# Relativistic Spin Hydrodynamics from Kinetic Theory

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# Section Outline :


Spin Polarization in Heavy Ion Collisions and Problem - I

Relativistic Hydrodynamics and Problem - II:

Relativistic Spin-hydrodynamics :

Relativistic Kinetic Theory with Spin:

Relativistic Spin Hydrodynamics with ERTA :

Relativistic Spin hydrodynamics with NRTA :

Summary and Outlook :

# Particle Polarization :

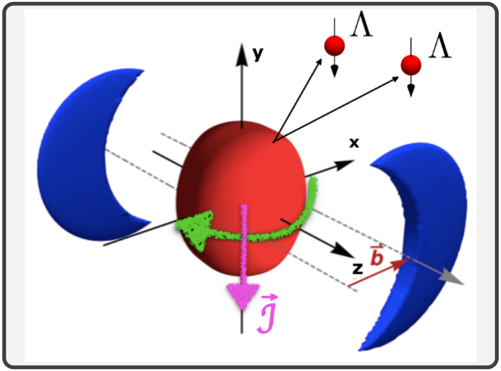
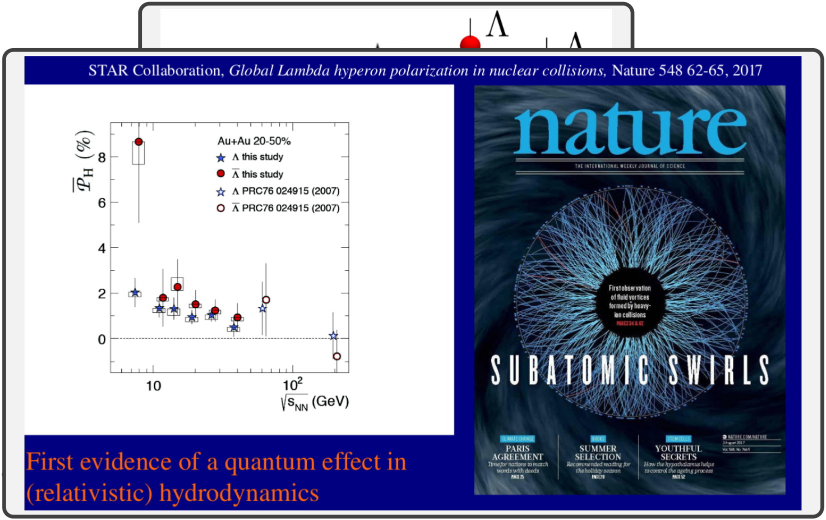



Figure 1: Origin of particle polarization. [W. Florkowski *et al*, PPNP 108 (2019) 103709]

- Large angular momentum → Local vorticities → spin alignment.

  [Z.-T. Liang and X.-N. Wang, Phys. Rev. Lett. 94, 102301 (2005); Phys. Lett. B 629, 20 (2005)]

# Particle Polarization :




Experimental evidence, [STAR Collaboration, Nature 548, 62 (2017), Phys. Rev. Lett. 123, 132301 (2019), Phys. Rev. Lett. 126, 162301 (2021)]

Theoretical models assuming equilibration of spin d.o.f. explains the data.

# Particle Polarization :

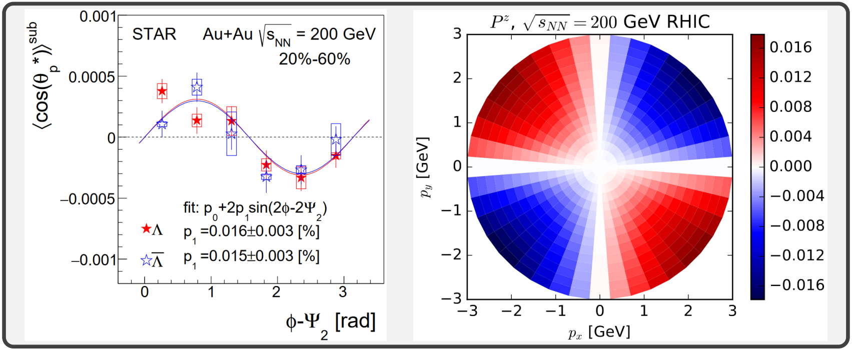



Figure 2: Observation (L) and prediction (R) of longitudinal polarization.

[Left: Phys. Rev. Lett. **123** 132301 (2019); Right: Phys. Rev. Lett. **120** 012302 (2018)]

- Inclusion of shear-induced polarization (SIP) solves the problem with extra constraints.
  [Fu et. al. Phys. Rev. Lett. 127, 142301 (2021); Becattini et. al. Phys. Lett. B 820 136519 (2021)]
- Still the resolution remains ambiguous.
  [Florkowski et. al., Phys. Rev. C 100, 054907 (2019); Phys.Rev.C 105, 064901 (2022)]
- Do dissipative forces play any role and solve the problem?

## Inclusion of Dissipation :

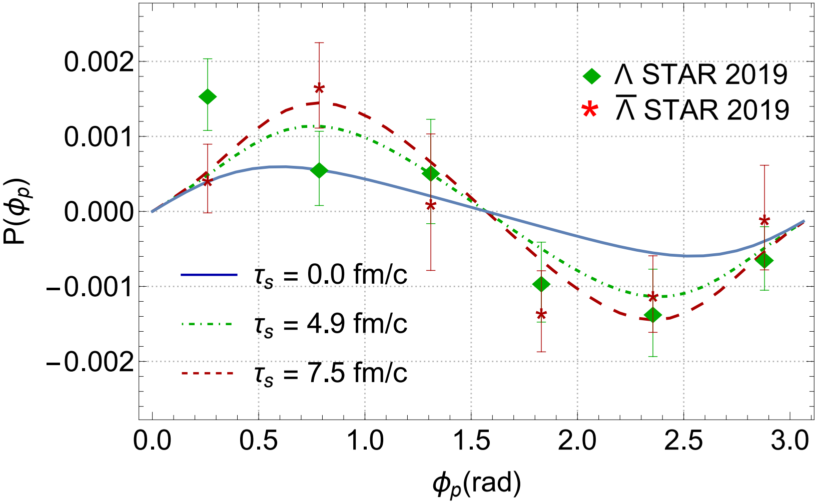



Figure 3: Polarization. $\tau_s = 7.5$ fm for $\overline{\Lambda}$ ($\chi^2_r = 0.6$) and, $\tau_s = 4.9$ fm for $\Lambda$ ($\chi^2_r = 1.5$).

