Simulations of Stochastic fluid dynamics near the QCD critical point

Chandrodoy Chattopadhyay

North Carolina State University

Physics Colloquium

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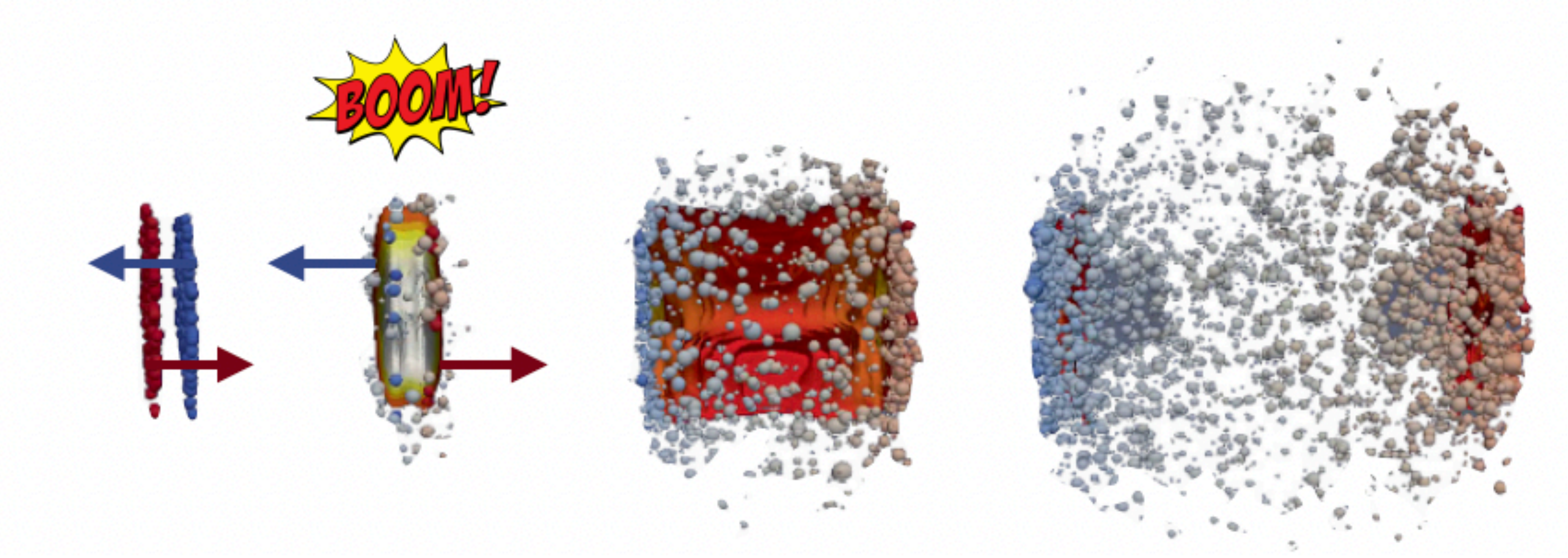
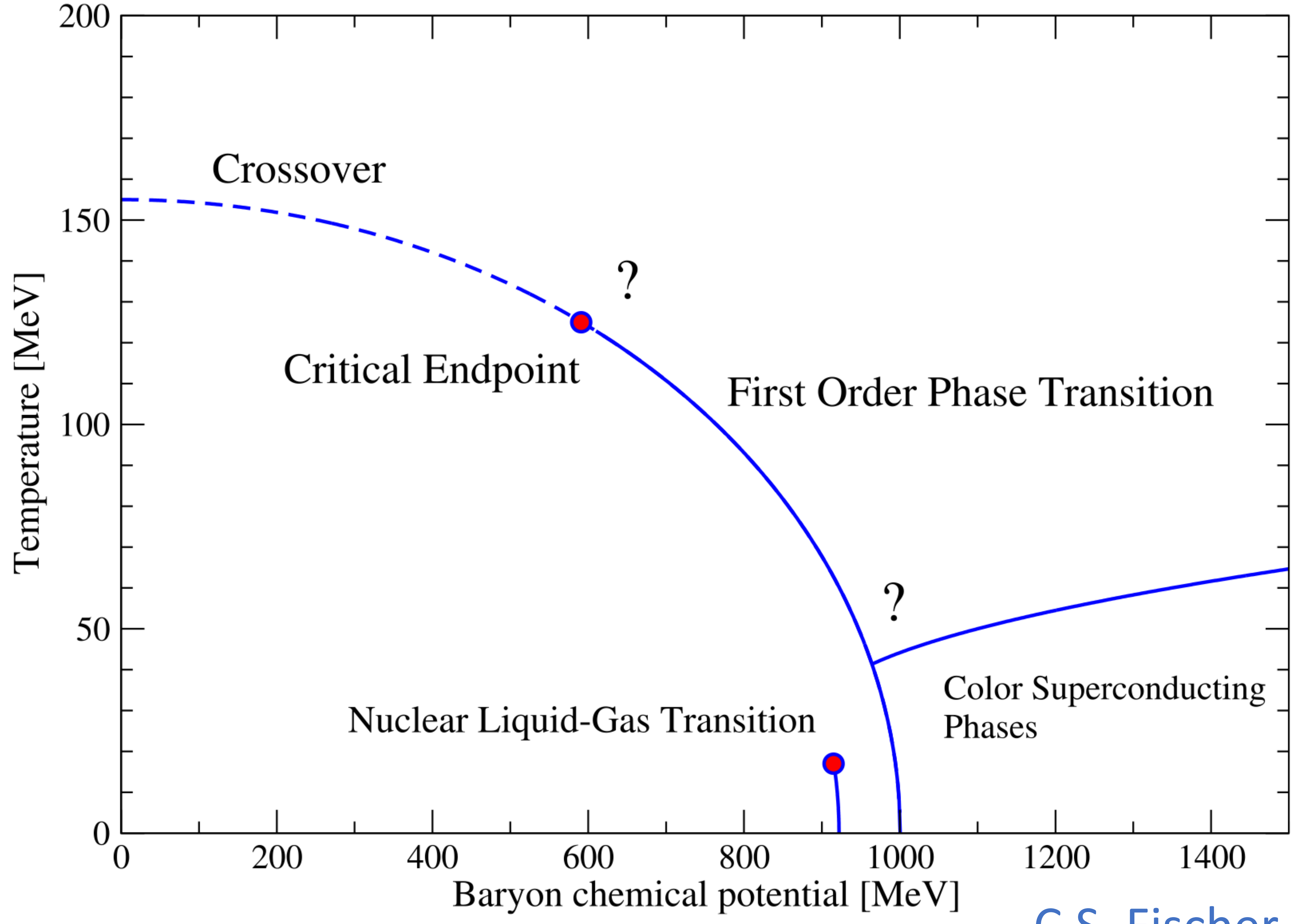
NC STATE

