Revisiting shear stress evolution in non-resistive Magnetohydrodynamics within the extended relaxation-time approximation

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Hydrodynamics and related observables in heavy-ion collisions

Based on *S. Singh, M. Kurian, V. Chandra,* ***Phy. Rev. D 110, 014004 (2024)***

Contents

Heavy Ion Collision

- The QGP phase can be modelled in the framework of relativistic hydrodynamics which is a relativisitc generalization of fluid mechanics.

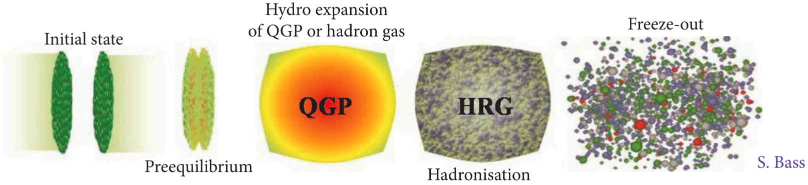


Figure: Various stages of Ultrarelativistic heavy-ion collisions

Conservation laws

- Conservation of particle number and the energy momentum tensor,

$$\partial_\mu T_f^{\mu\nu} = 0 \tag{1}$$

$$\partial_\mu N_f^{\mu} = 0 \tag{2}$$

- With the fluid velocity at each point being u^μ, the fluid energy momentum tensor and number current can be decomposed along and orthogonal to fluid velocity as:

$$T_f^{\mu\nu} = \epsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + 2u^{(\mu} h^{\nu)} + \pi^{\mu\nu} \tag{3}$$

$$N_f^{\mu} = n u^\mu + n^\mu \tag{4}$$

- Here $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is the orthogonal projector to u^μ.
- In Landau frame definition,

$$u^\mu = \frac{u_\nu T^{\mu\nu}}{\epsilon}$$

  And hence, the heat current $h^\mu = 0$, i.e, there is no energy dissipation.

Equations of motion

- The fluid equations of motion are now given by:

$$\dot{\varepsilon} + (\varepsilon + P)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0, \quad (5)$$
$$(\epsilon + P)\dot{u}^{\mu} - \nabla^{\mu}P + \Delta^{\mu}_{\nu}\partial_{\gamma}\pi^{\gamma\nu} = 0, \quad (6)$$
$$\dot{n} + n\theta + \partial_{\mu}n^{\mu} = 0 \quad (7)$$

- ϵ, n and P are related to each other via the equation of state.
- The evolution equations for $\pi^{\mu\nu}$ and n^{μ} is needed which can be derived from entropy-current analysis, requiring $\partial_{\mu}S^{\mu} \geq 0$.
- The evolution equation for $\pi^{\mu\nu}$ that we need to ensure the second law of thermodynamics is guaranteed is:

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_{\pi}} = 2\beta_{\pi}\sigma^{\mu\nu} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + 2\pi_{\gamma}^{\langle\mu}\omega^{\nu\rangle\gamma} - \tau_{\pi\pi}\pi_{\gamma}^{\langle\mu}\sigma^{\nu\rangle\gamma}$$
$$- \tau_{\pi n}n^{\langle\mu}\dot{u}^{\nu\rangle} + \lambda_{\pi n}n^{\langle\mu}\nabla^{\nu\rangle}\alpha + l_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle}. \quad (8)$$

- The transport coefficients must be determined from a microscopic theory.

Kinetic Theory Approach

- To solve the resulting equation of motion, we need to find the evolution equation of $\pi^{\mu\nu}$ as well as the other dissipative quantities. We do that using kinetic theory approach.
- We start with a distribution function for the particle, $f(x,p)$ and which can then be used to define:

$$T^{\mu\nu} = \int \mathrm{dP} p^\mu p^\nu f \qquad N^\mu = \int \mathrm{dP} p^\mu f \tag{9}$$

- With $f_0(x,p)$ being the local equilibrium distribution, the deviation of equilibrium can be written as $\delta f = f - f_0$.