- $\tau_s$ is in agreement with [Hidaka et. al., arXiv: 2312.08266, Wagnar et. al., arXiv: 2405.00533].
- We had to assume, $\varpi_{0i} \to 0$. [S. Banerjee *et. al.*, arXiv:2405.05089]

# Summary of the Problem I:

The first problem we wish to address is :

- Study the effects of spin-dependent relaxation time on spin-polrization.
  - $\rightarrow$ Formulate Dissipative Spin-hydrodynamics with extended RTA (ERTA).

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Summary and Outlook :

# Relativistic Hydrodynamics :

- The conserved currents of hydrodynamics are,

$$N^\mu = n_0 u^\mu + n^\mu, \qquad T^{\mu\nu} = \mathcal{E}_0 u^\mu u^\nu - (\mathcal{P} + \Pi)\,\Delta^{\mu\nu} + \pi^{\mu\nu}$$

  where, $u_\mu n^\mu = 0$, $u_\mu \pi^{\mu\nu} = 0$ and, $\pi^\mu_\mu = 0$.

- We have chosen the Landau frame $u_\mu T^{\mu\nu} = \mathcal{E} u^\nu$ and Landau matching conditions $\mathcal{E} = \mathcal{E}_0, n = n_0$.

- The number of unknown variables are, 4 + 8 + 3 = 15.

- However, the number of conservation laws are 5.

- So, apart from EoS, 9 more equations needed to close the system of equations.

# Relativistic Hydrodynamics :

- The conservation laws lead to the following dissipative hydro equations,

$$\dot{\mathcal{E}}_0 + (\mathcal{E}_0 + \mathcal{P} + \Pi)\,\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0$$
$$(\mathcal{E}_0 + \mathcal{P} + \Pi)\,\dot{u}^\alpha - \nabla^\alpha(\mathcal{P} + \Pi) + \Delta^\alpha_\nu \partial_\mu \pi^{\mu\nu} = 0$$
$$\dot{n}_0 + n_0\theta + \partial_\mu n^\mu = 0$$

  where, $\sigma^{\mu\nu} = (\nabla^\mu u^\nu + \nabla^\nu u^\mu)/2 - \Delta^{\mu\nu}\theta/3$ is the shear stress tensor.

- These equations are exact up to all order in gradients.

- Next we incorporate the order-by-order gradient corrections :

$$N^\mu = N^\mu_{(0)} + N^\mu_{(1)} + N^\mu_{(2)} + \cdots$$
$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + T^{\mu\nu}_{(2)} + \cdots$$

# Dissipative Relativistic Hydrodynamics up to $\mathcal{O}(\partial)$ :

- Truncating terms up to first order in spacetime gradients, we get the Navier-Stokes equations within Landau-Lifshitz frame and matching conditions as,

$$\pi^{\mu\nu} = 2\,\eta\,\sigma_{\mu\nu},$$
$$\Pi = -\zeta\,\theta,$$
$$n^{\mu} = \kappa\,(\nabla^{\mu}\xi)\,.$$

where, $\xi = \mu/T$. $\eta, \zeta$ and $\kappa$ are the $\mathcal{O}(\partial)$ transport coefficients.

- The details of the transport coefficients can only be obtained from a microscopic theory.

# Causality of $\mathcal{O}(\partial)$ Relativistic Hydrodynamics :

- To study the causality, perturb the hydrodynamic fields:

$$\mathcal{E} \to \mathcal{E}_0 + \delta\mathcal{E},$$
$$u^\mu \to u^\mu_0 + \delta u^\mu,$$

- We may assume a solution of the form $\to A = \widetilde{A}\, e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$.

- This leads to dispersion relation:

$$\omega = \left(\frac{\eta_0}{\mathcal{E}_0 + \mathcal{P}_0}\right) k^2.$$

- Propagation speed of the perturbation:

$$v_T^{\max} = \lim_{k\to\infty} \frac{d\omega}{dk} \to \infty$$

[P. Romatschke, IJMPE 19 (2010) 1-53]

# Dissipative Relativistic Hydrodynamics up to $\mathcal{O}(\partial^2)$ :

- Truncating terms up to second order in spacetime gradients, we get the evolution equations of the dissipative currents as,

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\,\beta_\pi\,\sigma_{\mu\nu} + \lambda_1 \pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} + \lambda_2 \pi^{\mu\nu}\theta + \lambda_3 \pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} + \lambda_4 \Pi\sigma^{\mu\nu},$$

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = \beta_\Pi\,\sigma_{\mu\nu} + \delta_1 \Pi\theta + \delta_2 \pi^{\mu\nu}\sigma_{\mu\nu},$$

$$\dot{n}^\mu + \frac{n^\mu}{\tau_n} = \beta_n\,(\nabla^\mu\xi) + \psi_1 n_\nu \omega^{\nu\mu} + \psi_2 n^\mu\theta + \psi_3 n_\nu \sigma_{\nu\mu} + \psi_4 \pi^{\mu\nu}\,(\nabla_\nu\xi).$$

- The dissipative currents can no longer be completely determined from other hydrodynamic variables and have to be promoted to independent variables.

- Higher order evolution equations can also be obtained. However, to completely specify the theory a microscopic theory is required $\rightarrow$ Kinetic Theory.