In collaboration with Josh Ott, Vladimir Skokov, and Thomas Schaefer

Introduction

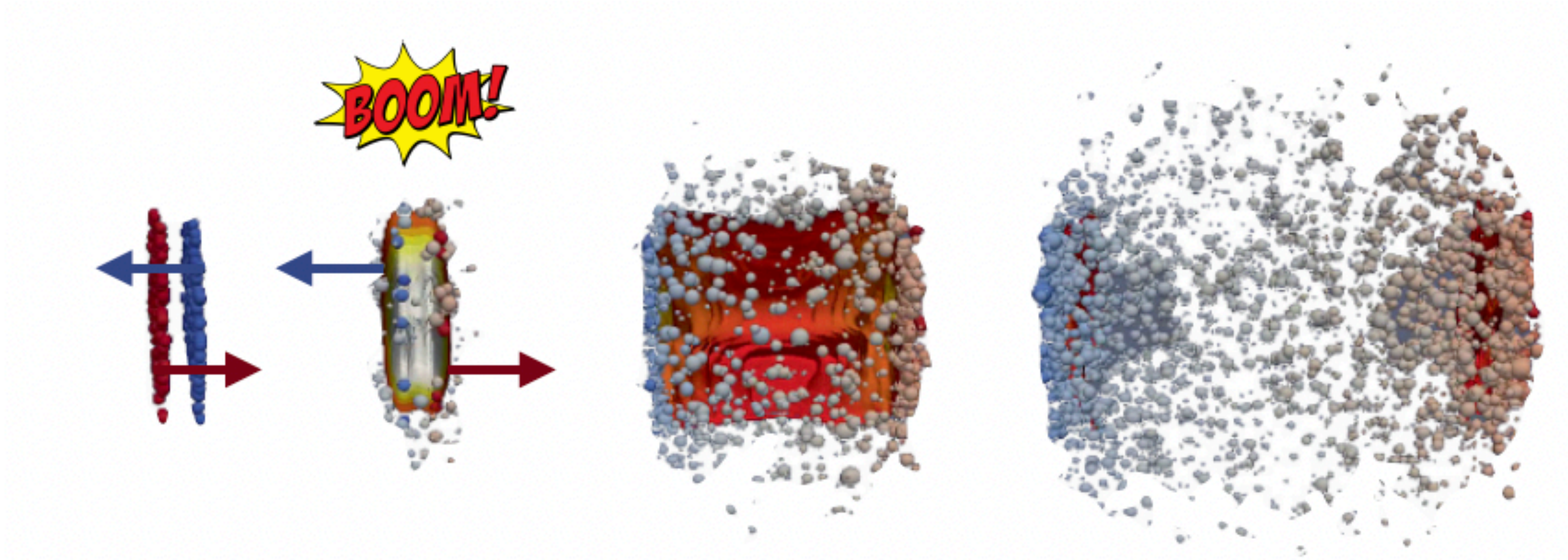


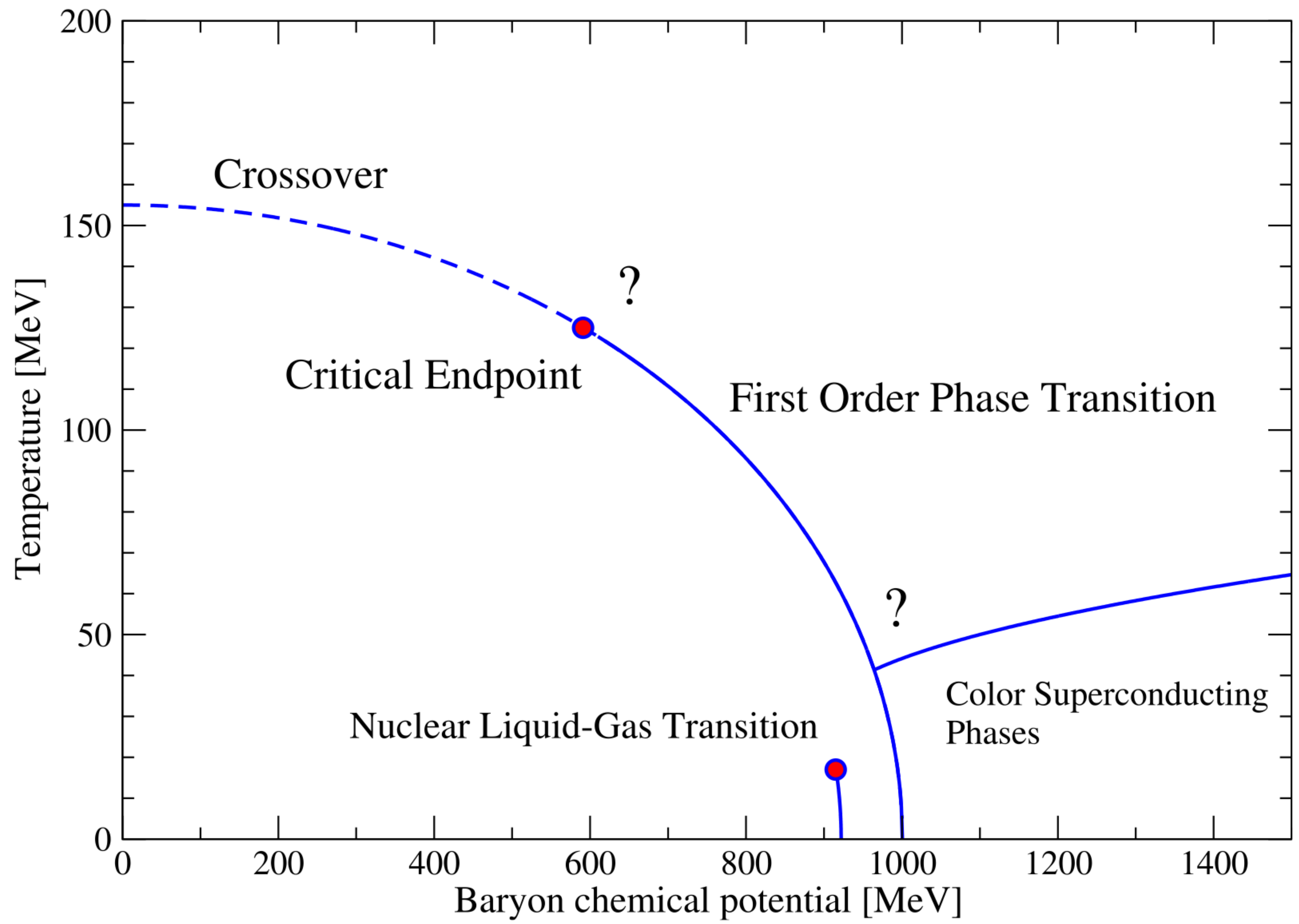


- Long-term goal: Identify signatures of a possible critical end point of QCD using heavy-ion collisions.
- Near a critical point, fluctuations become dominant. But fluctuations not equilibrated as fireball is rapidly expanding.
- Need for a dynamical theory of critical fluctuations.
- Fluid dynamics should still be applicable, but with appropriate modifications:
    - Inclusion of thermal fluctuations, slow dynamics of order parameter, and criticality in equation of state.

C.S. Fischer, Prog. Part. Nucl. Phys. 105, 1 (2019)

Introduction





- Dynamics of critical fluctuations are universal.
- Hence, study QCD critical dynamics using the simplest system from the same dynamic universality class.
- Universality class depends on
  - Order parameter being conserved/non-conserved.
  - Coupling of order parameter to other slow modes, eg, hydrodynamic modes.
- QCD critical point shares the same static universality class as the 3d Ising Model

The basic idea

- The properties of a fluid are defined by slow, macroscopic degrees of freedom: conserved densities, i.e., densities of energy, momentum, or any conserved charge.

- If a fluid is near a critical point, the dynamics of its order parameter becomes slow (critical slowing down). Must be included in the hydrodynamic description.

Hohenberg & Halperin

- These macroscopic fields fluctuate as they couple to microscopic degrees of freedom.

- The theory to be solved is then stochastic hydrodynamics coupled to an order parameter.

    - Such theories are classified by Hohenberg & Halperin: purely relaxational dynamics (Model A), critical diffusion (Model B), critical anti-ferromagnet (Model G), critical diffusion coupled to Navier-Stokes (Model H).

relevant to QCD

Rajagopal and Wilczek

Son and Stephanov

Previous works

- Use framework of non-critical stochastic hydro and include criticality in EOS and transport coefficients.
  - Deterministic approaches: The above framework can be used in linearized regime to write deterministic eqs for n-point equal time functions: Hydro+, Hydro++, hydro-kinetics.

