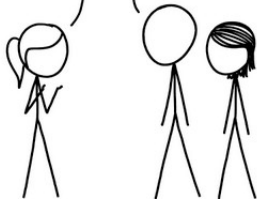


Relativistic spin-(magneto)hydrodynamics

THE SUN'S ATMOSPHERE IS A
SUPERHOT PLASMA GOVERNED BY
MAGNETOHYDRODYNAMIC FORCES...

AH, YES,
OF COURSE.



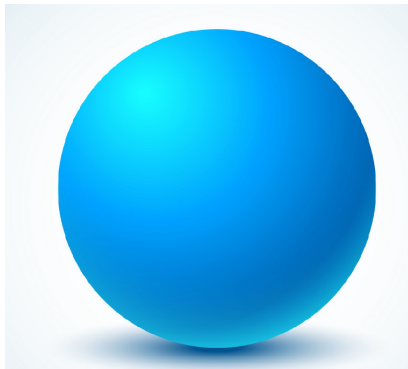
Magnetohydrodynamics combines the intuitive nature of Maxwell's equations with the easy solvability of the Navier-Stokes equations. It's so straightforward physicists add "relativistic" or "quantum" just to keep it from getting boring.

WHENEVER I HEAR THE WORD
"MAGNETOHYDRODYNAMIC" MY BRAIN
JUST REPLACES IT WITH "MAGIC."

xkcd.com

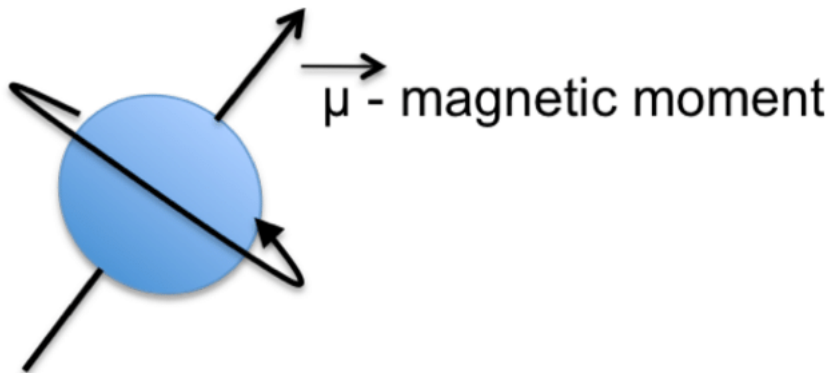
Hydro workshop 2024, Nantes, France

Decay of scalar particles



No anisotropy in the rest frame: isotropic decay products.

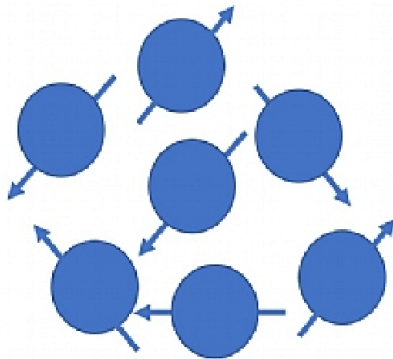
Decay of particles with spin



Preferred direction due to spin: anisotropic decay products

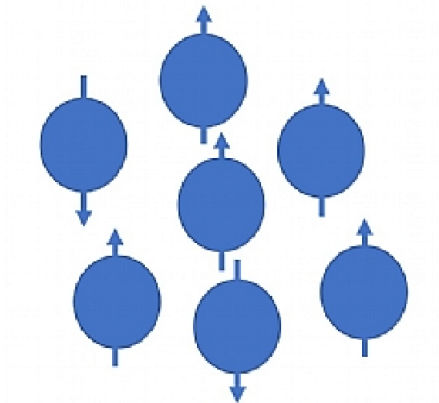
Basis for polarization observables.