# Causality of $\mathcal{O}(\partial^2)$ Relativistic Hydrodynamics :

- For a chargeless, conformal system, the dispersion relation becomes:

$$\omega = \left(\frac{\eta_0}{\mathcal{E}_0 + \mathcal{P}_0}\right) \frac{k^2}{(1 - \omega\tau_\pi)}.$$

- Propagation speed of the perturbation:

$$v_T^{\max} = \lim_{k\to\infty} \frac{d\omega}{dk} \to \sqrt{\frac{1}{\tau_\pi}\left(\frac{\eta_0}{\mathcal{E}_0 + \mathcal{P}_0}\right)}$$

- For $\tau_\pi > \eta_0/\left(\mathcal{E}_0 + \mathcal{P}_0\right)$, the system is causal $\implies$ MIS theories can be causal.
  [P. Romatschke, IJMPE 19 (2010) 1-53]

- More sophisticated methods of causality analysis exist.
  [M. P. Heller, PRL 130 (2023), 261601, L. Gavassino, PRL 132 (2024), 162301]

- The price to pay is to include non-thermodynamical variables.

# BDNK Theory and the Origin of Acausality :

- Under BDNK approach, the constitutive relations are:

$$\delta\mathcal{E} = \varepsilon_1\left(\dot{T}/T\right) + \varepsilon_2\,\theta + \varepsilon_3\,\dot{\xi} + \mathcal{O}\left(\partial^2\right)$$
$$\delta\mathcal{P} = \pi_1\left(\dot{T}/T\right) + \pi_2\,\theta + \pi_3\,\dot{\xi} + \mathcal{O}\left(\partial^2\right)$$
$$\delta n = \nu_1\left(\dot{T}/T\right) + \nu_2\,\theta + \nu_3\,\dot{\xi} + \mathcal{O}\left(\partial^2\right)$$
$$h^\mu = \theta_1\,\dot{u}^\mu + (\theta_2/T)\left(\nabla^\mu T\right) + \theta_3\left(\nabla^\mu \xi\right) + \mathcal{O}\left(\partial^2\right)$$
$$n^\mu = \gamma_1\,\dot{u}^\mu + (\gamma_2/T)\left(\nabla^\mu T\right) + \gamma_3\left(\nabla^\mu \xi\right) + \mathcal{O}\left(\partial^2\right)$$
$$\pi^{\mu\nu} = 2\eta\,\sigma^{\mu\nu} + \mathcal{O}\left(\partial^2\right).$$

[P. Kovtun, JHEP 10 (2019) 034, K. Jensen *et. al.* PRL 109 (2012), 101601]

# BDNK Theory and the Origin of Acausality :

- The replacement $D \to \nabla \implies$ Non-hyperbolic equations.
- BDNK resolves this from the macroscopic point of view.
- But microscopic theories like <u>*RTA*</u> requires $D \to \nabla$ for conservation laws.
- Novel RTA provides the appropriate framework in this regard.
  [G. S. Rocha *et.al.*, PRL 127 (2021) 4, 042301, PRD 106 (2022) 3, 036022]
- No such theory exists for spin-hydrodynamics.

# Summary of the Problem II :

The second problem we wish to address is :

- Construct a kinetic theory that is compatible with BDNK approach.
  - $\rightarrow$ Formulate Dissipative Spin-hydrodynamics with Novel RTA (NRTA).

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Summary and Outlook :

# Relativistic Spin-hydrodynamics :

- We first note that spin-polarization originates from the rotation of fluid.

- Hence, we will have to deal with three conserved currents :

$$\partial_\mu N^\mu = 0, \qquad \partial_\mu T^{\mu\nu} = 0, \qquad \partial_\lambda J^{\lambda,\mu\nu} = 0$$

where, $J = L + S$. Also, $L^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}$.

- For symmetric $T^{\mu\nu}$ we have, $\boxed{\partial_\lambda S^{\lambda,\mu\nu} = 0}$

$$N^\mu = N^\mu_{\text{eq}} + \delta N^\mu, \qquad T^{\mu\nu} = T^{\mu\nu}_{\text{eq}} + \delta T^{\mu\nu}, \qquad S^{\lambda,\mu\nu} = S^{\lambda,\mu\nu}_{\text{eq}} + \delta S^{\lambda,\mu\nu}$$

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- The dissipative parts require microscopic description $\rightarrow$ Kinetic Theory.

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## Kinetic Theory with Spin :

- To import spin in kinetic theory (KT), we start from the Wigner function ($\mathcal{W}_{\alpha\beta}$), that bridges the gap between QFT and KT.

$$\mathcal{W}_{\alpha\beta}(x,k) = \int d^4y e^{-ik\cdot y} \left\langle : \bar{\psi}_\beta(x_1)\psi_\alpha(x_2) \right\rangle :$$

- For spin-1/2 particles we set up kinetic equation of $\mathcal{W}_{\alpha\beta}$ using Dirac equation,

$$\left[\gamma\cdot\left(p+\frac{i}{2}\partial\right)-m\right]\mathcal{W}_{\alpha\beta} = \mathcal{C}\left[\mathcal{W}_{\alpha\beta}\right]$$

[Xin-Li Sheng, *PhD Thesis (2019)*, N. Weickgenannt et al, *PRL 127 (2021) 5, 052301*, *PRD 100, 056018 (2019)*.]

- The Wigner function can be decomposed as,

$$\mathcal{W}_{\alpha\beta} = \frac{1}{4}\left(\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\Sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right)_{\alpha\beta}$$

$\mathcal{F} \rightarrow$ scalar component,

$\mathcal{P} \rightarrow$ pseudoscalar component,

$\mathcal{V}_\mu \rightarrow$ vector component,

$\mathcal{A}_\mu \rightarrow$ axial vector component,

$\mathcal{S}_{\mu\nu} \rightarrow$ tensor component.

where, the $\gamma$-matrices are the $4\times 4$ Dirac $\gamma$-matrices and, $\Sigma^{\mu\nu} = i\gamma^{[\mu}\gamma^{\nu]}$.

# Kinetic Theory with Spin :

- For spin-hydrodynamics it suffices to consider only $\mathcal{F}$ and $\mathcal{A}_\mu$ components.

