{TV}(x_\gamma, z)}{f_{H_s}} = \frac{|q_s|}{x_\gamma} \frac{1 + 2C_V z^2}{1 + C_V \frac{2z^2}{x_\gamma}} [K_T + \text{NLO+NNLO}]$$

$$\frac{F_{TA}(x_\gamma, z)}{f_{H_s}} = \frac{|q_s|}{x_\gamma} \frac{1 + 2C_A^T z}{1 + C_A^T \frac{2z}{x_\gamma}} [K_T + \text{NLO+NNLO}]$$

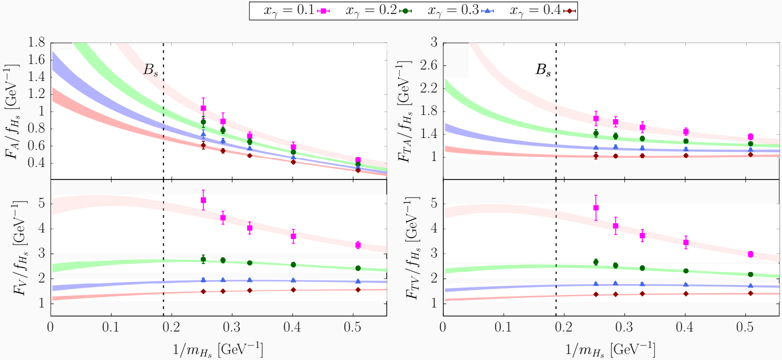
$z \equiv m_{H_s}^{-1}$.

We included in the fit also NLO $1/E_\gamma, 1/m_{H_s}$, and NNLO $1/E_\gamma^2$, $1/m_{H_s}^2$ corrections.

NNLO-terms not needed for a good χ^2/dof. They mainly serve to estimate systematic errors.

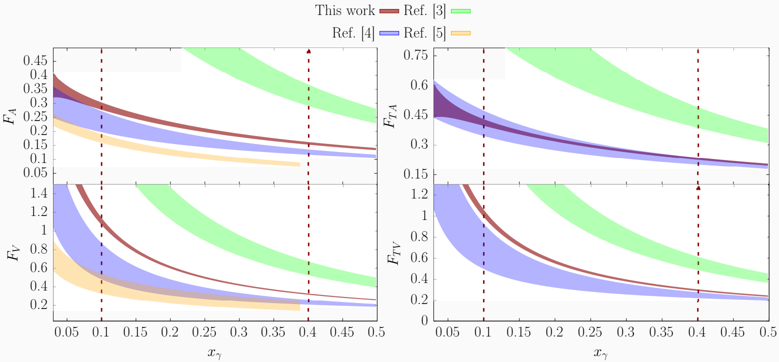
- The Ansatz takes into account LO and NLO constraints from heavy-mass and large-energy EFT, and contains the resonance corrections (relevant at small x_γ).
- Algebraic constraint at $x_\gamma = 1$, $F_{TV}(1) = F_{TA}(1)$, incorporated in the fit Ansatz.

The form factors at the physical point $m_{B_s} \simeq 5.367$ GeV



- Observed steeper m_{H_s}-dependence of the form factors at small x_γ ✓. [Determination of f_{H_s} and f_{B_s} in backup].
- We performed more than 500 fits, by including or not some of the NLO and NNLO fit parameters, and imposing or not $K = K_T$ and $C_A = C_A^T$.
- Different fits combined using AIC or by including in the final average (and with a uniform weight) only those fits having $\chi^2/dof < 1.4$ (the two strategies give consistent results, second criterion used to give final numbers).

Comparison with previous calculations



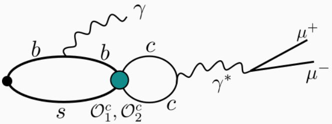
- Ref. [3] = Janowski, Pullin , Zwicky , JHEP '21 , light-cone sum rules.
- Ref. [4] = Kozachuk, Melikhov, Nikitin , PRD '18 , relativistic dispersion relations.
- Ref. [5] = Guadagnoli, Normand, Simula, Vittorio, JHEP '23, VMD/quark-model/lattice.

With a few exceptions, our results for the form factors differ significantly from the earlier estimates (which also differ from each other).

Estimating uncertainties from the missing penguin operators

We did not compute from first-principles the contributions from four-quark and chromomagnetic operators $\mathcal{O}_{i=1-6,8}$.

- It is expected that among these contributions the dominant one in $\bar{B}_s \to \mu^+\mu^-\gamma$ at $q^2 > (4.2\ \mathrm{GeV})^2$ is the charming-penguin diagram stemming from $\mathcal{O}_{1-2}$ due to $J^P = 1^-$ charmonium resonances.



This contribution can be included as a shift of the Wilson coefficient C_9:

$$C_9 \to C_9^{\mathrm{eff}}(q^2) = C_9 - \Delta C_9(q^2)$$

$\delta_V = |k_V| - 1 = 0$ holds in the factorization approximation.