Several random decays

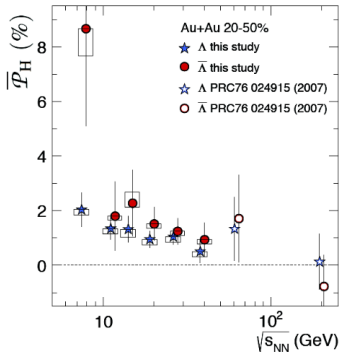


Averaging over random decays should lead to isotropic decay products.

Decay of spin polarized particles



Averaging over decay of spin-polarized particles should lead to anisotropic decay products.



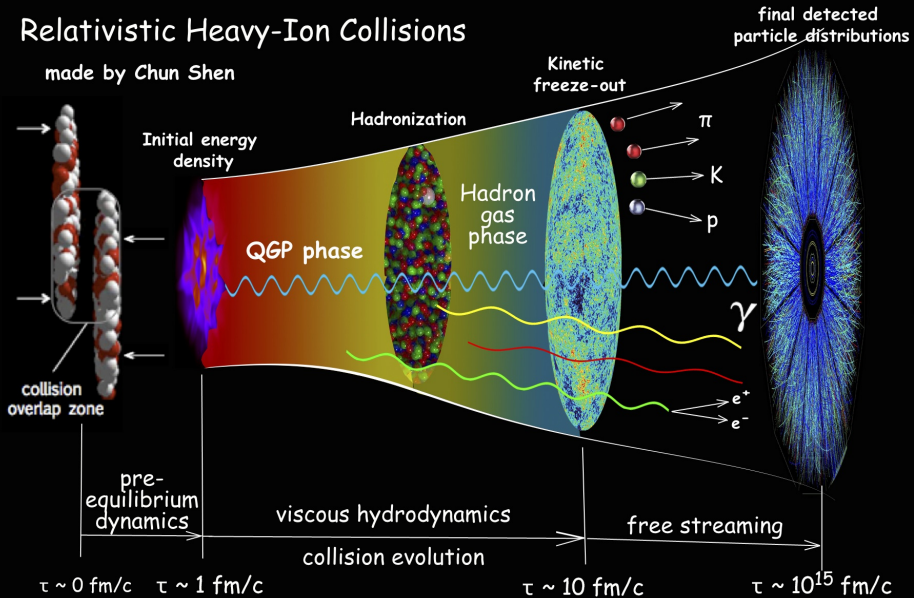
First evidence of a quantum effect in (relativistic) hydrodynamics



Adapted from F. Becattini
'Subatomic Vortices'

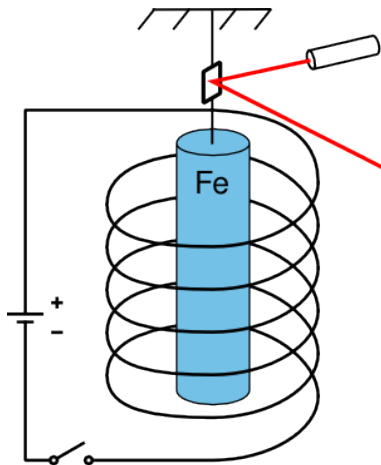
Relativistic Heavy-Ion Collisions

made by Chun Shen



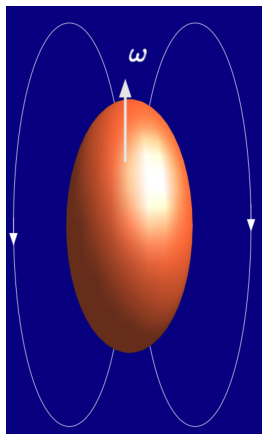
Other effects related to spin polarization.

Einstein-de Haas effect



Electron spins get aligned in external magnetic field which is compensated by rotation of the ferromagnetic material.

Converse: Barnett effect



Second Series.

October, 1915

Vol. VI., No. 4

THE
PHYSICAL REVIEW.