  [Xin-Li Sheng, *PhD Thesis (2019)*]

| | ***Scalar Component*** | ***Axial Component*** |
|---|---|---|
| Kin. Eq. | $k^\mu \partial_\mu \mathcal{F}(x,k) = \mathcal{C}_\mathcal{F}$ | $k^\mu \partial_\mu \mathcal{A}^\nu(x,k) = \mathcal{C}^\nu_\mathcal{A}$ |
| RTA | $\mathcal{C}_\mathcal{F} = \frac{(k\cdot u)}{\tau_{\rm eq}}\left[\mathcal{F}_{\rm eq}(x,k) - \mathcal{F}(x,k)\right]$ | $\mathcal{C}^\nu_\mathcal{A} = \frac{(k\cdot u)}{\tau_{\rm eq}}\left[\mathcal{A}^\nu_{\rm eq}(x,k) - \mathcal{A}^\nu(x,k)\right]$ |
| Dist. fn. | $\mathcal{F}^\pm(x,k) = 2m\int_{p,s} f^\pm(x,p,s)\ \delta^{(4)}(k\mp p)$ | $\mathcal{A}^\mu_\pm(x,k) = 2m\int_{p,s} s^\mu f^\pm(x,p,s)\ \delta^{(4)}(k\mp p)$ |

[S.B., W. Florkowski, A. Jaiswal, A. Kumar and, R. Ryblewski, *PLB 814 (2021) 136096*, *PRD 103 (2021) 1, 014030*]

Momentum measure $\rightarrow \quad \int_p (\cdots) \rightarrow \int \mathrm{dP}(\cdots), \quad \int \mathrm{dP} = d^3p/\,(2\pi)^3\, p^0.$

Spin measure $\rightarrow \quad \int_s (\cdots) \rightarrow \int \mathrm{dS}(\cdots), \quad \int \mathrm{dS} = (m/\pi\mathfrak{s}) \int d^4s\,\delta(s\cdot s + \mathfrak{s}^2).$

# Relativistic Kinetic Equation :

- We take the equilibrium (extended) phase-space distribution function to be :

$$f^{\pm}_{\rm eq}(x,p,s) \equiv f^{\pm}_{0,s} = e^{-\beta(u\cdot p)\pm\xi}\left(1+\frac{1}{2}\omega_{\mu\nu}s^{\mu\nu}\right)+\mathcal{O}(\omega^2)$$

[F. Becatinni et al., *Annals Phys. 338 (2013) 32-49*, W. Florkowski et al., *PRD 97 (2018) 11, 116017*]

- Near local equilibrium $f(x,p,s)$ is expanded using Chapman-Enskog :

$$f^{\pm}(x,p,s) = f^{\pm}_{\rm eq}(x,p,s) + \delta f^{\pm}(x,p,s).$$

[de Groot, van Leewan, van Weert, *'Relativistic Kinetic Theory - Principle and Applications (1980)'.*]

- The conserved currents are expressed in kinetic theory as,

$$\boxed{N^{\mu}=\int_{p,s} p^{\mu}\left(f^{+}-f^{-}\right); \qquad T^{\mu\nu}=\int_{p,s} p^{\mu}p^{\nu}\left(f^{+}+f^{-}\right); \qquad S^{\lambda,\mu\nu}=\int_{p,s} p^{\lambda}s^{\mu\nu}\left(f^{+}+f^{-}\right)}$$

- Under RTA, conservation laws require, $\displaystyle\int_{p,s}(u\cdot p)\,\delta f/\tau_{\rm R} = 0$,

$\displaystyle\int_{p,s} p^{\nu}\,(u\cdot p)\,\delta f/\tau_{\rm R} = 0$ and, $\displaystyle\int_{p,s}(u\cdot p)\,s^{\mu\nu}\delta f/\tau_{\rm R} = 0.$

# Dissipative Currents in Spin-hydrodynamics:

- The dissipative quantities are defined as,

$$
\begin{aligned}
n^\mu &= \Delta^\mu_\alpha \int dP \int dS\, p^\alpha \left(\delta f^+ - \delta f^-\right) \\
\Pi &= -\frac{\Delta_{\alpha\beta}}{3} \int dP \int dS\, p^\alpha p^\beta \left(\delta f^+ + \delta f^-\right) \\
\pi^{\mu\nu} &= \Delta^{\mu\nu}_{\alpha\beta} \int dP \int dS\, p^\alpha p^\beta \left(\delta f^+ + \delta f^-\right) \\
\delta S^{\lambda,\mu\nu} &= \int dP \int dS\, p^\lambda s^{\mu\nu} \left(\delta f^+ + \delta f^-\right)
\end{aligned}
$$

where, $\Delta^{\mu\nu}_{\alpha\beta} = (1/2)(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\nu_\beta \Delta^\mu_\alpha) - (1/3)\Delta^{\mu\nu}\Delta_{\alpha\beta}$ is a traceless symmetric projection operator.

# Dissipative Currents in Spin-hydrodynamics with RTA:

- The non-equilibrium parts give the transport coefficients:

$$
\begin{aligned}
\delta N^{\mu} &= \tau_{\rm eq}\beta_n(\nabla^{\mu}\xi),\\
\delta T^{\mu\nu} &= \tau_{\rm eq}\big[-\beta_{\Pi}\,\Delta^{\mu\nu}\,\theta+2\,\beta_{\pi}\,\sigma^{\mu\nu}\big],\\
\delta S^{\lambda,\mu\nu} &= \tau_{\rm eq}\Big[B_{\Pi}^{\lambda,\mu\nu}\theta+B_{n}^{\phi\lambda,\mu\nu}(\nabla_{\phi}\xi)+B_{\pi}^{\alpha\beta\lambda,\mu\nu}\sigma_{\alpha\beta}+B_{\Sigma}^{\rho\gamma\phi\lambda,\mu\nu}(\nabla_{\rho}\omega_{\gamma\phi})\Big]
\end{aligned}
$$

- By choosing the Landau frame and matching conditions we found the following relations:

$$
\dot{\xi}=\xi_{\theta}\,\theta,\qquad \dot{\beta}=\beta_{\theta}\,\theta,\qquad \beta\dot{u}_{\mu}=-\nabla_{\mu}\beta+\frac{n_0\,\tanh\xi}{(\mathcal{E}+\mathcal{P})}(\nabla_{\mu}\xi)
$$

$$
\dot{\omega}^{\mu\nu}=\mathcal{D}_{\Pi}^{\mu\nu}\theta+\mathcal{D}_{\rm n}^{\mu\nu\alpha}(\nabla_{\alpha}\xi)+\mathcal{D}_{\pi}^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}+\mathcal{D}_{\Sigma}^{\lambda\mu\nu\alpha\beta\gamma}(\nabla_{\alpha}\omega_{\beta\gamma}),
$$

[S.B., W. Florkowski, A. Jaiswal, A. Kumar and, R. Ryblewski, *PLB 814 (2021) 136096*, *PRD 103, 014030 (2021)*]

- But such first-order theory is not causal and we had to assume $\tau_{\rm R}=\tau_{\rm eq}(x)$.