In analogy with previous works [Guadagnoli et al, JHEP '17, '23] we model $\Delta C_9(q^2)$ as

$$\Delta C_9(q^2) = \frac{9\pi}{\alpha_{\mathrm{em}}^2} \bar{C} \sum_V |k_V| e^{i\delta_V} \frac{m_V\, B(V \to \mu^+\mu^-)\, \Gamma_V}{q^2 - m_V^2 + i m_V \Gamma_V}$$

$$\bar{C} = C_1 + C_2/3 \simeq -0.2$$

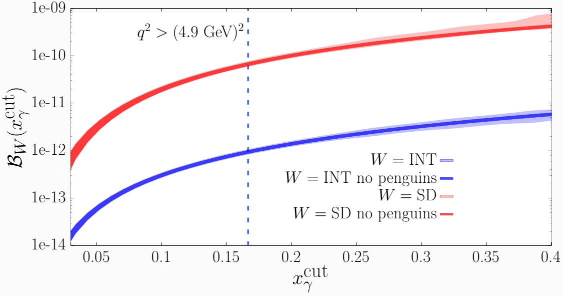
$V_{c\bar{c}}$	$M_{V_{c\bar{c}}}$ [GeV]	Γ [MeV]	$\mathcal{B}(V_{c\bar{c}} \to \mu^+\mu^-)$
J/ψ	3.096900(6)	0.0926(17)	0.05961(33)
$\Psi(2S)$	3.68610(6)	0.294(8)	$8.0(6) \cdot 10^{-3}$
$\Psi(3770)$	3.7737(4)	27.2(1.0)	$^*9.6(7) \cdot 10^{-6}$
$\Psi(4040)$	4.039(1)	80(10)	$^*1.07(16) \cdot 10^{-5}$
$\Psi(4160)$	4.191(5)	70(10)	$^*6.9(3.3) \cdot 10^{-6}$
$\Psi(4230)$	4.2225(24)	48(8)	$3.2(2.9) \cdot 10^{-5}$
$\Psi(4415)$	4.421(4)	62(20)	$2(1) \cdot 10^{-5}$
$\Psi(4660)$	4.630(6)	72^{+14}_{-12}	not seen

We assume uniformly distributed phases $\delta_V \in [0, 2\pi]$ and $|k_V| = 1.75(75)$.

The branching fractions

$$\mathcal{B}(x_\gamma^{\rm cut}) = \int_0^{x_\gamma^{\rm cut}} {\rm d}x_\gamma \, \frac{{\rm d}\mathcal{B}}{{\rm d}x_\gamma} \qquad x_\gamma^{\rm cut} \equiv 1 - \frac{q^2_{\rm cut}}{m^2_{B_s}}$$

- $E_\gamma^{\rm cut} = x_\gamma^{\rm cut} m_{B_s}/2$ is the upper-bound on the measured photon energy.



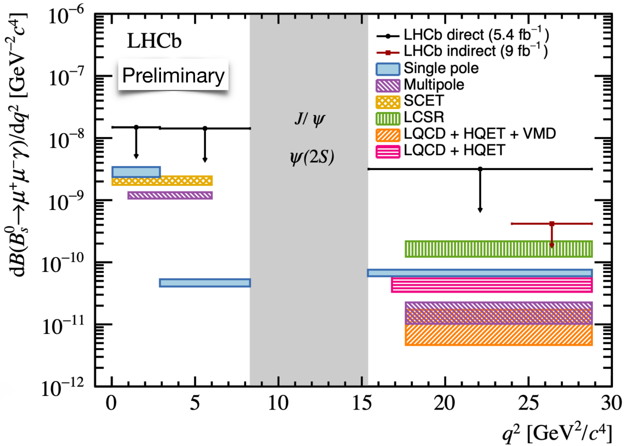
- SD contribution dominated by vector form factor F_V. Tensor form-factor contributions suppressed by small Wilson coefficient $C_7 \ll C_9, C_{10}$.
- At $x_\gamma^{\rm cut} \sim 0.4$ our estimate of charming-penguins uncertainties is around 30%.

Comparison with published LHCb upper-bound for $x_\gamma^{\rm cut} \sim 0.166$.

$$\boxed{\mathcal{B}_{\rm SD}^{\rm LHCb}(0.166) < 2 \times 10^{-9} \,, \qquad \mathcal{B}_{\rm SD}(0.166) = 6.9(9) \times 10^{-11} \qquad \text{[This work]}}$$

New preliminary results from LHCb

Taken from I. Bachiller talk at ”LA THUILE 2024”



⟸ This work

New LHCb measurement with explicit detection of the photon in the final state, gives an upper-bound, for $q^2_{\rm cut} \sim 15$ GeV^2, which is roughly one order of magnitude larger than previous bound.

Conclusions

- We have presented a first-principles lattice calculation of the form factors F_V, F_A, F_{TV}, F_{TA} entering the $\bar{B}_s \to \mu^+\mu^-\gamma$ decay, in the electroquenched approximation.
- Systematic errors have been controlled thanks to the use of gauge configurations produced by the ETM Collaboration, which correspond to four values of the lattice spacing $a \in [0.057 : 0.09]$ fm, and through the use of five different heavy-strange masses $m_{H_s} \in [m_{D_s} : 2m_{D_s}]$.
- Presently our result for the branching fractions have uncertainties ranging from $\sim 15\%$ at $\sqrt{q^2_{\rm cut}} = 4.9$ GeV to $\sim 30\%$ at $\sqrt{q^2_{\rm cut}} = 4.2$ GeV.
- At small $q^2_{\rm cut}$ uncertainty dominated by the charming-penguins which we included using a phenomenological parameterization.