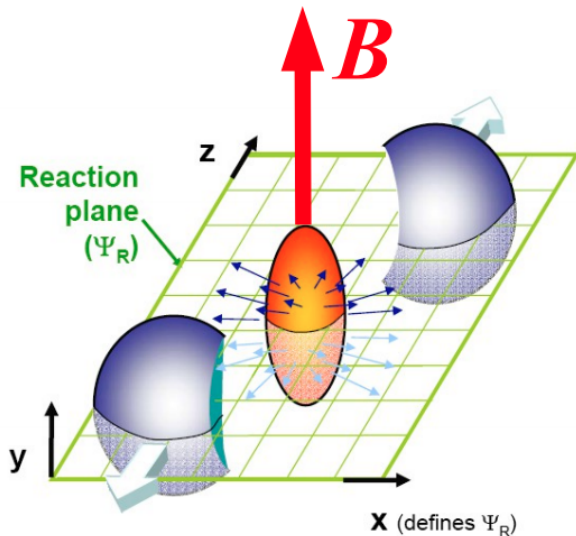
MAGNETIZATION BY ROTATION.¹

BY S. J. BARNETT.

§1. In 1909 it occurred to me, while thinking about the origin of terrestrial magnetism, that a substance which is magnetic (and therefore, according to the ideas of Langevin and others, constituted of atomic or molecular orbital systems with individual magnetic moments fixed in magnitude and differing in this from zero) must become magnetized by a sort of molecular gyroscopic action on receiving an angular velocity.

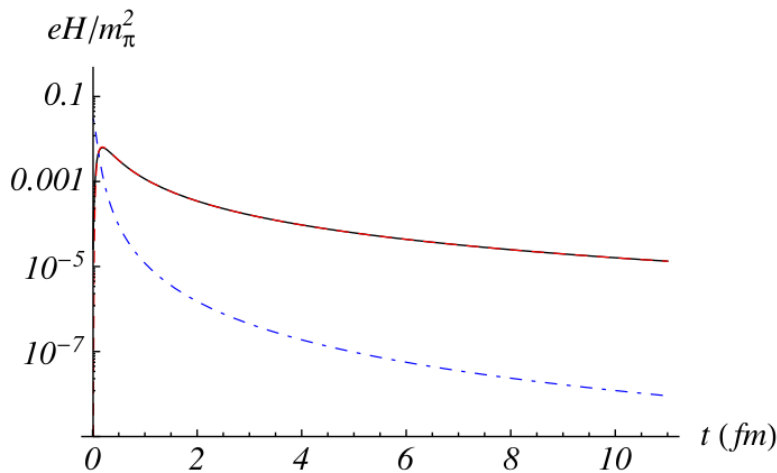
Spontaneous magnetization when spun around. Transformation of orbital angular momentum into spin alignment. Angular velocity decreases with appearance of magnetic field. Explanation appeals to spin-orbit coupling.

Generation of magnetic field in heavy ion collisions



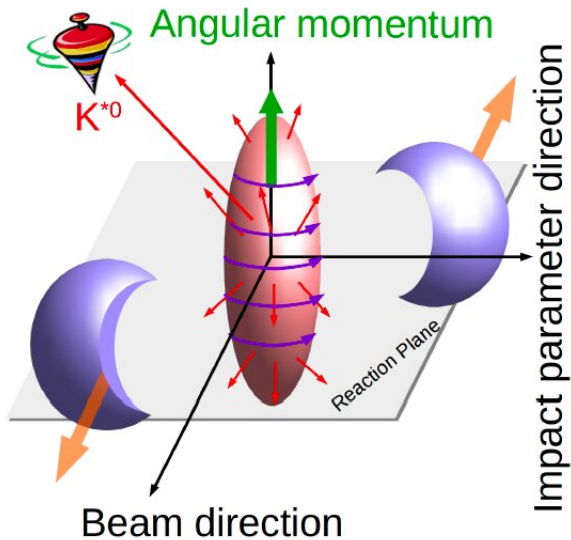
[Adapted from D. Kharzeev @ CPOD 2013.]