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int \mathrm{dP} p^\alpha p^\beta \delta f \tag{10}$$

Boltzmann Equation and the RTA collision kernel

- To get the form of δf, the evolution of $f(x,p)$ is needed via the Boltzmann equation.

$$p^\mu \partial_\mu f = \mathcal{C}[f] \tag{11}$$

- The collision term which encodes the details about various collisional processes happening in the system can be approximated using the relaxation time approximation (RTA) as:

$$\mathcal{C}[f] = -\frac{(u \cdot p)}{\tau_R}(f - f_0) \tag{12}$$

- Conservation laws implies that for $\mathcal{C}[f]$,

$$\int \mathrm{dP}\mathcal{C}[f] = 0 = \int \mathrm{dP} p^\mu \mathcal{C}[f] \tag{13}$$

- The relaxation time τ_R is momentum independent for RTA since a momentum dependent $\tau_R(p)$ leads to violation of conservation laws with Landau matching condition:

$$\int \mathrm{dP}\mathcal{C}[f] \neq 0 \neq \int \mathrm{dP} p^\mu \mathcal{C}[f] \tag{14}$$

Extended relaxation time: Motivation

- $\tau_R(x,p)$ reflects how quickly particles at that momentum equilibriate with rest of the fluid.
- According to some studies[1], various physical scenarios lead to various forms of momentum dependence for the relaxation time.
- If energy loss of particles grows linearly with momentum, $\frac{dp}{dt} \propto p$, relaxation time is expected to be independent of p
- If energy loss of particles approaches a constant value, relaxation time follows $\tau_R \propto p$
- For scalar $\lambda\phi^4$ theory, relaxation time follows $\tau_R \propto p$.
- For QCD, the momentum dependence should lie between these two cases.
- For example, in case of QCD radiation energy loss taken into account, we expect $\tau_R \propto p^{0.5}$
- This leads to the formulation of Hydrodynamics using a momentum-dependent relaxation time approach.

[1] K. Dusling, G.D. Moore & D. Teany, ***Phy Rev C 81, 034907 (2010)***

Extended relaxation time

- The collision kernel for extended relaxation time reads [2]:

$$C[f] = -\frac{(u \cdot p)}{\tau_R(x,p)}(f - f_0^*) \tag{15}$$

Where f_0^* is the equilibrium distribution function in the "thermodynamic" frame.

$$f_0^* = e^{-\beta^*(u^* \cdot p)+\alpha^*} \tag{16}$$

- Where $u^{*\mu}$,β^* and α^* are related with the usual variables by:

$$u^{\mu *} = u^\mu + \delta u^\mu, \qquad \mu^* = \mu + \delta\mu, \qquad T^* = T + \delta T, \tag{17}$$

- This lets us use a momentum dependent relaxation time to determine the transport coefficients. The form of momentum dependent $\tau_R(x,p)$ is taken as:

$$\tau_R(x,p) = \frac{\kappa}{T}\left(\frac{u \cdot p}{T}\right)^\ell \tag{18}$$

[2] D. Dash, S. Bhadury, S. Jaiswal, A. Jaiswal, ***Physics Letters B, 831 (2022)***

Gradient expansion

- We use a Champan-Enskog like gradient expansion to get the form of δf upto second order.

$$\Delta f = -\frac{\tau_R}{(u\cdot p)} p^\mu \partial_\mu f_0 - \frac{\tau_R}{(u\cdot p)} p^\mu \partial_\mu \Big[-\frac{\tau_R}{(u\cdot p)} p^\mu \partial_\mu f_0 + \delta f^*_{(1)} \Big] + \Delta f^*_{(2)}. \tag{19}$$

Where $\Delta f^*_{(2)}$ is given by $\Delta f^*_{(2)} = f^*_0 - f_0$.

- This can be obtained by Taylor expanding f^*_0 about T, μ and u^μ:

$$\Delta f^*_{(2)} = \Big[-\frac{(\delta u_{(2)} \cdot p)}{T} + \frac{(u\cdot p - \mu)}{T^2} \delta T_{(2)} + \frac{\delta \mu_{(2)}}{T} \Big] f_0. \tag{20}$$

- With the three conditions, viz. Landau frame condition, $\epsilon = \epsilon_0$ and $n = n_0$, we obtain the form of δu^μ, δT and $\delta \mu$.