Outlook:

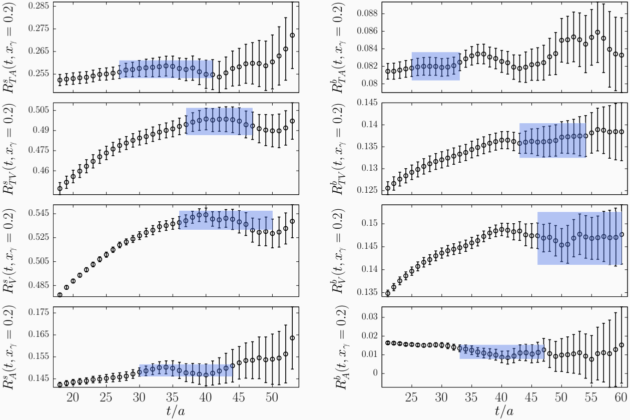
- Evaluate electro-unquenching effects.
- Evaluate charming-penguins contributions from first-principles.
- Simulate on finer lattice spacings to be able to reach higher m_{H_s} and reduce the impact of the mass-extrapolation.

Thank you for the attention!

Backup

Extraction of the form factors from lattice data

Illustrative example on the finest lattice spacing $a \sim 0.057$ fm for $x_\gamma = 0.2$ and $m_h/m_c = 2$.



- We analyze separately the two contributions corresponding to the emission of the real photon from the strange or the heavy quark.
- $x_\gamma = 2E_\gamma/m_{H_s}$ kept fixed increasing the heavy-meson mass ($E_\gamma \propto m_{H_s}$).

Heavy-quark/large energy EFT scaling relations

- Elegant scaling laws were derived in the limit of large photon energies E_γ and large m_{H_s} [Beneke et al, EPJC 2011, JHEP 2020]. Up to $\mathcal{O}(E_\gamma^{-1}, m_{H_s}^{-1})$ one has

$$\frac{F_V(x_\gamma, m_{H_s})}{f_{H_s}} = \frac{|q_s|}{x_\gamma}\left(\frac{R(E_\gamma,\mu)}{\lambda_B(\mu)} + \xi(x_\gamma, m_{H_s}) + \frac{1}{m_{H_s} x_\gamma} + \frac{|q_b|}{|q_s|}\frac{1}{m_h}\right)$$

$$\frac{F_A(x_\gamma, m_{H_s})}{f_{H_s}} = \frac{|q_s|}{x_\gamma}\left(\frac{R(E_\gamma,\mu)}{\lambda_B(\mu)} + \xi(x_\gamma, m_{H_s}) - \frac{1}{m_{H_s} x_\gamma} - \frac{|q_b|}{|q_s|}\frac{1}{m_h}\right)$$

$$\frac{F_{TV}(x_\gamma, m_{H_s}, \mu)}{f_{H_s}} = \frac{|q_s|}{x_\gamma}\left(\frac{R_T(E_\gamma,\mu)}{\lambda_B(\mu)} + \xi(x_\gamma, m_{H_s}) + \frac{1 - x_\gamma}{m_{H_s} x_\gamma} + \frac{|q_b|}{|q_s|}\frac{1}{m_{H_s}}\right)$$

$$\frac{F_{TA}(x_\gamma, m_{H_s}, \mu)}{f_{H_s}} = \frac{|q_s|}{x_\gamma}\left(\frac{R_T(E_\gamma,\mu)}{\lambda_B(\mu)} + \xi(x_\gamma, m_{H_s}) - \frac{1 - x_\gamma}{m_{H_s} x_\gamma} + \frac{|q_b|}{|q_s|}\frac{1}{m_{H_s}}\right)$$

- λ_B is 1st inverse-moment of B_s LCDA. R, R_T are radiative corrections. ξ is a power-suppressed term $\propto 1/E_\gamma, 1/m_{H_s}$, f_{H_s} the decay constant of H_s meson.
- Photon emission from b ($\propto |q_b|$) power-suppressed w.r.t. to emission from s.
- Tensor form factors are scale and scheme dependent. On the lattice we obtained them in $\overline{\mathrm{MS}}$ scheme at $\mu = 5~\mathrm{GeV}$.

The global fit Ansatz

We extrapolate to the physical B_s through a combined fit of the form factors
[$z = 1/m_{H_s}$, fit parameters are in red]:

$$\frac{F_V(x_\gamma,z)}{f_{H_s}} = \frac{|q_s|}{x_\gamma}\,\frac{1}{1+\textcolor{red}{C_V}\frac{2z^2}{x_\gamma}}\left(\textcolor{red}{K} + (1+\textcolor{red}{\delta_z})\frac{z}{x_\gamma} + \frac{1}{z^{-1}-\textcolor{red}{\Lambda_H}} + \textcolor{red}{A_m}z + \textcolor{red}{A_{x_\gamma}}\frac{z}{x_\gamma}\right)$$

$$\frac{F_A(x_\gamma,z)}{f_{H_s}} = \frac{|q_s|}{x_\gamma}\,\frac{1}{1+\textcolor{red}{C_A}\frac{2z}{x_\gamma}}\left(\textcolor{red}{K} - (1+\textcolor{red}{\delta_z})\frac{z}{x_\gamma} - \frac{1}{z^{-1}-\textcolor{red}{\Lambda_H}} + \textcolor{red}{A_m}z + (\textcolor{red}{A_{x_\gamma}} + 2\textcolor{red}{K}\textcolor{red}{C_A})\frac{z}{x_\gamma}\right)$$