[A. Daher *et. al.*, *PRD 107 (2023) 5, 054043*]

# Boltzmann Equation

- Spacetime Evolution

$$p^\mu \partial_\mu f + F^\mu \partial_\mu^{(p)} f = C[f]$$

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◦ Gravitational Forces : $F^\mu = -\Gamma^\mu_{\alpha\beta} p^\alpha p^\beta$.

◦ Electromagnetic Forces : $F^\mu = qF^{\mu\nu} p_\nu$.

◦ Mean-Field Forces : $F^\mu = M\,(\partial^\mu M)$.

# Boltzmann Equation

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  - Gravitational Forces : $F^\mu = -\Gamma^\mu_{\alpha\beta} p^\alpha p^\beta .$
  - Electromagnetic Forces : $F^\mu = q F^{\mu\nu} p_\nu .$
  - Mean-Field Forces : $F^\mu = M \, (\partial^\mu M).$

- Global Equilibrium solution:

$$f_0 = (\exp g + r)^{-1}$$

where, $r = 0, \pm 1$ and $g \equiv \sum_n \alpha_n \phi_n$

- Under local equilibrium, $\alpha_n \to \alpha_n(x^\mu)$

# The Collision Kernel

- For $2 \leftrightarrow 2$ collisions:

$$C[f] = \int dP\, dP'\, dK'\, \underbrace{\mathcal{W}_{\mathbf{k}\mathbf{k}' \leftrightarrow \mathbf{p}\mathbf{p}'}} \times \left( f_{\mathbf{p}} f_{\mathbf{p}'} - f_{\mathbf{k}} f_{\mathbf{k}'} \right)$$

Transition Amplitude:

- For $2 \leftrightarrow 2$ collisions:

$$C[f] = \int dP\, dP'\, dK'\, \underbrace{\mathcal{W}_{\mathbf{kk}' \leftrightarrow \mathbf{pp}'}} \times \left(f_{\mathbf{p}} f_{\mathbf{p}'} - f_{\mathbf{k}} f_{\mathbf{k}'}\right)$$

Transition Amplitude:

- Chapman-Enskog Expansion:

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}} = f_{0\mathbf{k}} \left(1 + \phi_{\mathbf{k}}\right)$$

# The Collision Kernel

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Transition Amplitude: ↗

- Chapman-Enskog Expansion:

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}} = f_{0\mathbf{k}} \left(1 + \phi_{\mathbf{k}}\right)$$

- Linearized Collision Operator :

$$C[f] \to \hat{L}\phi_{\mathbf{k}} = \int dP\, dP'\, dK'\, \mathcal{W}_{\mathbf{k}\mathbf{k}' \leftrightarrow \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \times \left(\phi_{\mathbf{p}} + \phi_{\mathbf{p}'} - \phi_{\mathbf{k}} - \phi_{\mathbf{k}'}\right)$$

Collisional Invariants: ↗

[S. R. de Groot *et. al.*, Relativistic Kinetic Theory, C. Cercignani *et. al.*, The Relativistic Boltzmann Equation]

# The Conservation Laws

- Collisional invariants remain conserved during collisions.

- Each collisional invariant correspond to a conservation law.

- For a non-rotating, unpolarizable fluid :
  - $\phi \equiv 1 \longrightarrow$ Number Conservation.
  - $\phi \equiv E_{\mathbf{k}} \longrightarrow$ Energy Conservation.
  - $\phi \equiv \vec{k}\ (\sim k^{\langle\mu\rangle}) \longrightarrow$ Linear Momentum Conservation.

- Thus, a collision kernel should satisfy:

$$\hat{L}\ 1 = 0, \qquad \hat{L}\ E_{\mathbf{k}} = 0, \qquad \hat{L}\ k^{\langle\mu\rangle} = 0.$$

- The linearized collision kernel satisfies the property:

$$\int dK\, \psi_{\mathbf{k}} \hat{L} \phi_{\mathbf{k}} = \int dK\, \phi_{\mathbf{k}} \hat{L} \psi_{\mathbf{k}}$$

$$\Longrightarrow \int dK\, \hat{L}\phi_{\mathbf{k}} = 0, \qquad \int dK\, k^{\mu} \hat{L}\phi_{\mathbf{k}} = \int dK \left(u^{\mu} E_{\mathbf{k}} + k^{\langle\mu\rangle}\right) \hat{L}\phi_{\mathbf{k}} = 0.$$

[S. R. de Groot *et. al.*, Relativistic Kinetic Theory, C. Cercignani *et. al.*, The Relativistic Boltzmann Equation]

# New Collision Kernels

- We will work with two types linearized collision kernels.
  1. Extended Relaxation Time Approximation (ERTA):

$$\boxed{\hat{L}_{\rm ERTA}\,\phi_{\mathbf{k}} = -\frac{E_{\mathbf{k}}}{\tau_{\rm R}}\left(\phi_{\mathbf{k}} - \phi^{*}_{\mathbf{k}}\right) f_{0\mathbf{k}}}$$

where,

$$\phi^{*}_{1,\mathbf{k}} = -\frac{(k\cdot\delta u)}{T} + \frac{(E_{\mathbf{k}}-\mu)\,\delta T}{T^{2}} + \frac{\delta\mu}{T}$$

$$\delta u_{\mu} = u^{*}_{\mu} - u_{\mu}\,, \qquad \delta T = T^{*} - T, \qquad \delta\mu = \mu^{*} - \mu.$$

[D. Dash *et. al.*, PLB 831 (2022) 137202]