Shear stress evolution within ERTA

- The δf with the necessary counter terms inside $u^{*\mu}$, β^* and μ^* leads to:

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \tau_{\pi\pi}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \tau_{\pi n} n^{\langle\mu}\dot{u}^{\nu\rangle} + \lambda_{\pi n} n^{\langle\mu}\nabla^{\nu\rangle}\alpha + l_{\pi n}\nabla^{\langle\mu} n^{\nu\rangle}. \quad (21)$$

- This evolution equation is second order in the gradient expansion of the hydrodynamic fields
- The central result of the current work is that in the massless MB limit, **the evolution of the number diffusion n^μ is coupled to the evolution of the shear stress tensor** whereas they are decoupled in the RTA limit [3].

$$\tau_\pi = \frac{\bar{\kappa}\Gamma(5+2\ell)}{T\Gamma(5+\ell)}, \quad \ell > -\frac{5}{2} \quad (22)$$

$$\eta_0 = \frac{e^\alpha d_g \bar{\kappa}\Gamma(5+\ell)}{30\pi^2\beta^3}, \quad \ell > -5 \quad (23)$$

$$\tau_{\pi\pi} = \frac{2\Gamma(6+2\ell)}{7\Gamma(5+\ell)}, \quad \ell > -\frac{5}{2} \quad (24)$$

$$l_{\pi n} = \frac{T\ell\Gamma(5+\ell)\left\{\Gamma(5+\ell)\Gamma(4+\ell) - 48\Gamma(4+2\ell)\right\}}{15(\ell^2-\ell+4)\Gamma(3+\ell)\Gamma(5+2\ell)}, \quad \ell > -2 \quad (25)$$

$$\tau_{\pi n} = \frac{4T\ell\Gamma(5+\ell)\left\{\Gamma(5+\ell)\Gamma(4+\ell) - 48\Gamma(4+2\ell)\right\}}{15(\ell^2-\ell+4)\Gamma(3+\ell)\Gamma(5+2\ell)}, \quad \ell > -2 \quad (26)$$

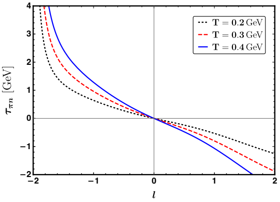
(27)

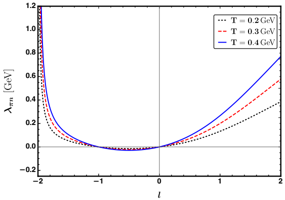
[3] A. Jaiswal, B. Friman, K. Redlich, ***Phy Lett B 751 (2015)***

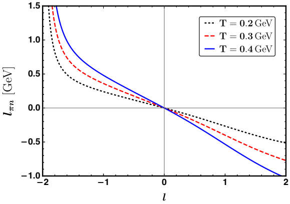
Transport coefficients

$$\lambda_{\pi n} = \frac{\ell(\ell+1)T\Gamma(5+\ell)\left\{-\Gamma(4+\ell)\Gamma(5+\ell)+48\Gamma(4+2\ell)\right\}}{60(\ell^2-\ell+4)\Gamma(3+\ell)\Gamma(5+2\ell)}, \quad \ell > -2 \tag{28}$$

- In the massless MB limit, the coefficients of the term $\tau_{\pi n} n^{\langle\mu}\dot{u}^{\nu\rangle}$, $l_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle}$ and $\lambda_{\pi n} n^{\langle\mu}\nabla^{\nu\rangle}\alpha$ are plotted below [4]:







[4] S. Singh, M. Kurian, V. Chandra, ***Phy. Rev. D 110, 014004 (2024)***

Exact calculations of transport coefficients

- Some recent studies [5],[6] analytically extracted a full set of eigenvalues and eigenfunctions of the relativistic linearized Boltzman collision operator for $\lambda\phi^4$ theory.

$$\hat{L}\phi_k = \frac{g}{2}\int dK' dP dP' f_{0k'}(2\pi)^5\delta^{(4)}(k+k'-p-p')(\phi_p+\phi_{p'}-\phi_k-\phi_{k'}). \tag{29}$$