    Stephanov, Yin, X. An, Akamatsu, Teaney, Mazeliaukas, F. Yan, H. U. Yee, Martinez, Schaefer…
  - Extend them to critical regime by replacing susceptibilities and relaxation-rates by their critical expectations. Numerical studies of one-dimensional expanding systems.

    M. Nahrgang et al., G. Pihan et al. , M. Bluhm, L. Du, Heinz and others
- Use of ε-expansions, functional renormalization group. Berges, Schlichting et al, Schweitzer, von Smekal, Chen, Tan, Fu, Roth, Ye
- Not many studies of direct simulation of critical fluid dynamics. A novel approach to simulate stochastic dynamics based on Metropolis has been recently formulated.

  Florio, Grossi, Soloviev, Teaney, Schaefer, Skokov, Basar, Bhambure, Singh, Newhall et al

Outline of this talk

- Main goal: Discuss numerical simulations of Model H, i.e., critical dynamics of a conserved order parameter coupled to fluid dynamic variables.

- Part I: critical diffusion of a conserved order parameter (Model B)

    - Simulation of diffusive dynamics using a Metropolis algorithm

    - Dynamic scaling in Model B

- Part II: Coupling of the conserved order parameter to hydrodynamic modes (Model H)

    - Modification to dynamic scaling behavior compared to Model B

    - Renormalization of shear viscosity of the fluid

- Part I: critical diffusion of a conserved order parameter (Model B)

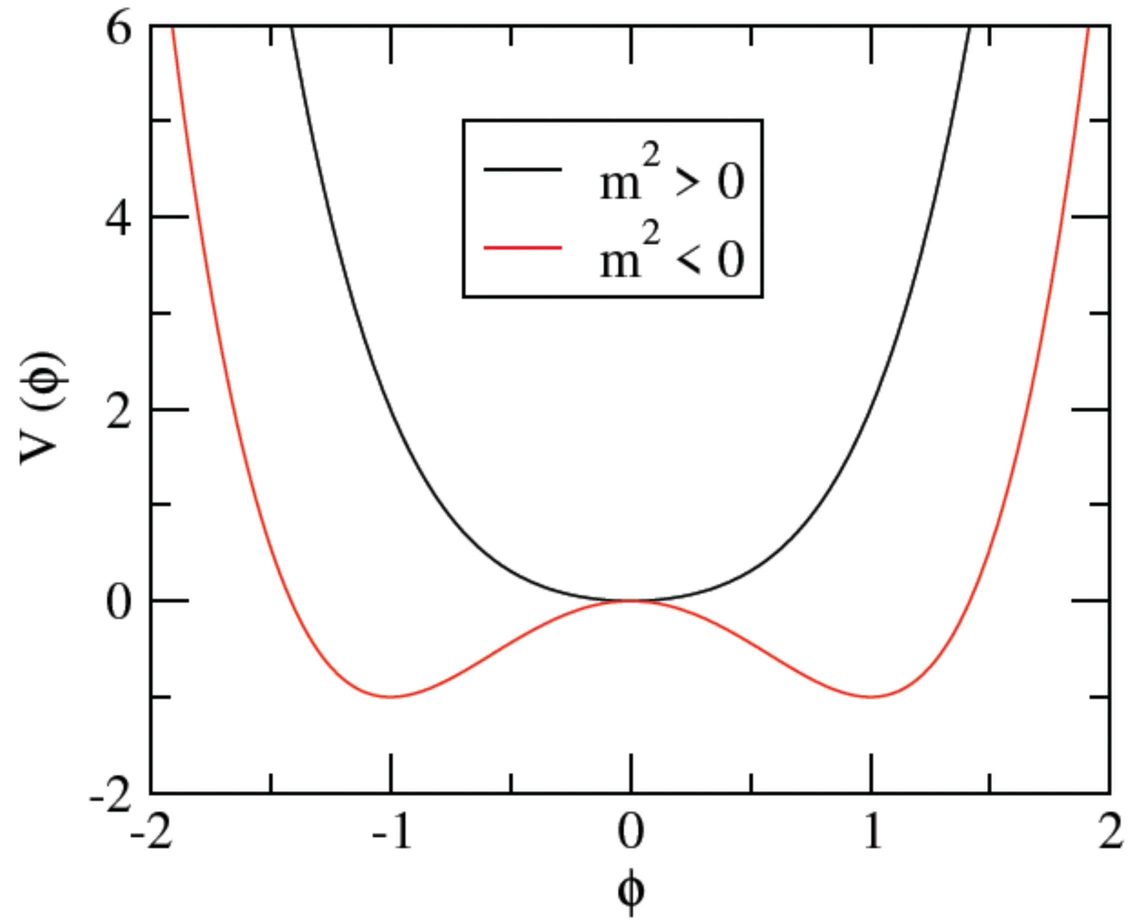
Based on C.C., J. Ott, T. Schaefer, V. Skokov
(PRD 108 (2023) 074004)

Model B

- Consider the Ising model. Coarse grain the spin (microscopic) degrees of freedom to obtain an order parameter $\phi(x)$ (magnetization density).