$$\frac{F_{TV}(x_\gamma,z)}{f_{H_s}} = \frac{|q_s|}{x_\gamma}\,\frac{1+2\textcolor{red}{C_V}z^2}{1+\textcolor{red}{C_V}\frac{2z^2}{x_\gamma}}\left(\textcolor{red}{K_T} + (\textcolor{red}{A_m^T}+1)z + \textcolor{red}{A_{x_\gamma}^T}\frac{z}{x_\gamma} + (1+\textcolor{red}{\delta_z'})z\frac{1-x_\gamma}{x_\gamma}\right)$$

$$\frac{F_{TA}(x_\gamma,z)}{f_{H_s}} = \frac{|q_s|}{x_\gamma}\,\frac{1+2\textcolor{red}{C_A^T}z}{1+\textcolor{red}{C_A^T}\frac{2z}{x_\gamma}}\left(\textcolor{red}{K_T} + (\textcolor{red}{A_m^T}+1)z + \textcolor{red}{A_{x_\gamma}^T}\frac{z}{x_\gamma} - (1+\textcolor{red}{\delta_z'} - 2\textcolor{red}{K_T}\textcolor{red}{C_A^T})z\frac{1-x_\gamma}{x_\gamma}\right)$$

- Fit structure takes into account constraints from the scaling laws valid at large E_γ and m_{H_s}, and contains the resonance corrections (relevant at small x_γ).
- We included in the fit also NNLO $1/E_\gamma^2$, $1/m_{H_s}^2$ corrections.
- Some of the constraints appearing in the large energy/mass EFT have been relaxed as they are valid neglecting $\mathcal{O}(m_s)$ and radiative corrections to the power-suppressed terms.

Fit parameters from global fit

Pole parameters:

$$C_V = (0.57(3)\ \text{GeV})^2\ , \qquad C_A = 0.70(7)\ \text{GeV}\ , \qquad C_A^T = 0.77(4)\ \text{GeV}$$

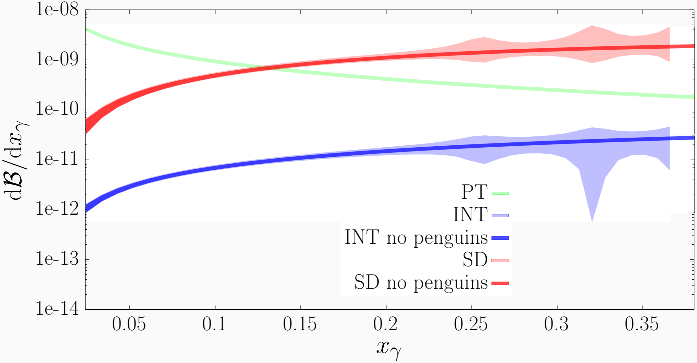
Expectations from pure VMD:

$$C_V^{\text{VMD}} = \lambda_2 \simeq (0.5\ \text{GeV})^2, \qquad C_A^{\text{VMD}} = C_A^{T,\text{VMD}} = \Lambda_1 \simeq 0.5\ \text{GeV}$$

- In vector channels, where VMD is expected to be a reasonable approximation, substantial agreement between C_V and C_V^{VMD}.
- In the axial channels, VMD does not work very well: many resonances of masses $m_{\text{res}} \sim m_{H_s} + \mathcal{O}(\Lambda_{\text{QCD}})\ldots$
- ... which is the reason why for F_A and F_{TA} two different parameters C_A, C_A^T have been introduced. C_A and C_A^T of order $\mathcal{O}(\Lambda_{QCD})$, as expected.
- For K and K_T we obtain:

$$K = 1.46(10)\ \text{GeV}^{-1}\ , \qquad K_T = 1.39(6)\ \text{GeV}^{-1}$$

The differential branching fractions



- For $x_\gamma \gtrsim 0.15$, the SD is dominant over the PT contribution.

- For $x_\gamma \gtrsim 0.2$, charming-penguin uncertainties become dominant, due to the presence of charmonium states which overlap with the $x_\gamma-$region considered.

- INT contribution is always about two orders of magnitude smaller than SD.

The $N_f = 2+1+1$ ETMC gauge ensembles

For this calculation we made use of the Wilson-Clover twisted-mass ensembles generated by the Extended Twisted Mass Collaboration (ETMC) using $N_f = 2+1+1$ active flavours

ensemble	β	V/a^4	a (fm)	$a\mu_\ell$	m_π (MeV)	L (fm)	N_g
A48	1.726	$48^3 \cdot 128$	0.09075 (54)	0.00120	174.5 (1.1)	4.36	109
B64	1.778	$64^3 \cdot 128$	0.07957 (13)	0.00072	140.2 (0.2)	5.09	400
C80	1.836	$80^3 \cdot 160$	0.06821 (13)	0.00060	136.7 (0.2)	5.46	72
D96	1.900	$96^3 \cdot 192$	0.05692 (12)	0.00054	140.8 (0.2)	5.46	100

- Iwasaki action for gluons.
- Wilson-clover twisted mass fermions at maximal twist for quarks (automatic $\mathcal{O}(a)$ improvement).
- valence quark masses m_s and m_c set imposing $M_{\eta_{ss'}} = 689.89(49)\ \text{MeV}$, $M_{\eta_c} = 2.984(4)\ \text{GeV}$.

Determination of f_{H_s}

We determined the decay constant corresponding to the five simulated values of the heavy-strange mass m_{H_s} on the same ensembles used to determine the form factors.