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$$\boxed{\hat{L}_{\text{ERTA}}\,\phi_{\mathbf{k}} = -\frac{E_{\mathbf{k}}}{\tau_{\text{R}}}\left(\phi_{\mathbf{k}} - \phi^{*}_{\mathbf{k}}\right) f_{0\mathbf{k}}}$$

where,

$$\phi^{*}_{1,\mathbf{k}} = -\frac{(k\cdot\delta u)}{T} + \frac{(E_{\mathbf{k}}-\mu)\,\delta T}{T^2} + \frac{\delta\mu}{T}$$

$$\delta u_\mu = u^{*}_{\mu} - u_\mu, \qquad \delta T = T^{*} - T, \qquad \delta\mu = \mu^{*} - \mu.$$

[D. Dash *et. al.*, PLB 831 (2022) 137202]

  2. Novel Relaxation Time Approximation (NRTA):

$$\boxed{\hat{L}_{\text{NRTA}} \sim \left(-\mathbb{1} + \sum_{n=1}^{5} |\lambda_n\rangle\langle\lambda_n|\right)}$$

where, $|\lambda_n\rangle$ are degenerate, orthogonal eigenvectors of $\hat{L}_{\text{NRTA}}$.

[G. S. Rocha *et. al.*, PRL 127 (2021), 042301]

# Solving The Boltzmann Equation

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[S. R. de Groot *et. al.*, Relativistic Kinetic Theory, C. Cercignani *et. al.*, The Relativistic Boltzmann Equation]

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[S. R. de Groot *et. al.*, Relativistic Kinetic Theory, C. Cercignani *et. al.*, The Relativistic Boltzmann Equation]

- Both ERTA and NRTA have this feature.

$$\text{ERTA}: \quad a \to (E_{\mathbf{k}} - \mu)(\delta T/T^2) + (\delta\mu/T), \quad \text{and}, \quad b_\mu \to -(\delta u_\mu/T).$$

$$\text{NRTA}: \quad a \to \Phi_{\mathrm{o}}, \quad \text{and}, \quad b_\mu \to \Phi_{\mathrm{o}}^{\langle\mu\rangle}.$$

[D. Dash *et. al.*, PLB 831 (2022) 137202, G. S. Rocha *et. al.*, PRL 127 (2021), 042301]

# Solving The Boltzmann Equation (Contd.)

- Two popular approaches are considered :

1. Chapman-Enskog-like iterative solution:

$$\phi_{n,\mathbf{k}}\, f_{0\mathbf{k}} = \phi^{*}_{n,\mathbf{k}}\, f_{0\mathbf{k}} - \left(\frac{\tau_{\mathrm{R}}}{E_{\mathbf{k}}}\right)(k\cdot\partial)\, f_{(n-1)\,\mathbf{k}},$$

   We will use this to solve the Extended RTA case.

   [D. Dash *et. al.*, PLB 831 (2022) 137202]

2. Moment method:

$$\phi_{\mathbf{k}} = \sum_{n,\ell=0}^{\infty} \Phi_n^{\langle\mu_1\cdots\mu_\ell\rangle}\, k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle}\, P_n^{(\ell)}(\beta E_{\mathbf{k}})$$

   We will use this to solve the Novel RTA case. Here $P_n^{(\ell)}$ are orthogonal polynomials satisfying the property:

$$\frac{\ell!}{(2\ell+1)!!}\left\langle (E_{\mathbf{k}}/\tau_{\mathrm{R}})\,(k\cdot\Delta\cdot k)^{\ell}\, P_n^{(\ell)} P_m^{(\ell)}\right\rangle_{0\mathbf{k}} = A_n^{(\ell)}\delta_{nm},$$

$$A_n^{(\ell)} = \frac{\ell!}{(2\ell+1)!!}\left\langle (E_{\mathbf{k}}/\tau_{\mathrm{R}})\,(k\cdot\Delta\cdot k)^{\ell}\, P_n^{(\ell)} P_n^{(\ell)}\right\rangle_{0\mathbf{k}}$$

   [G. S. Rocha *et. al.*, PRL 127 (2021), 042301]

# Inclusion of Spin

- Phase-space is extended to include spin degrees of freedom :

$$f_{\mathbf{k}}(x,k) \longrightarrow f_s(x,k,s)$$

$$f_{0\mathbf{k}} \longrightarrow f_{0,s} = f_{0\mathbf{k}} \exp(s:\omega) \approx f_{0\mathbf{k}} \left[1 + \frac{1}{2}(s:\omega)\right] + \mathcal{O}(\omega^2)$$

- Homogeneous part for spin-polarizable particles :

$$\phi_{\mathrm{h}} = a + b_\mu k^\mu + c_{\mu\nu} s^{\mu\nu}$$

- The solutions are modified as :

1. Chapman-Enskog-like iterative solution (ERTA):

$$\phi_{n,s} f_{0,s} = \phi^*_{n,s}\, f_{0,s} - \frac{\tau_{\mathrm{R}}}{(u\cdot p)}\,(p\cdot\partial)\, f_{(n-1),s},$$

2. Moment method (NRTA):

$$\phi_s = \sum_{n,\ell=0}^{\infty} \left(\Phi_n^{\langle\mu_1\cdots\mu_\ell\rangle} + s_{\mu\nu}\Psi_n^{\mu\nu,\langle\mu_1\cdots\mu_\ell\rangle}\right) k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle} P_n^{(\ell)}(\beta E_{\mathbf{k}})$$

[S. Bhadury, 2408.14462]

## Section Outline :

Spin Polarization in Heavy Ion Collisions and Problem - I

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Relativistic Kinetic Theory with Spin:

Relativistic Spin Hydrodynamics with ERTA :

Relativistic Spin hydrodynamics with NRTA :

Summary and Outlook :

# Field Redefinition - ERTA

- The thermodynamic variables (starred) are determined via field re-definitions :

$$\langle q_1\, \phi_s\rangle_0 + \langle \bar{q}_1\, \bar{\phi}_s\rangle_{\bar{0}} = 0, \qquad \langle q_2\, \phi_s\rangle_0 + \langle \bar{q}_2\, \bar{\phi}_s\rangle_{\bar{0}} = 0$$

$$\left\langle q_3\, k^{\langle\mu\rangle}\, \phi_s\right\rangle_0 + \left\langle \bar{q}_3\, k^{\langle\mu\rangle}\, \bar{\phi}_s\right\rangle_{\bar{0}} = 0, \qquad \langle q_4\, s^{\mu\nu}\, \phi_s\rangle_0 + \langle \bar{q}_4\, s^{\mu\nu}\, \bar{\phi}_s\rangle_{\bar{0}} = 0$$

where we use the notations:

$$\langle(\cdots)\rangle_0 = \int dK\, dS\, (\cdots)\, f_{0\mathbf{k}}, \qquad \langle(\cdots)\rangle_{\bar{0}} = \int dK\, dS\, (\cdots)\, \bar{f}_{0\mathbf{k}}$$

- The thermodynamic variables are :

$$X^* = X + \delta X$$

where, $X \equiv T, \mu, u_\mu, \omega_{\mu\nu}$.