Magnetic field time evolution



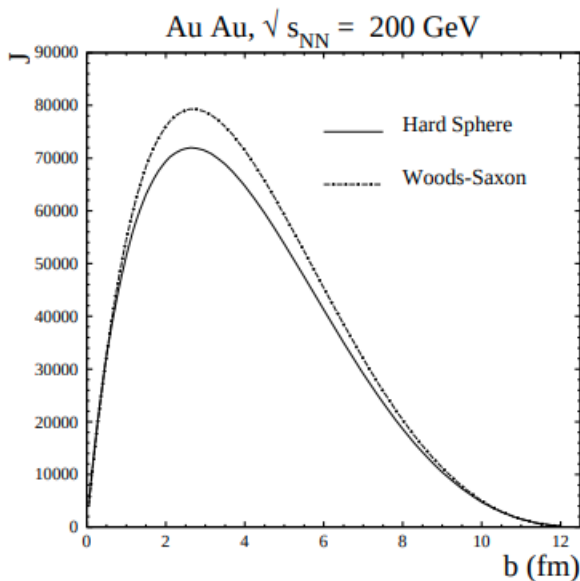
[K. Tuchin, Int. J. Mod. Phys. E23, 1430001 (2014).]

Global angular momentum in heavy ion collisions

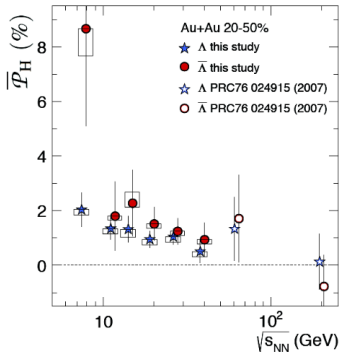


[B. Mohanty, ICTS News 6, 18-20 (2020).]

Angular momentum generation in non-central collisions



[F. Becattini, et al., Phys. Rev. C77, 024906 (2008).]



First evidence of a quantum effect in (relativistic) hydrodynamics



Adapted from F. Becattini
'Subatomic Vortices'

Hydrodynamics with angular momentum conservation

- Apply relativistic (magneto)-hydrodynamics to understand particle polarization.
- Hydrodynamic evolution should ensure total angular momentum conservation.
- Formulation of relativistic magneto-hydrodynamics with angular momentum conservation necessary.
- Polarization processes are generally dissipative in nature.
- Dissipation should also be incorporated in the formulation.
- Relativistic kinetic theory framework.
- Geometrical approach with torsionful curved background.

Angular momentum conservation: particles

- Angular momentum of a particle with momentum \vec{p} :

$$\vec{L} = \vec{x} \times \vec{p} \quad \Rightarrow \quad L_i = \varepsilon_{ijk} x_j p_k$$

- One can obtain the dual tensor:

$$L_{ij} \equiv \varepsilon_{ijk} L_k \quad \Rightarrow \quad L_{ij} = x_i p_j - x_j p_i$$

- We know that both definitions are equivalent.

- In absence of external torque, $\frac{d\vec{L}}{dt} = 0$, we also have: $\partial_i L_{ij} = 0$.

- Relativistic generalization: $L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ and $\partial_\mu L^{\mu\nu} = 0$.

- This treatment valid for point particles.