- f_{H_s} determined using two different estimators, which only differ by $\mathcal{O}(a^2)$ cut-off effects.
- 1st estimator: f_{H_s} determined from mesonic pseudoscalar two-point correlation function (std method). We refer to this determination as $f^{\mathrm{2pt}}_{H_s}$.
- 2nd estimator: from the zero-momentum correlation function:

$$\int \mathrm{d}^4 y \; \hat{T}\langle 0|J^i_{\mathrm{em}}(y)J^i_A(0)|\bar{H}_s(\mathbf{0})\rangle \propto f_{H_s}$$

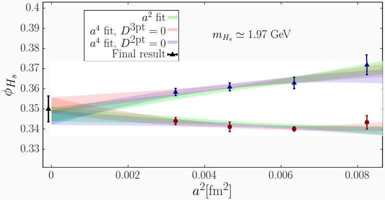
- $J^\nu_A = \bar{s}\gamma^\nu\gamma_5 h$ is the axial current. We refer to this determination as $f^{\mathrm{3pt}}_{H_s}$.

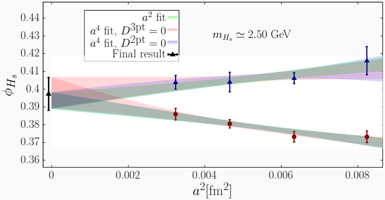
Combined continuum-extrapolation of $f^{\mathrm{2pt}}_{H_s}$ and $f^{\mathrm{3pt}}_{H_s}$ using the Ansatz:

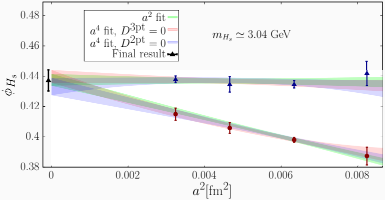
$$\phi^{\mathrm{2pt}}_{H_s} \equiv f^{\mathrm{2pt}}_{H_s}\sqrt{m_{H_s}} = A + B^{\mathrm{2pt}}a^2 + D^{\mathrm{2pt}}a^4$$

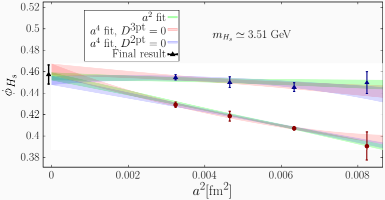
$$\phi^{\mathrm{3pt}}_{H_s} \equiv f^{\mathrm{3pt}}_{H_s}\sqrt{m_{H_s}} = A + B^{\mathrm{3pt}}a^2 + D^{\mathrm{3pt}}a^4$$

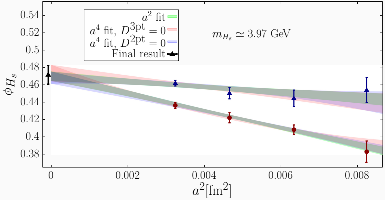
Continuum-limit extrapolation of $\phi_{H_s} = f_{H_s}\sqrt{m_{H_s}}$









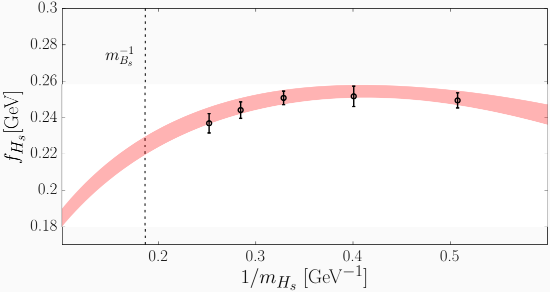


Extrapolation to the physical B_s mass

To extrapolate to the physical B_s mass, we employed the following HQET Ansatz

$$\phi(m_{H_s}) = \underbrace{C_{\gamma^0\gamma^5}(m_h, m_h)}_{\text{HQET/QCD matching}} \; \exp\underbrace{\left\{\int_0^{\alpha_s(m_h)} \frac{\gamma_{\tilde{J}}(\alpha_s)}{2\beta(\alpha_s)}\frac{d\alpha_s}{\alpha_s}\right\}}_{\text{HQET-evolutor}} \left(A + \frac{B}{m_{H_s}}\right)$$

- A and B are free fit parameters.
- m_h should be identified with the pole mass m_h^{pole} (notoriously affected by renormalon ambiguities). We used in place of the pole mass the meson mass: $m_{H_s} - m_h^{\text{pole}} \simeq \mathcal{O}(\Lambda_{\text{QCD}})$.



We obtain: $f_{B_s} = 224.5\,(5.0)$ MeV

FLAG average: 230.3 (1.3) MeV

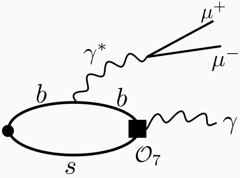
Determination of the form factor $\bar{F}_T$