- The statics of the system near the critical point (small ϕ) is governed by an effective free-energy functional (Ginzburg-Landau)

$$F[\phi] = \int d^3x \left[\frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4}\phi^4\right]$$



- Dynamics: If the order parameter is conserved, its evolution may be modeled as

$$\frac{\partial\phi}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0,$$

the current

$$\vec{j} = -\Gamma\,\vec{\nabla}\frac{\delta F}{\delta\phi} + \vec{\xi}$$

Diffusion

Noise

Noise ensures fluctuation-dissipation

$$\langle \xi^i(t,\vec{x})\,\xi^j(t',\vec{x}')\rangle = 2\,\Gamma\,T\,\delta^{ij}\,\delta(t-t')\,\delta^3(\vec{x}-\vec{x}')$$

Model B in mean-field approximation

- In the free-energy functional set $\lambda = 0$

$$F[\phi] = \int d^3x \left[\frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4\right]$$

- Evolution of ϕ becomes linear. The equal-time correlator $N_k(t) = \langle\phi(t,\vec{k})\,\phi(t,-\vec{k})\rangle$ satisfies

$$\frac{\partial N_k}{\partial t} = -2\Gamma_k(N_k - N_k^{eq})$$

Equilibrium correlator $N_k^{eq} = \dfrac{T}{k^2+m^2}$ and relaxation-rate $\Gamma_k = \Gamma\, k^2(k^2+m^2)$

- Near $m^2 = 0$, mean-field predicts $\Gamma_k \sim k^z$ with a dynamic exponent $z = 4$.

- Later: interactions, coupling of ϕ to hydro modes lead to modifications from $z = 4$.

Model B: the non-linear case

- Interactions renormalize m^2. For chosen values of (T, λ) it is possible to tune m^2 to hit the critical point.

$$F[\phi] = \int d^3x \left[\frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4\right]$$

- To determine m_c^2 for an infinite system from finite volume calculations. Quantities like $\langle M^2\rangle$, $\langle M^4\rangle$ show peaks whose location depends on L.

- At the true critical point, leading order finite volume effects on the Binder cumulant U cancel

$$U \equiv 1 - \frac{\langle M^4\rangle}{3(\langle M^2\rangle)^2}$$

- Model B configs have long thermalization time $\tau_R \sim L^z$ with $z \approx 4$.

- Determine m_c^2 using Model A (purely relaxational dynamics), lies in same static universality class, easier to thermalize $\tau_R \sim L^2$.

T. Schaefer and V. Skokov PRD 014006 (2022)

Metropolis step for Model B

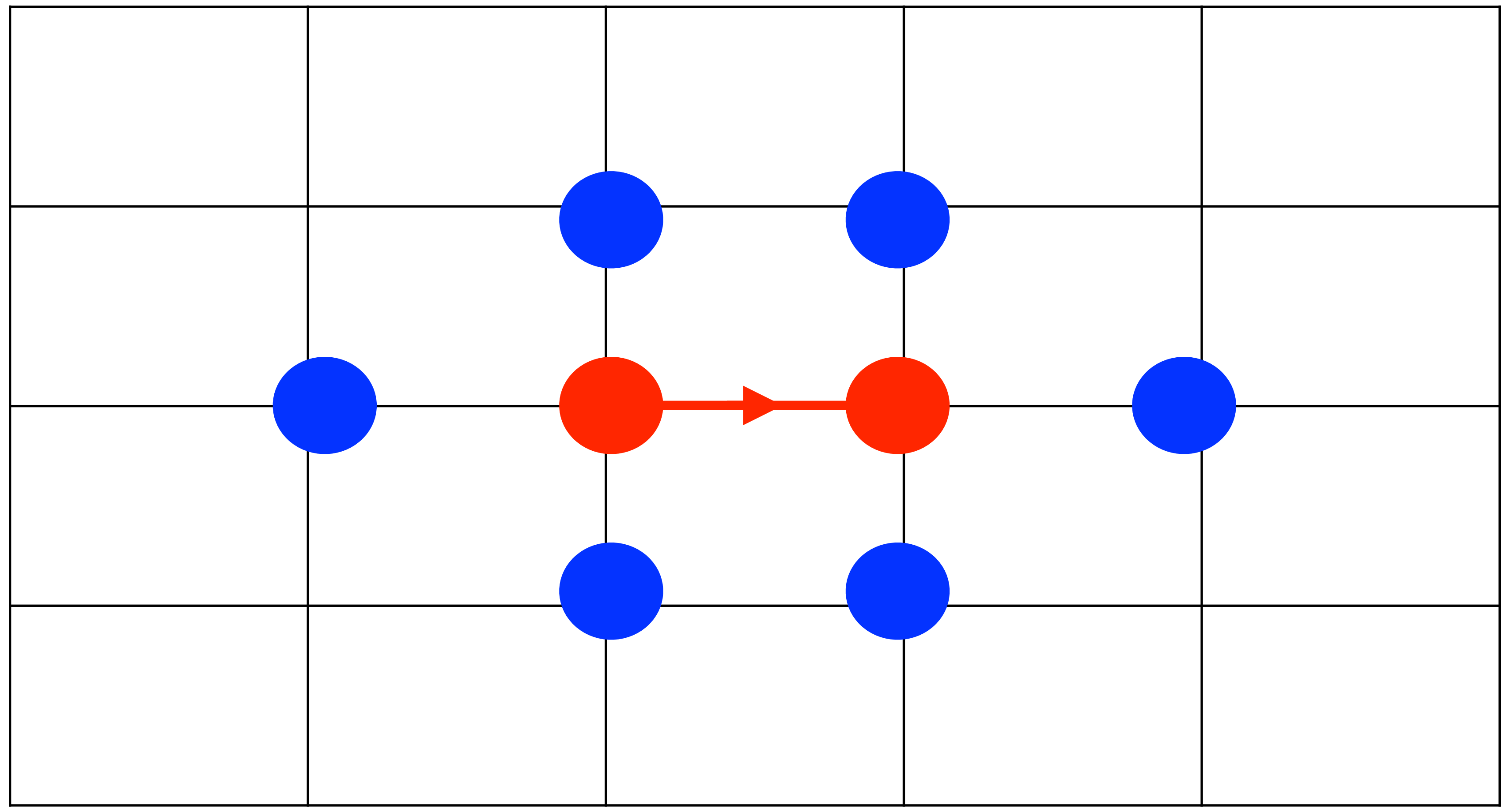
- Choose trial updates at $\vec{x}$ and $\vec{x} + \hat{\mu}$ (conserves ϕ)