- For fluids, particle momenta \rightarrow “generalized fluid momenta”

The energy-momentum tensor

Angular momentum conservation: fluid

- The orbital angular momentum for relativistic fluids is defined as

$$L^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}$$

- Keeping in mind the energy-momentum conservation, $\partial_\mu T^{\mu\nu} = 0$:

$$\partial_\lambda L^{\lambda,\mu\nu} = T^{\mu\nu} - T^{\nu\mu}$$

- Obviously, for symmetric $T^{\mu\nu}$, orbital angular momentum is automatically conserved. Classically $T^{\mu\nu}$ symmetric.
- For medium constituent with intrinsic spin, different story

$$J^{\lambda,\mu\nu} = L^{\lambda,\mu\nu} + S^{\lambda,\mu\nu}$$

- Ensure total angular momentum conservation: $\partial_\lambda J^{\lambda,\mu\nu} = 0$.
- Basis for formulation of spin Hydrodynamics.

[Florkowski et. al., Prog.Part.Nucl.Phys. 108 (2019) 103709; Bhadury et. al., Eur.Phys.J.ST 230 (2021) 3, 655-672]

Pseudo-gauge transformations

- Ignoring the L - S coupling terms,

$$J^{\lambda,\mu\nu} = L^{\lambda,\mu\nu} + S^{\lambda,\mu\nu}$$

- With $\partial_\mu T^{\mu\nu} = 0$, and $\partial_\lambda L^{\lambda,\mu\nu} = T^{\mu\nu} - T^{\nu\mu}$,

$$\partial_\lambda J^{\lambda,\mu\nu} = 0 \quad \implies \quad \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}$$

- Hence the final hydrodynamic equations can be written as

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}$$

- Also holds with the following redefinition

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \frac{1}{2} \partial_\lambda \left(\Phi^{\lambda,\mu\nu} - \Phi^{\mu,\lambda\nu} - \Phi^{\nu,\lambda\mu} \right)$$

$$\tilde{S}^{\lambda,\mu\nu} = S^{\lambda,\mu\nu} - \Phi^{\lambda,\mu\nu}$$

- Freedom due to space-time symmetry; including torsion fixes this.

[Gallegos et. al., SciPost Phys. 11, 041 (2021); Hongo et. al., JHEP 11 (2021) 150]

Our Formulation: Relativistic Kinetic Theory

Relativistic kinetic theory

- Kinetic theory: calculation of macroscopic quantities by means of statistical description in terms of distribution function.
- Let us consider a system of relativistic particles of rest mass m with momenta \mathbf{p} and energy p^0

$$p^0 = \sqrt{\mathbf{p}^2 + m^2}$$

- For large no. of particles, $f(x, p)$ gives a distribution of the four-momenta $p = p^\mu = (p^0, \mathbf{p})$ at each space-time point.
- $f(x, p) \Delta^3 x \Delta^3 p$ gives average no. of particles in the volume element $\Delta^3 x$ at point x with momenta in the range $(\mathbf{p}, \mathbf{p} + \Delta \mathbf{p})$.
- Statistical assumptions:
 - No. of particles contained in $\Delta^3 x$ is large ($N \gg 1$).
 - $\Delta^3 x$ is small compared to macroscopic volume ($\Delta^3 x / V \ll 1$).
- The equilibrium distribution: $f_{eq}(x, p, s) = [\exp(\beta \cdot p - \xi) \pm 1]^{-1}$

Extended phase-space for spin degrees of freedom

- The phase-space for single particle distribution function gets extended $f(x, p, s)$.
- The equilibrium distribution for Fermions is given by

$$f_{eq}(x, p, s) = \frac{1}{\exp [\beta \cdot p - \xi - \frac{1}{2} \omega : s] + 1} \quad \begin{cases} \beta \cdot p \equiv \beta_\mu p^\mu \\ \omega : s \equiv \omega_{\mu\nu} s^{\mu\nu} \end{cases}$$

- Quantities $\beta^\mu = u^\mu/T$, $\xi = \mu/T$, $\omega_{\mu\nu}$ are functions of x .
- ξ , β^μ , $\omega^{\mu\nu}$: Lagrange multipliers for conserved quantities.
- $s^{\mu\nu}$: Particle spin, similar to particle momenta p^μ .
- Hydrodynamics: average over particle momenta and spin.
- Classical treatment of spin.