- Then we find using the field-redefinitions (up to $\mathcal{O}\left(\partial\right)$):

$$\delta u_\mu = \beta \mathcal{C}_1 \left(\nabla_\mu \xi\right), \qquad \delta T = \mathcal{C}_2\, \theta, \qquad \delta\mu = \mathcal{C}_3\, \theta,$$

$$\delta\omega^{\mu\nu} = \mathcal{D}_{\Pi}^{\mu\nu}\, \theta + \mathcal{D}_{n}^{\mu\nu\gamma} \left(\nabla_\gamma \xi\right) + \mathcal{D}_{\pi}^{\mu\nu\alpha\beta}\, \sigma_{\alpha\beta} + \mathcal{D}_{\Sigma}^{\mu\nu\gamma\alpha\beta} \left(\nabla_\gamma \omega_{\alpha\beta}\right).$$

- Even with arbitrary frame and matching conditions, we had to use, $D \to \nabla$.

- While the first-order theory is still acausal, we can now have, $\tau_{\mathrm{R}}(x, p, s)$.

# Constitutive Equations & Dissipative Currents - ERTA

- Under a general field re-definition the constitutive relations are given by :

$$N^\mu = (n_0 + \delta n)\, u^\mu + n^\mu,$$
$$T^{\mu\nu} = (\mathcal{E}_0 + \delta\mathcal{E})\, u^\mu u^\nu - (\mathcal{P} + \delta\mathcal{P})\, \Delta^{\mu\nu} + 2h^{(\mu} u^{\nu)} + \pi^{\mu\nu},$$
$$S^{\lambda,\mu\nu} = S_0^{\lambda,\mu\nu} + \delta S^{\lambda,\mu\nu}.$$

- The dissipative currents are :

$$\delta n = \nu\, \theta \qquad \delta\mathcal{E} = e\, \theta, \qquad \delta\mathcal{P} = \rho\, \theta,$$
$$n^\mu = \kappa_n^{\mu\nu} \left(\nabla_\nu \xi\right), \qquad h^\mu = \kappa_h^{\mu\nu} \left(\nabla_\nu \xi\right),$$
$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu},$$
$$\delta S^{\lambda,\mu\nu} = B_\Pi^{\mu\nu}\, \theta + B_n^{\mu\nu\gamma} \left(\nabla_\gamma \xi\right) + B_\pi^{\mu\nu\alpha\beta}\, \sigma_{\alpha\beta} + B_\Sigma^{\mu\nu\gamma\alpha\beta} \left(\nabla_\gamma \omega_{\alpha\beta}\right)$$

- The expressions of the transport coefficients can be obtained assuming:

$$\tau_{\mathrm{R}}(x, p, s) = \tau_{\mathrm{eq}}(x)\, (\beta \cdot p)^{\ell_1}\, (u \cdot s)^{\ell_2}$$

[S. Bhadury, 2408.14462]

# Frame Invariant Transport Coefficients - ERTA

- The entropy production is given by,

$$\partial_\mu \mathcal{H}_\mu = -\beta \Pi \theta - \mathcal{Q}^\mu \left(\nabla_\mu \xi\right) + \beta \pi^{\mu\nu} \sigma_{\mu\nu} - \mathcal{S}^{\lambda\mu\nu} \left(\nabla_\lambda \omega_{\mu\nu}\right)$$

- The frame-invariant transport coefficients are:

$$\Pi = -\zeta\,\theta = \delta\mathcal{P} - \left(\frac{\chi_b}{\beta}\right)\delta\mathcal{P} + \left(\frac{\chi_a}{\beta}\right)\delta n,$$

$$\mathcal{Q}^\mu = \kappa^{\mu\nu}\left(\nabla_\nu \xi\right) = n^\mu - \left(\frac{n_0}{\mathcal{E}_0 + \mathcal{P}_0}\right) h^\mu,$$

$$\pi^{\mu\nu} = 2\eta\,\sigma^{\mu\nu},$$

$$\mathcal{S}^{\lambda\mu\nu} = \beta_\Sigma^{\lambda\mu\nu\gamma\alpha\beta}\left(\nabla_\gamma \omega_{\alpha\beta}\right) = \frac{1}{2}\left(u^\rho D_\Sigma^{\alpha\beta\lambda\mu\nu} \delta S_{\rho,\alpha\beta} - \delta S^{\lambda,\mu\nu}\right)$$

[S. Bhadury, 2408.14462]

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# The Linearized collision kernel - NRTA

- We work with $\mu = 0$, as NRTA is not built for pair production and annihilation.