- Where the eigenfunctions and thier eigenvalues are given by:

$$\hat{L}L_{nk}^{(2m+1)}k^{\langle\mu_1}\ldots k^{\mu_\ell\rangle} = -\frac{g\mathcal{M}}{2}\left[\frac{n+m-1}{n+m+1}+\delta_{\ell 0}\delta_{n0}\right]L_{nk}^{(2m+1)}k^{\langle\mu_1}\ldots k^{\mu_m\rangle}, \tag{30}$$

- Expanding ϕ_k in terms of these eigenfunctions and keeping the terms with zero eigenvalues leads to the collision kernel being:

$$\hat{L}\phi_k = -\frac{g\mathcal{M}}{2}\left[\phi_k - c_0 - c_1 L_{1k}^{(1)} - c_0^\mu k_{<\mu>}\right] \tag{31}$$

- Recovering the RTA limit from the exact theory leads to the form of momentum-dependent relaxation time being:

$$\tau_R(p) = \frac{2(u\cdot p)}{g\mathcal{M}}$$

Which implies $\ell = 1$ and $\kappa = 4\pi^2/(ge^\alpha)$ in the corresponding ERTA framework.

[5] Gabriel S. Rocha, Caio V.P. de Brito, and Gabriel S. Denicol, ***Phys. Rev. D 108, 036017 (2023)***

[6] Gabriel S. Rocha, Gabriel S. Denicol, and Jorge Noronha, ***Phys. Rev. Lett. 127, 042301 (2021)***

Comparision with self interacting $\lambda\phi^4$ theory

Coefficients	RTA results $(l=0)$	ERTA results $(l=1)$	$\lambda\phi^4$ results (exact)
τ_π	τ_c	$\frac{24d_g}{gn_0\beta^2}$	$\frac{72}{gn_0\beta^2}$
η	$\frac{4P\tau_c}{5}$	$\frac{16d_g}{g\beta^3}$	$\frac{48}{g\beta^3}$
κ	$\frac{n_0\tau_c}{12}$	$\frac{d_g}{g\beta^2}$	$\frac{3}{g\beta^2}$
$\delta_{\pi\pi}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
$\tau_{\pi\pi}$	$\frac{10}{7}$	2	2
$l_{\pi n}$	0	$-\frac{4}{3\beta}$	$-\frac{4}{3\beta}$
$\tau_{\pi n}$	0	$-\frac{16}{3\beta}$	$-\frac{16}{3\beta}$
$\lambda_{\pi n}$	0	$\frac{2}{3\beta}$	$\frac{5}{6\beta}$

Table: Comparison of the ERTA coefficients with exact results from $\lambda\phi^4$ theory.

Electromagnetic field

- The electromagnetic field tensor $F^{\mu\nu}$ and its dual $\tilde{F}^{\mu\nu}$ can also be decomposed into components parallel and perpendicular to the fluid velocity:

$$F^{\mu\nu} = E^{\mu}u^{\nu} - E^{\nu}E^{\mu} + \epsilon^{\mu\nu\alpha\beta}u_{\alpha}B_{\beta} \tag{32}$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = B^{\mu}u^{\nu} - B^{\nu}u^{\mu} - \epsilon^{\mu\nu\alpha\beta}u_{\alpha}E_{\beta} \tag{33}$$

  Where $E^{\mu} = F^{\mu\nu}u_{\nu}$ and $B^{\mu} = \tilde{F}^{\mu\nu}u_{\nu}$.

- In the non-resistive limit, we take the electric field $E^{\mu} \to 0$ so that induced current doesn't blow up. This leads to following evolution equation obtained from Maxwell's equations:

$$\epsilon^{\mu\nu\alpha\beta}\left(u_{\alpha}\partial_{\mu}B_{\beta} + B_{\beta}\partial_{\mu}u_{\alpha}\right) = J^{\nu} \tag{34}$$

$$\dot{B}^{\mu} + B^{\mu}\theta = u^{\mu}\partial_{\nu}B^{\nu} + B^{\nu}\nabla_{\nu}u^{\mu}. \tag{35}$$

Equations of motion for MHD

- In this non-resistive limit,

$$T^{\mu\nu}_{em} = \frac{B^2}{2}(u^\nu u^\nu - \Delta^{\mu\nu} - 2b^\mu b^\nu) \tag{36}$$