Bhadury et. al., PLB 814, 136096 (2021); PRD 103, 01430 (2021).

Conserved currents and spin-hydrodynamics

- Express hydrodynamic quantities in terms of $f(x, p, s)$.

$$T^{\mu\nu}(x) = \int dP dS p^\mu p^\nu [f(x, p, s) + \bar{f}(x, p, s)]$$

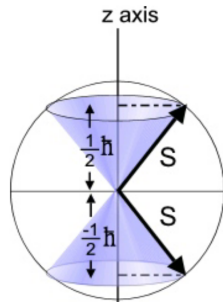
$$N^\mu(x) = \int dP dS p^\mu [f(x, p, s) - \bar{f}(x, p, s)]$$

$$S^{\lambda, \mu\nu}(x) = \int dP dS p^\lambda s^{\mu\nu} [f(x, p, s) + \bar{f}(x, p, s)]$$

$$dP \equiv \frac{d^3 p}{E_p (2\pi)^3}, \quad dS \equiv m \frac{d^4 s}{\pi \mathfrak{s}} \delta(s \cdot s + \mathfrak{s}^2) \delta(p \cdot s)$$

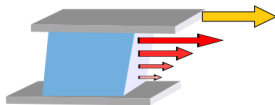
$$\int dS = 2; \quad \mathfrak{s}^2 = \frac{1}{2} \left(\frac{1}{2} + 1 \right) = \frac{3}{4}; \quad s^\mu \equiv \frac{1}{2m} \epsilon^{\mu\nu\alpha\beta} p_\nu s_{\alpha\beta}$$

- Classical treatment of spin: internal angular momentum.
- Equations of motion: $\partial_\mu T^{\mu\nu} = 0$, $\partial_\mu N^\mu = 0$, $\partial_\lambda S^{\lambda, \mu\nu} = 0$.
- Non-dissipative spin hydrodynamics: $f(x, p, s) = f_{eq}(x, p, s)$.
- Dissipative spin-hydrodynamics: Boltzmann equation for $f(x, p, s)$.

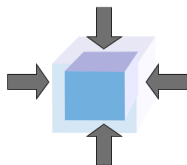


Dissipative effects

- ▶ Shear viscosity: fluid's resistance to shear forces



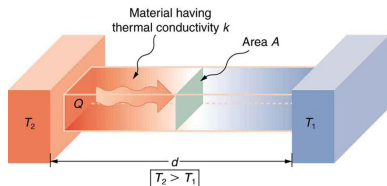
- ▶ Bulk viscosity: fluid's resistance to compression



- Charge/heat conductivity: fluid's resistance to flow of charge/heat.

- Dissipation to spin current new:

Bhadury et. al., PLB 814, 136096 (2021); PRD 103, 01430 (2021).



- The particle four-current and its conservation is given by

$$N^\mu = nu^\mu + n^\mu, \quad \partial_\mu N^\mu = 0$$

- Total stress-energy tensor of the system: $T^{\mu\nu} = T_f^{\mu\nu} + T_{\text{int}}^{\mu\nu} + T_{\text{em}}^{\mu\nu}$

$$T_f^{\mu\nu} = \epsilon u^\mu u^\nu - (P + \Pi) \Delta^{\mu\nu} + \pi^{\mu\nu},$$

$$T_{\text{int}}^{\mu\nu} = -\Pi^\mu u^\nu - F^\mu_\alpha M^{\nu\alpha}$$

$$T_{\text{em}}^{\mu\nu} = -F^{\mu\alpha} F^\nu_\alpha + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}$$