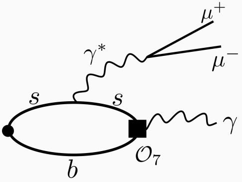
The form factor $\bar{F}_T$, is the smallest of all the form factors (and barely relevant within present accuracy). It can be computed from the knowledge of the following hadronic tensor

$$H^{\mu\nu}_{\bar{T}}(p,k) = i\int d^4x\; e^{i(p-k)x}\;\hat{\mathrm{T}}\langle 0|J^{\nu}_{\bar{T}}(0)J^{\mu}_{\mathrm{em}}(x)|\bar{B}_s(\mathbf{0})\rangle = -\varepsilon^{\mu\nu\rho\sigma}k_\rho p_\sigma\,\frac{\bar{F}_T}{m_{B_s}}$$

where (Z_T is the renormalization constant of tensor current)

$$J^{\nu}_{\bar{T}} = -iZ_T(\mu)\bar{s}\,\sigma^{\nu\rho}\,b\,\frac{k_\rho}{m_{B_s}}$$



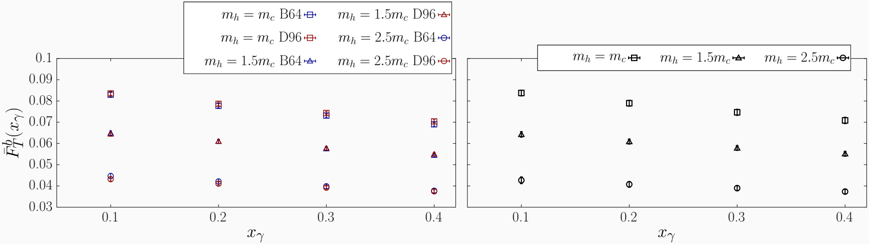


- When the virtual photon γ^* is emitted by a strange quark, the presence of $J^P = 1^-$ $s\bar{s}$ intermediate states forbid the analytic continuation of the relevant correlation functions from Minkowskian to Euclidean spacetime (where we perform MC simulations).

The $b-$quark contribution to $\bar{F}_T$

Let us start discussing the simpler contribution $\bar{F}_T^b$, due to the emission of γ^* from a b-quark.

- In this case the calculation proceeds as in the case of the other form factors F_W, $W = \{V, A, TV, TA\}$, i.e. the hadronic tensor $H^{\mu\nu}_{\bar{T}_b}$ can be directly evaluated from Euclidean spacetime simulations.
- We performed simulations for three value of the heavy-strange meson mass $m_{H_s} \in [m_{D_s} : 1.8 m_{D_s}]$ (or in terms of the heavy quark mass m_h for $m_h/m_c = 1, 1.5, 2.5$), and two values of the lattice spacings (the two gauge ensembles are called B64 and D96). Very small cut-off effects observed.



raw-data after continuum-limit extrapolation

Mass extrapolation of $\bar{F}_T^b$ (I)

The extrapolation of $\bar{F}_T^b(x_\gamma)$ to the physical mass $m_{B_s} = 5.367\ \mathrm{GeV}$ is carried out using a VMD inspired Ansatz.

- $\bar{F}_T^b$ is expected to be dominated by $J^P = 1^-\ b\bar{b}$ resonance contributions (e.g. $\Upsilon(1S)$, $\Upsilon(2S)$, $\Upsilon(3S)$, $\ldots$), which can be approximated as stable states.
- Using an unphysical heavy quark mass $m_h < m_b$ these states will be fictitious $h\bar{h}$, $J^P = 1^-$, intermediate states.
- The contribution to $\bar{F}_T^b$ of a given resonance "n" of mass m_n and electromagnetic decay constant f_n is given by

$$\bar{F}_{T,n}^b(x_\gamma) = \frac{q_b\, f_n\, m_n\, g_n^+(0)}{E_n(E_n + E_\gamma - m_{H_s})} + \text{regular terms}$$

where $E_n = \sqrt{m_n^2 + E_\gamma^2}$ and (η is the polarization of the vector resonance)

$$\langle n(-\boldsymbol{k},\eta)|\, \bar{s}\sigma^{\mu\nu} h \,|\bar{H}_s(\mathbf{0})\rangle = i\eta_\beta^* \epsilon^{\mu\nu\beta\gamma} g_n^+(p_\gamma^2)(p+q_n)_\gamma + \ldots$$

with $q_n = (E_n, -\boldsymbol{k})$, $p_\gamma = p - q_n$.

Mass extrapolation of $\bar{F}_T^b$ (II)

In the heavy-quark limit the following scaling laws hold

$$f_n \propto \frac{1}{\sqrt{m_h}} + \ldots \propto \frac{1}{\sqrt{m_{H_s}}} + \ldots \,, \qquad \frac{m_n}{m_{H_s}} = 2 + \frac{\Lambda_T^n}{m_{H_s}} + \ldots$$

- $\Lambda_T^n \simeq \mathcal{O}(\Lambda_{\mathrm{QCD}})$ and ellipses indicate NLO terms in the heavy-quark expansion.
- Using these relations $\bar{F}_{T,n}^b$ can be approximated by

$$\bar{F}_{T,n}^b(x_\gamma) = \frac{q_b}{m_{H_s}} \frac{f_n\, g_n^+(0)}{1 + \frac{x_\gamma}{2} + \frac{\Lambda_T^n}{m_{H_s}}} \left(1 + \mathcal{O}\left(x_\gamma, \frac{\Lambda_{\mathrm{QCD}}}{m_{H_s}}\right)\right)$$