- With the maxwell's equations, the conservation of total $T^{\mu\nu} = T^{\mu\nu}_{em} + T^{\mu\nu}_{f}$ leads to:

$$\partial_\mu T^{\mu\nu}_f = F^{\mu\lambda} J_{f,\lambda} \tag{37}$$

- The fluid equations of motion are now given by:

$$\dot{\varepsilon} + (\varepsilon + P)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0, \tag{38}$$

$$(\epsilon + P)\dot{u}^\mu - \nabla^\mu P + \Delta^\mu_\nu \partial_\gamma \pi^{\gamma\nu} = -Bb^{\nu\lambda} n_\lambda, \tag{39}$$

$$\dot{n} + n\theta + \partial_\mu n^\mu = 0 \tag{40}$$

- The Boltzmann equation in the presence of the external magnetic fields and using the extended relaxation time is given by:

$$p^\mu \partial_\mu f - qB^{\sigma\nu} p_\nu \frac{\partial f}{\partial p^\sigma} = -\frac{u \cdot p}{\tau_R(x,p)}(f - f_0^*) \tag{41}$$

Transport coefficients in MHD

- Again, doing a Chapman Enskog-like gradient expansion, we get the $Delta f_{(2)}$ needed to derive $\pi^{\mu\nu}$ evolution as:

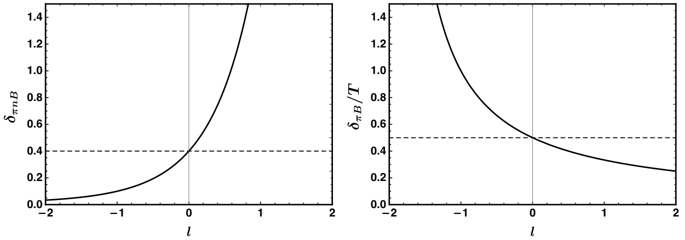
$$\Delta f = -\frac{\tau_R}{(u\cdot p)} p^\gamma \partial_\gamma f_0 - \frac{\tau_R}{(u\cdot p)} p^\gamma \partial_\gamma \delta f_{(1)} + \frac{\tau_R}{(u\cdot p)} qBb^{\sigma\nu} p_\nu \frac{\partial}{\partial p^\sigma} \delta f_{(1)} + \frac{\tau_R}{(u\cdot p)} qBb^{\sigma\nu} p_\nu \frac{\partial f_0}{\partial p^\sigma} + \Delta f^*_{(2)}, \quad (42)$$

- Using the above Δf, the second order shear evolution equation is given by:

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \tau_{\pi\pi}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \tau_{\pi n} n^{\langle\mu}\dot{u}^{\nu\rangle} + \lambda_{\pi n} n^{\langle\mu}\nabla^{\nu\rangle}\alpha + l_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} + \delta_{\pi B}\Delta^{\mu\nu}_{\eta\beta} qBb^{\gamma\eta} g^{\beta\rho}\pi_{\gamma\rho} - qB\tau_{\pi n B}\dot{u}^{\langle\mu}b^{\nu\rangle\sigma}n_\sigma - qB\lambda_{\pi n B} n_\sigma b^{\sigma\langle\mu}\nabla^{\nu\rangle}\alpha - q\tau_0\delta_{\pi n B}\nabla^{\langle\mu}\left(B^{\nu\rangle\sigma}n_\sigma\right) \quad (43)$$

Results

- The plots for two of these coefficients, $\delta_{\pi B}$ and $\delta_{\pi n B}$ against the momentum dependence parameter l is:



- We see that both of these coefficients tend to their limiting case values for RTA at $\delta_{\pi B} \to \frac{\beta}{2}$ and $\delta_{\pi n B} \to 2/5$ respectively [7].