$$\phi^{\text{trial}}(t + \Delta t, \vec{x}) = \phi(t, \vec{x}) - q_\mu, \qquad \phi^{\text{trial}}(t + \Delta t, \vec{x} + \hat{\mu}) = \phi(t, \vec{x} + \hat{\mu}) + q_\mu$$

$$q_\mu = \sqrt{2\,\Gamma\,T\,\Delta t}\;\xi_\mu$$

- Compute the change in free energy due to these updates

$$F[\phi] = \int d^3x \left[\frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4\right]$$



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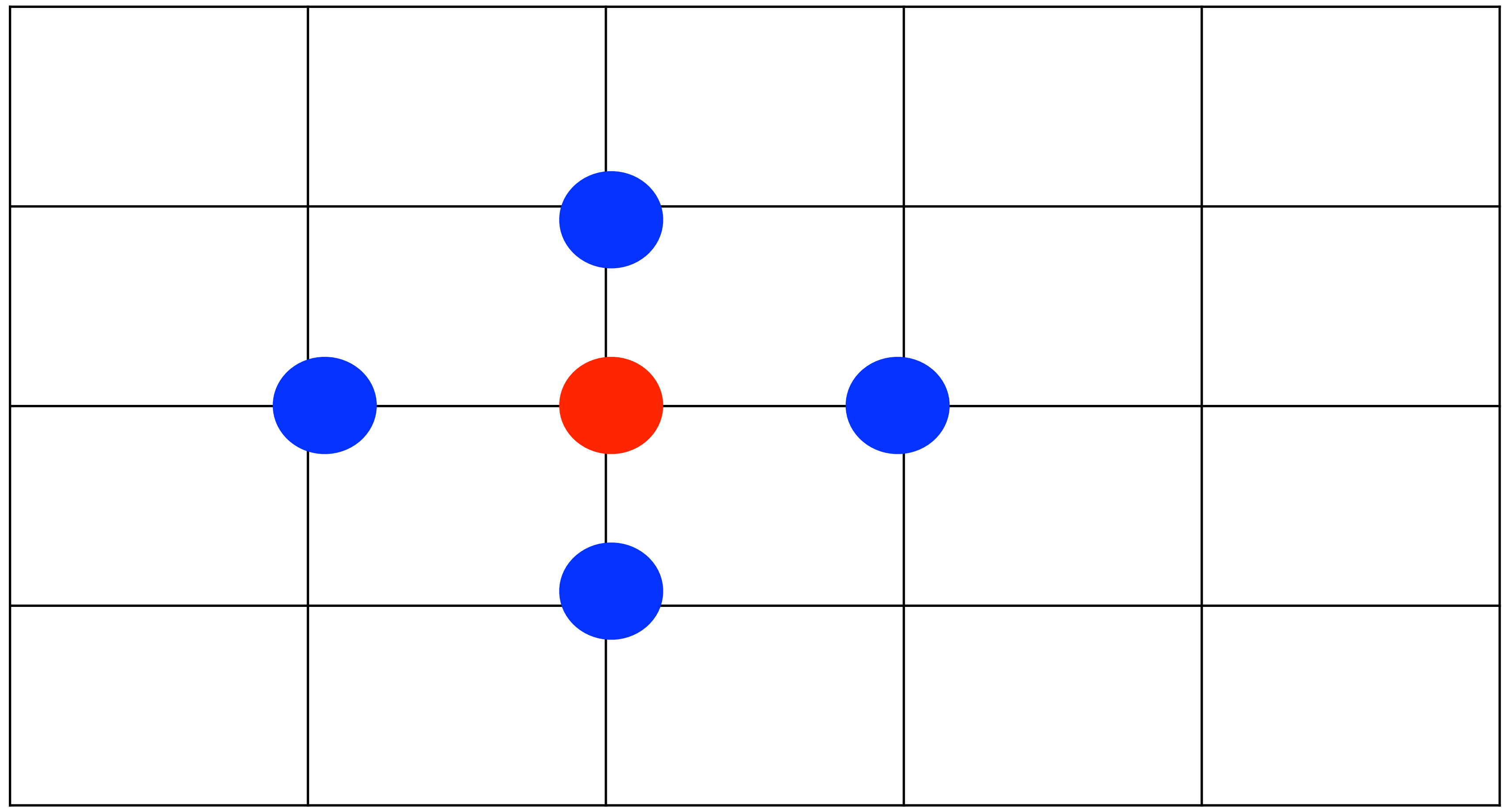
$$q_\mu = \sqrt{2\,\Gamma\, T\, \Delta t}\; \xi_\mu$$