- Maxwell's equation: $\partial_\mu H^{\mu\nu} = J^\nu$ and $H^{\mu\nu} = F^{\mu\nu} + M^{\mu\nu}$,

$$\partial_\mu T_{\text{em}}^{\mu\nu} = F^\nu_\alpha J^\alpha$$

- Current generating external field, $J^\mu = J_f^\mu + J_{\text{ext}}^\mu$ where $J_f^\mu = \mathbf{q}N^\mu$,

$$\partial_\mu T^{\mu\nu} = -f_{\text{ext}}^\nu, \quad f_{\text{ext}}^\nu = F^\nu_\alpha J_{\text{ext}}^\alpha$$

Equations of motion

- Divergence of matter part of energy-momentum tensor,

$$\partial_\nu T_f^{\mu\nu} = F^\mu{}_\alpha J_f^\alpha + \frac{1}{2} (\partial^\mu F^{\nu\alpha}) M_{\nu\alpha}$$

- Next, consider total angular momentum conservation:

$$J^{\lambda,\mu\nu} = L^{\lambda,\mu\nu} + S^{\lambda,\mu\nu}$$

- In presence of external torque its divergence leads to,

$$\partial_\lambda J^{\lambda,\mu\nu} = -\tau_{\text{ext}}^{\mu\nu}, \quad \tau_{\text{ext}}^{\mu\nu} = x^\mu f_{\text{ext}}^\nu - x^\nu f_{\text{ext}}^\mu$$

- Torque due to moment of external force; “pure” torque ignored.
- The orbital part of angular momentum and its divergence is

$$L^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}, \quad \partial_\lambda L^{\lambda,\mu\nu} = -\tau_{\text{ext}}^{\mu\nu}$$

- Spin part of the total angular momentum is conserved

$$\partial_\lambda S^{\lambda,\mu\nu} = 0$$

- Along with particle four-current conservation, $\partial_\mu N^\mu = 0$.

Boltzmann equation

- Boltzmann equation (BE) in relaxation-time approximation (RTA)

$$\left(p^\alpha \frac{\partial}{\partial x^\alpha} + m \mathcal{F}^\alpha \frac{\partial}{\partial p^\alpha} + m \mathcal{S}^{\alpha\beta} \frac{\partial}{\partial s^{\alpha\beta}} \right) f = C[f] = - (u \cdot p) \frac{f - f_{\text{eq}}}{\tau_{\text{eq}}}$$

- The force term is:

$$\mathcal{F}^\alpha = \frac{\mathbf{q}}{m} F^{\alpha\beta} p_\beta + \frac{1}{2} \left(\partial^\alpha F^{\beta\gamma} \right) m_{\beta\gamma}, \quad m^{\alpha\beta} = \chi s^{\alpha\beta}$$

- There is a “pure” torque term:

$$\mathcal{S}^{\alpha\beta} = 2 F^{\gamma[\alpha} m^{\beta]}_\gamma - \frac{2}{m^2} \left(\chi - \frac{\mathbf{q}}{m} \right) F_{\phi\gamma} s^{\phi[\alpha} p^{\beta]} p^\gamma$$

- We ignore this “pure” torque term for now.
- Employ the Boltzmann equation to obtain $\delta f = \delta f_1$.
- Evolution equations for dissipative spin-magnetohydrodynamics.

Einstein-de Haas and Barnett effects

- One can define the polarization-magnetization tensor as

$$M^{\mu\nu} = m \int dP dS m^{\mu\nu} (f - \bar{f})$$

- The equilibrium polarization-magnetization tensor is

$$M_{eq}^{\mu\nu} = m \int dP dS m^{\mu\nu} (f_{eq} - \bar{f}_{eq})$$

- Magnetic dipole moment $m^{\mu\nu} = \chi s^{\mu\nu}$.
- χ : resembles the gyromagnetic ratio.
- Integrating over the momentum and spin degrees of freedom,

$$M_{eq}^{\mu\nu} = a_1 \omega^{\mu\nu} + a_2 u^{[\mu} u_{\gamma} \omega^{\nu]\gamma}$$

- In global equilibrium, $\omega^{\mu\nu}$ corresponds to rotation of the fluid.
- Rotation produces magnetization (Barnett effect) and vice versa (Einstein-de Haas effect).