[7] A. Panda, A.Dash, R. Biswas, V.Roy, ***JHEP 03, 216 (2021)***

The Navier Stokes limit

- In the Navier Stokes' limit which gives $\pi^{\mu\nu}$ in the first order theory, without magnetic fields:

$$\pi^{\mu\nu} = 2\eta_0\sigma^{\mu\nu} \tag{44}$$

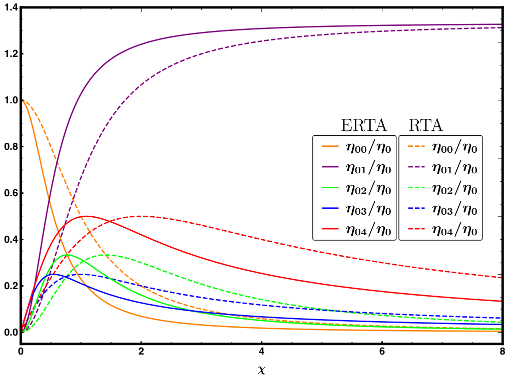
- In the first order, the shear evolution in case of MHD becomes:

$$\left(\frac{g^{\mu\gamma}g^{\nu\rho}}{\tau_\pi} - \delta_{\pi B}\Delta^{\mu\nu}_{\eta\beta}qBb^{\gamma\eta}g^{\beta\rho}\right)\pi_{\gamma\rho} = 2\beta_\pi\sigma^{\mu\nu}. \tag{45}$$

- With a finite magnetic field, the shear viscosity splits into five components:

$$\pi^{\mu\nu} = \Big[2\eta_{00}\left(\Delta^{\mu\alpha}\Delta^{\nu\beta}\right) + \eta_{01}\left(\Delta^{\mu\nu} - \frac{3}{2}\Xi^{\mu\nu}\right)\left(\Delta^{\alpha\beta} - \frac{3}{2}\Xi^{\alpha\beta}\right) - 2\eta_{02}\left(\Xi^{\mu\alpha}b^\nu b^\beta + \Xi^{\nu\alpha}b^\mu b^\beta\right) - 2\eta_{03}\left(\Xi^{\mu\alpha}b^{\nu\beta} + \Xi^{\nu\alpha}b^{\mu\beta}\right) + 2\eta_{04}\left(b^{\mu\alpha}b^\nu b^\beta + b^{\nu\alpha}b^\mu b^\beta\right)\Big]\sigma_{\alpha\beta}. \tag{46}$$

The Navier Stokes limit (comparision with RTA MHD results)



Where, $\chi = \frac{qB\tau_0(x)}{T}$.
As we see, the momentum dependence of the ERTA has a significant effect on the various shear viscosity coefficients even in the first order.

Conclusion

- This study shows that there is a significant impact of momentum dependence of the relaxation time on the dynamics of the fluid in both with and without magnetic field.
- Incorporating these affects via the modified transport coefficients should lead to a more accurate simulation of the expanding fireball in heavy ion collisions.
- Further work can be done in recognizing the momentum dependence parameter ℓ for various theories.
- The Magnetohydrodynamics of ERTA can be studied in the resistive case as an extension of this work.

Current work..

- Extension of the previous work is being conducted where the goal is to derive the evolution equation for number diffusion.
- This requires solving the system of equations given by the matching conditions to get δu^μ, $\delta\mu$ and δT.
- An example of this matching condition is the landau frame condition which will give us the necessary counter-terms that will be needed to satisfy the Landau frame condition even when the relaxation time is momentum dependent:

$$
\begin{aligned}
-\frac{I_{31}}{T}\delta u^\mu + (I_{30}-\mu I_{20})\frac{\delta T}{T^2}u^\mu + I_{20}\frac{\delta\mu}{T}u^\mu = {} & Au^\mu + B\nabla^\mu\alpha + C\sigma^\mu_\alpha\dot{u}^\alpha + D\dot{\sigma}^\mu_\alpha u^\alpha \\
& + E\sigma^\mu_\alpha\nabla^\alpha\alpha + F\sigma^\mu_\alpha\nabla^\mu\beta - \xi K_{31}\Delta^\mu_\alpha\partial_k\pi^{\alpha k} + G\dot{u}_\alpha(\nabla^\alpha u^\mu) + HD(\nabla^\mu\alpha) \\
& + I\nabla_\alpha\alpha(\nabla^\mu u^\alpha) + J\nabla^\alpha\sigma^\mu_\alpha
\end{aligned} \tag{47}
$$

- Where the coefficients are expressed in terms of various thermodynamic integrals.
- The other two matching conditions which needs to be satisified by addition of these counter-terms are $n = n_0$ and $\epsilon = \epsilon_0$.
- Various values of ℓ will be obtained depending on the theory and these transport coefficients will be predicted for those physical theories.

Thank you!