- The change in free energy $\Delta F(\vec{x}, \vec{x} + \hat{\mu}) = \Delta F(\vec{x}) + \Delta F(\vec{x} + \hat{\mu}) + q_\mu^2$

$$\Delta F(x) = \left(d + \frac{m^2}{2}\right)\left(\phi_{\text{trial}}^2(x) - \phi^2(x)\right) + \frac{\lambda}{4}\left(\phi_{\text{trial}}^4(x) - \phi^4(x)\right)$$

$$-\left(\phi_{\text{trial}}(x) - \phi(x)\right)\sum_{\hat{\mu}=1}^{d}\left(\phi(x + \hat{\mu}) - \phi(x - \hat{\mu})\right)$$



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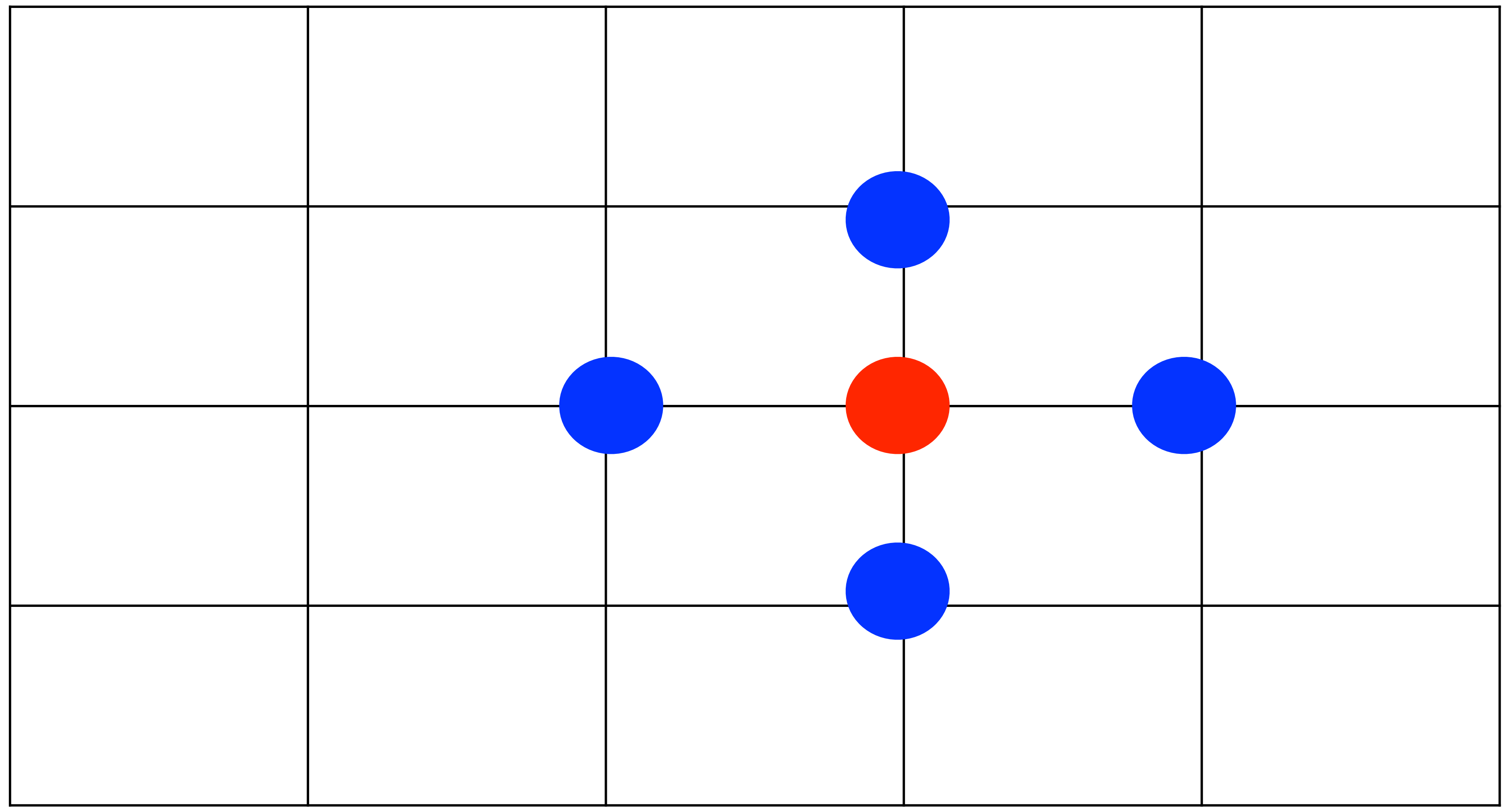