Our work in this direction within kinetic theory

- Ideal spin-hydrodynamics:

- W. Flokowski, B. Friman, A. Jaiswal and E. Speranza, Physical Review C 97, 041901 (2018).
- W. Flokowski, B. Friman, A. Jaiswal, R. Ryblewski and E. Speranza, Physical Review D 97, 116017 (2018).

- Dissipative spin-hydrodynamics:

- S. Bhadury, W. Flokowski, A. Jaiswal, A. Kumar, and R. Ryblewski, Physics Letters B 814, 136096 (2021).
- S. Bhadury, W. Flokowski, A. Jaiswal, A. Kumar, and R. Ryblewski, Physical Review D 103, 014030 (2021).



- Relativistic Spin Magnetohydrodynamics: S. Bhadury, W. Flokowski, A. Jaiswal, A. Kumar, and R. Ryblewski, Phys. Rev. Lett., 129, 192301 (2022).

Other aspects of spin-hydrodynamics formulation

- Starting from the symmetries of the Lagrangian of a given theory, one can construct conserved currents using Noether's theorem.
- Energy-momentum tensor-variation of Lagrangian with metric $g^{\mu\nu}$: conservation is a consequence of diffeomorphism invariance.
- Conserved charge current-variation with gauge field A^μ : consequence of local gauge symmetry.
- Spin-current can be constructed similarly.
- Price to pay: introduce torsion in metric, non-Riemannian geometry.
- Spin current: variation w.r.t torsion.
- Relativistic spin-hydrodynamics: formulation from action principle; Lagrangian formalism.
- Kubo relations for dissipation in spin current.
- Effect of Pseudogauge transformations on Angular momentum algebra of spin operators [S. Dey *et. al.*, *Physics Letters B* 843, 137994 (2023)].



Heavy-ion phenomenology with spin-hydrodynamics

- Global and local (longitudinal) polarization measured in HICs; see talk by Radoslaw Ryblewski.
- Global polarization relatively well understood from spin hydrodynamic prediction of thermal vorticity.
- Sign problem in longitudinal spin polarization of the Λ hyperons.
- Thermal vorticity employed at freezeout.
- Assumes global equilibrium: not a good approximation.
- Large spin-relaxation time obtained from fit with experimental data of longitudinal polarization [S. Banerjee *et. al.*, arXiv:2405.05089].
- Hydrodynamic evolution required for $\omega^{\mu\nu}$.
- Numerical solution of relativistic spin-hydrodynamics.



Other approaches in spin hydrodynamics formulation

- Other parallel approaches from Wigner function [N. Weickgenannt, X.-l. Sheng, E. Speranza, Q. Wang and D. Rischke, PRD 100 (2019) 056018].
- Chiral kinetic theory [S. Shi, C. Gale and S. Jeon, PRC 103 (2021) 044906].
- Lagrangian method [D. Montenegro and G. Torrieri, PRD 100 (2019) 056011].
- Formulation with torsion in metric [A. D. Gallegos, U. Gürsoy and A. Yarom, SciPost Phys. 11, 041 (2021); M. Hongo, X.-G. Huang, M. Kaminski, M. Stephanov, H.-U. Yee, JHEP 11 (2021) 150].
- Spin polarization as a signal of QCD critical point: [S. K. Singh, J. Alam, PRD 074042, 107 (2023); EPJC 83 (2023) 7, 585].
- Useful reviews on spin hydro: [W. Florkowski, R. Ryblewski and A. Kumar, Prog.Part.Nucl.Phys. 108 (2019) 103709; S. Bhadury, J. Bhatt, A. Jaiswal and A. Kumar, Eur.Phys.J.ST 230 (2021) 3, 655-672].
- **Relativistic spin-magnetohydrodynamics: relatively less explored.**
- Much work needed in this direction.

Thank you!