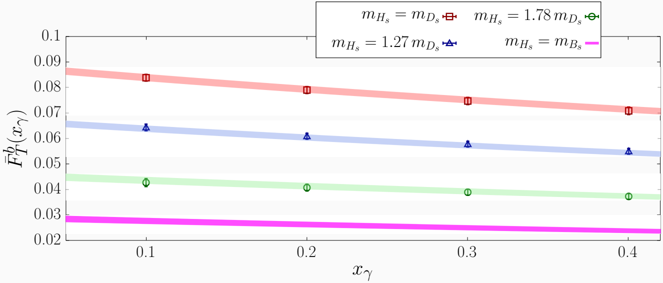
- Our strategy is to replace the tower of resonance contributions, with a single effective-pole

$$\bar{F}_T^b(x_\gamma, m_{H_s}) = \frac{1}{m_{H_s}} \frac{A + B\,x_\gamma}{1 + \frac{x_\gamma}{2} + \frac{\Lambda_T}{m_{H_s}}}$$

- A , B and Λ_T are free-fit parameters. Our Ansatz assumes $g_n^+ \propto \sqrt{m_{H_s}}$, which is consistent with our data.

Final results for $\bar{F}_T^b$

We have performed a global fit of the x_γ- and m_{H_s}-dependence of our lattice data, using the Ansatz in the previous slide.



- Our VMD-inspired Ansatz (which contains only 3 free-parameters) perfectly captures the x_γ and m_{H_s} dependence of the data.
- The magenta band corresponds to the extrapolated results at $m_{B_s} = 5.367$ GeV. Effective-pole located at $2m_{H_s} + \Lambda_T \simeq 10.4(1)$ GeV.
- As anticipated, this contribution turns out to be one order of magnitude suppressed w.r.t. F_{TV} and F_{TA}.

The strange-quark contribution $\bar{F}_T^s$

The hadronic tensor $H^{\mu\nu}_{\bar{T}_s}$ cannot be analytically continued to Euclidean spacetime

$[J^s_{\text{em}} = q_s \bar{s}\gamma^\mu s,\ \hat{H}$ is the Hamiltonian$]$

$$H^{\mu\nu}_{\bar{T}_s}(p,k) = i\int_{-\infty}^{\infty} dt\, e^{i(m_{B_s}-E_\gamma)t}\,\langle 0|J^\nu_{\bar{T}}(0)\,J^s_{\text{em}}(0,-\boldsymbol{k})|\bar{B}_s(\mathbf{0})\rangle$$

$$= \langle 0|J^\nu_{\bar{T}}(0)\frac{1}{\hat{H}-E_\gamma-i\varepsilon}J^{s,\mu}_{\text{em}}(0,-\boldsymbol{k})|\bar{B}_s(0)\rangle$$

$$+ \langle 0|J^{s,\mu}_{\text{em}}(0,-\boldsymbol{k})\frac{1}{\hat{H}+E_\gamma-m_{B_s}-i\varepsilon}J^\nu_{\bar{T}}(0)|\bar{B}_s(0)\rangle = H^{\mu\nu}_{\bar{T}_s,1}(p,k) + H^{\mu\nu}_{\bar{T}_s,2}(p,k)$$

- Analytic continuation $t \to -it$ possible only if the following positivity-conditions are met

$$\langle n|\hat{H}-E_\gamma|n\rangle > 0, \qquad \langle n|\hat{H}+E_\gamma-m_{B_s}|n\rangle > 0$$

- $|n\rangle$ is any of the intermediate-states that can propagate between the electromagnetic and tensor currents.
- The second condition is equivalent to $q^2 < m_n^2$ (m_n is the rest-energy of the intermediate state $|n\rangle$)...
- ...which is violated because the smallest m_n here is $2m_K$. In the case of the $b-$quark this is instead m_Υ. The first condition is instead always satisfied.

The spectral-density representation

The main idea for circumventing the problem of analytic continuation is to consider the spectral-density representation of the hadronic tensor $[E = m_{B_s} - E_\gamma]$

$$H^{\mu\nu}_{\bar{T}_s,2}(E, \boldsymbol{k}) = \lim_{\varepsilon \to 0^+} \int_{E^*}^{\infty} \frac{dE'}{2\pi} \frac{\rho^{\mu\nu}(E', \boldsymbol{k})}{E' - E - i\varepsilon} = \mathrm{PV} \int_{E^*}^{\infty} \frac{dE'}{2\pi} \frac{\rho^{\mu\nu}(E', \boldsymbol{k})}{E' - E} + \frac{i}{2} \rho^{\mu\nu}(E, \boldsymbol{k})$$

- The spectral-density $\rho^{\mu\nu}$ is related to the Euclidean correlation function $C^{\mu\nu}(t, \boldsymbol{k})$, which we can directly compute on the lattice, via

$$\underbrace{C^{\mu\nu}(t, \boldsymbol{k})}_{\text{lattice input}} = \int_{E^*}^{\infty} \frac{dE'}{2\pi} e^{-E' t} \rho^{\mu\nu}(E', \boldsymbol{k})$$

- Unfortunately, determining $\rho^{\mu\nu}$ from $C^{\mu\nu}(t, \boldsymbol{k})$, which is computed on the lattice at a discrete set of times and with a finite accuracy, is not possible (inverse Laplace transform problem).