- We have 10 collisional invariants and hence : $\hat{L}\phi_s \equiv -\mathbb{1} + \sum_{n=1}^{10} |\lambda_n\rangle \langle\lambda_n|\phi_s\rangle$

- The orthonormal basis are :

$$|\lambda_1\rangle = \frac{E_{\mathbf{k}}}{\sqrt{\left\langle \left(E_{\mathbf{k}}^3/\tau_{\mathrm{R}}\right)\right\rangle_0}},$$

$$|\lambda_{2-4}\rangle = \frac{k^{\langle\mu\rangle}}{\sqrt{(1/3)\left\langle \left(E_{\mathbf{k}}/\tau_{\mathrm{R}}\right) k_{\langle\alpha\rangle} k^{\langle\alpha\rangle}\right\rangle_0}},$$

$$|\lambda_{5-7}\rangle = \sqrt{\frac{3}{\langle (E_{\mathbf{k}}/\tau_{\mathrm{R}})(\widetilde{s}\cdot\widetilde{s})\rangle_0}}\left[\widetilde{s}^{\,\mu} - \frac{\left\langle \left(E_{\mathbf{k}}^2/\tau_{\mathrm{R}}\right)\widetilde{s}^{\,\mu}\right\rangle_0}{\left\langle \left(E_{\mathbf{k}}^3/\tau_{\mathrm{R}}\right)\right\rangle_0} E_{\mathbf{k}} - \frac{\left\langle \left(E_{\mathbf{k}}/\tau_{\mathrm{R}}\right) k^{\langle\gamma\rangle}\widetilde{s}^{\,\mu}\right\rangle_0 k_{\langle\gamma\rangle}}{\left\langle (1/3)\left(E_{\mathbf{k}}/\tau_{\mathrm{R}}\right) k^{\langle\alpha\rangle} k_{\langle\alpha\rangle}\right\rangle_0}\right],$$

$$|\lambda_{8-10}\rangle = \sqrt{\frac{3}{\langle (E_{\mathbf{k}}/\tau_{\mathrm{R}})(\widetilde{s}:\widetilde{s})\rangle_0}}\left[\widetilde{s}^{\,\mu\nu} - \frac{\left\langle \left(E_{\mathbf{k}}^2/\tau_{\mathrm{R}}\right)\widetilde{s}^{\,\mu\nu}\right\rangle_0}{\left\langle \left(E_{\mathbf{k}}^3/\tau_{\mathrm{R}}\right)\right\rangle_0} E_{\mathbf{k}} - \frac{\left\langle \left(E_{\mathbf{k}}/\tau_{\mathrm{R}}\right) k^{\langle\mu\rangle}\widetilde{s}^{\,\mu\nu}\right\rangle_0 k_{\langle\mu\rangle}}{\left\langle (1/3)\left(E_{\mathbf{k}}/\tau_{\mathrm{R}}\right) k^{\langle\alpha\rangle} k_{\langle\alpha\rangle}\right\rangle_0}\right].$$

- Then we get,

$$\hat{L}E_{\mathbf{k}} = 0, \qquad \hat{L}k^{\langle\mu\rangle} = 0, \qquad \hat{L}s^{\mu\nu} = 0.$$

# Field Redefinition - NRTA

- We work with $\mu = 0$, as NRTA is not built for pair production and annihilation.

- Now we use a new set of notations :

$$\langle(\cdots)\rangle_0 = \int dK\, dS\, (\cdots)\, f_{0s}, \qquad \langle(\cdots)\rangle_{0\mathbf{k}} = \int dK\, (\cdots)\, \bar{f}_{0\mathbf{k}}$$

- The thermodynamic variables (starred) are determined via field re-definitions :

$$\int dK dS\, q_1 \phi_s\, f_{0\mathbf{k}} = 0, \qquad \int dK dS\, q_2 k^{\langle\mu\rangle} \phi_s\, f_{0\mathbf{k}} = 0, \qquad \int dK dS\, q_3 s^{\mu\nu} \phi_s\, f_{0\mathbf{k}} = 0$$

- The homogeneous parts of the solution are :

$$\Phi_0 = -\sum_{n=1}^{\infty} \Phi_n \frac{\left\langle q_1 P_n^{(0)} \right\rangle_{0\mathbf{k}}}{\left\langle q_1 P_0^{(0)} \right\rangle_{0\mathbf{k}}}, \qquad \Phi_0^{\langle\mu\rangle} = -\sum_{n=1}^{\infty} \Phi_n^{\langle\mu_1\rangle} \frac{\left\langle q_2\, k^{\langle\mu\rangle} k_{\langle\mu_1\rangle} P_n^{(1)} \right\rangle_{0\mathbf{k}}}{\left\langle q_2\, (k \cdot \Delta \cdot k)\, P_0^{(1)} \right\rangle_{0\mathbf{k}}},$$

$$\Psi_0^{\mu\nu} = -\sum_{n=1}^{\infty} \Psi_n^{\mu\nu} \left\langle q_3\, P_n^{(0)} \right\rangle_{0\mathbf{k}} / \left\langle q_3 P_0^{(0)} \right\rangle_{0\mathbf{k}}.$$

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# Summary and Outlook :

- **Summary :**
  1. ERTA provides a simple solution to make the relaxation time dependent on $p$ and $s$.
  2. ERTA does not lead to first-order causal theory of spin hydrodynamics.
  3. NRTA gives the option of constructing first-order causal spin-hydrodynamics.
  4. NRTA cannot describe a system with pair production and annihilation.

- **Outlook :**
  1. For the NRTA, we still need to find :

$$\Phi_n, \; \Phi_n^{\langle\mu\rangle}, \; \Psi_n^{\mu\nu} \quad \text{for} \quad n \geq 1$$
$$\Phi_n^{\langle\mu_1\cdots\mu_\ell\rangle} \quad \text{for} \quad n \geq 0, \; \ell \geq 2$$
$$\Psi_n^{\mu\nu,\langle\mu_1\cdots\mu_\ell\rangle} \quad \text{for} \quad n \geq 0, \; \ell \geq 1.$$

  2. Second-order Spin hydrodynamics from kinetic theory should be formulated.

***Thank you.***

# Back Up :

- The frame-invariant transport coefficients are:

$$f_i \equiv \pi_i - \left(\frac{\partial \mathcal{P}_0}{\partial \mathcal{E}_0}\right)_{n_0} \varepsilon_i + \left(\frac{\partial \mathcal{P}_0}{\partial n_0}\right)_{\mathcal{E}_0} \nu_i,$$

$$l_i \equiv \gamma_i - \left(\frac{n_0}{\mathcal{E}_0 + \mathcal{P}_0}\right) \theta_i.$$

- Out of the 16 parameters, only three one-derivative transport coefficients $\zeta(f_1, f_2, f_3),\ \eta,\ \kappa(l_1, l_2)$ are independent.