- The regularized quantity that we can evaluate, exploiting the Hansen-Lupo-Tantalo method [PRD 99 '19], is a smeared version of the hadronic tensor, obtained by considering non-zero values of the Feynman's ε

$$H^{\mu\nu}_{\bar{T}_s,2}(E, \boldsymbol{k}; \varepsilon) = \int_{E^*}^{\infty} \frac{dE'}{2\pi} \frac{\rho^{\mu\nu}(E', \boldsymbol{k})}{E' - E - i\varepsilon}$$

The smeared form factor

The evaluation of the hadronic tensor at finite ε leads to a smeared form factor $\bar{F}_T^s(x_\gamma;\varepsilon)$. In the limit of vanishing ε one has

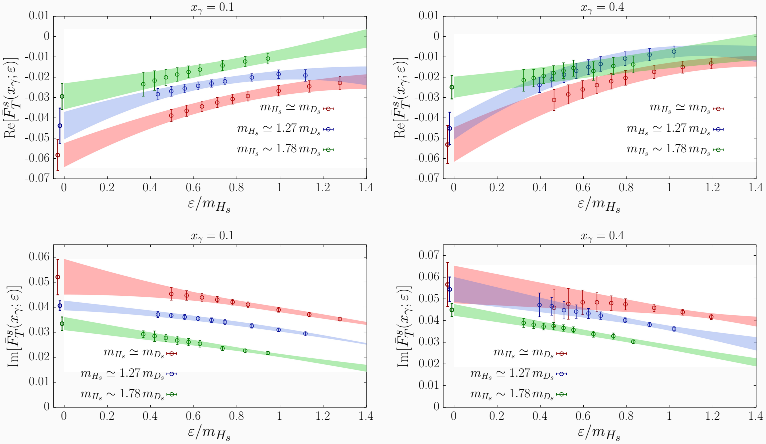
$$\lim_{\varepsilon\to0^+} \bar{F}_T^s(x_\gamma;\varepsilon) = \bar{F}_T^s(x_\gamma)$$

- As we have shown in [Frezzotti et al. PRD 108 '23], the corrections to the vanishing ε limit are of the form

$$\bar{F}_T^s(x_\gamma;\varepsilon) = \bar{F}_T^s(x_\gamma) + A_1\,\varepsilon + A_2\varepsilon^2 + \mathcal{O}(\varepsilon^3)$$

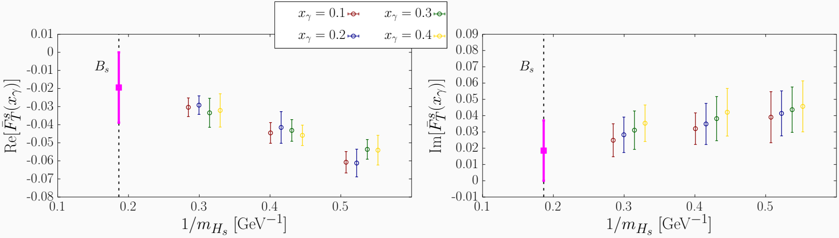
- The onset of the polynomial regime depends on the typical size $\Delta(E)$ of the interval around E on which the hadronic tensor is significantly varying, and one needs $\varepsilon \ll \Delta(E)$.
- We evaluated $\bar{F}_T(x_\gamma;\varepsilon)$ for several values of $\varepsilon/m_{H_s} \in [0.4:1.3]$, and then performed a polynomial extrapolation in ε.

The vanishing-ε extrapolation



Both the real and imaginary part of the smearead form factor $\bar{F}^s_T(x_\gamma;\varepsilon)$ show an almost linear behaviour at small ε. Besides the polynomial extrapolations, we have performed additional model-dependent, non-polynomial, extrapolations, to have a conservative estimate of the possible systematics associated to the vanishing-ε limit.

$\bar{F}_T^s$ at the physical mass $m_{B_s} \simeq 5.367\ \mathrm{GeV}$



- Very small x_γ dependence observed.
- To have a conservative error estimate, we take the results at the largest simulated mass $m_{H_s} \simeq 1.78\, m_{D_s}$ as a bound for the value of the form factor at the physical point, $m_{H_s} = m_{B_s}$.

From the form factors to the branching fractions

The differential branching fraction for $\bar{B}_s \to \mu^+\mu^-\gamma$ can be decomposed as a sum of three terms

$$\frac{\mathrm{d}\mathcal{B}}{dx_\gamma} = \frac{\mathrm{d}\mathcal{B}_{\mathrm{PT}}}{dx_\gamma} + \frac{\mathrm{d}\mathcal{B}_{\mathrm{INT}}}{dx_\gamma} + \frac{\mathrm{d}\mathcal{B}_{\mathrm{SD}}}{dx_\gamma} \qquad \left[q^2 = m_{B_s}^2(1 - x_\gamma)\right]$$

- $\mathrm{d}\mathcal{B}_{\mathrm{PT}}/\mathrm{dx}_\gamma$ is the point-like contribution ($\propto f_{B_s}^2$).
- It suffers from an IR-divergence ($\mathrm{d}\mathcal{B}/\mathrm{dx}_\gamma \propto 1/\mathrm{x}_\gamma$ at small x_γ), which is then cancelled by the virtual-photon correction to $\bar{B}_s \to \mu^+\mu^-$ through the Block-Nordsieck mechanism.
- $\mathrm{d}\mathcal{B}_{\mathrm{INT}}/\mathrm{dx}_\gamma$ is the interference contribution and depends linearly on the form factors.
- $\mathrm{d}\mathcal{B}_{\mathrm{SD}}/\mathrm{dx}_\gamma$ is the structure-dependent contribution and is quadratic in the form factors.

Both the interference and structure-dependent contributions are infrared finite.