Metropolis step for Model B

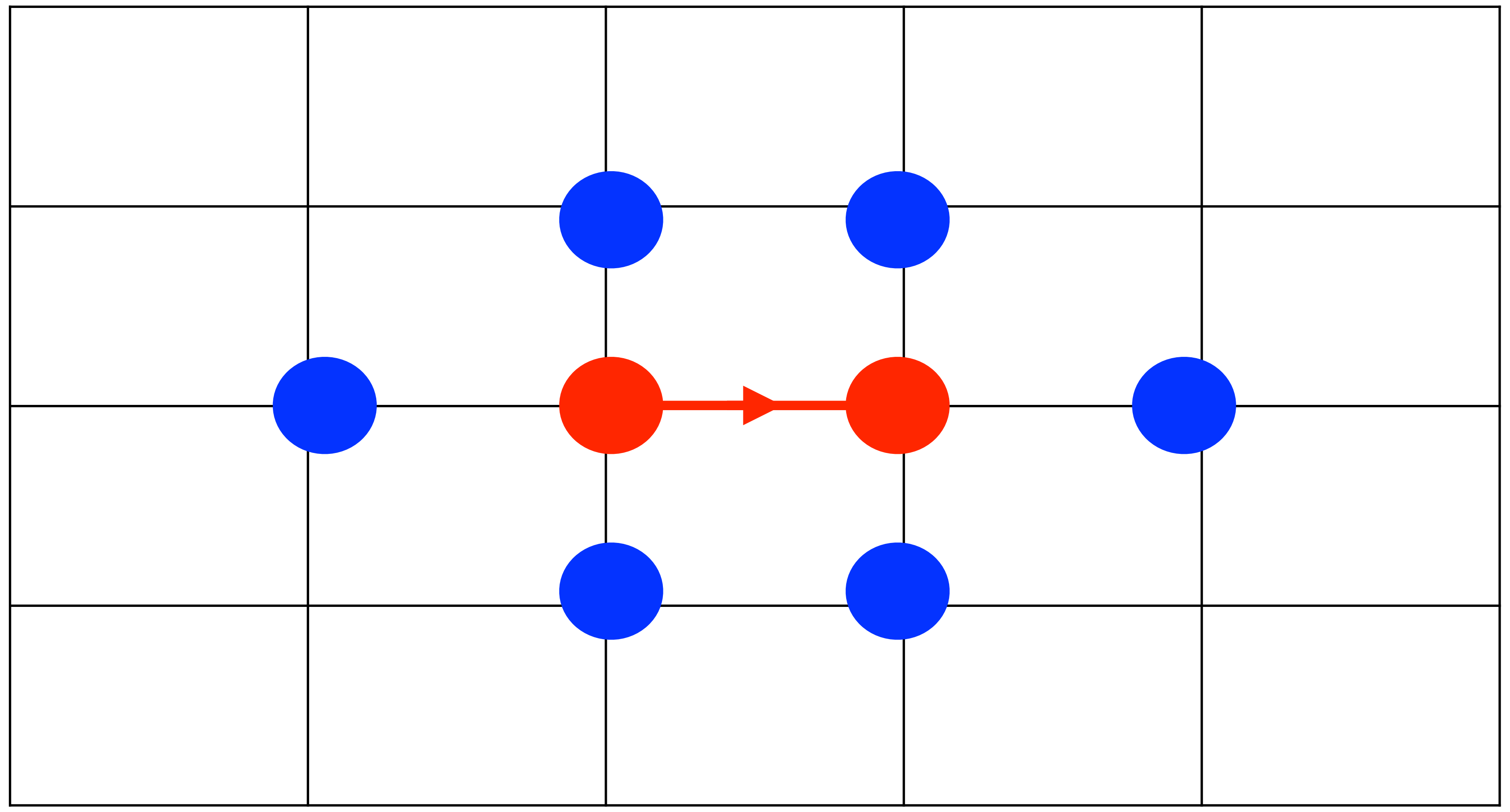
- Choose trial updates at $\vec{x}$ and $\vec{x} + \hat{\mu}$

$$\phi^{\text{trial}}(t + \Delta t, \vec{x}) = \phi(t, \vec{x}) - q_\mu, \qquad \phi^{\text{trial}}(t + \Delta t, \vec{x} + \hat{\mu}) = \phi(t, \vec{x} + \hat{\mu}) + q_\mu$$

$$q_\mu = \sqrt{2\Gamma T \Delta t}\, \xi_\mu$$

- The change in free energy $\Delta F(\vec{x}, \vec{x} + \hat{\mu}) = \Delta F(\vec{x}) + \Delta F(\vec{x} + \hat{\mu}) + q_\mu^2$

- Accept with probability $P = \min(1, \exp(-\Delta F/T))$



The Metropolis scheme

- The Metropolis update reproduces the flux on average, and also its variance

$$\langle \vec{q} \rangle = -\Delta t\, \Gamma\, \overrightarrow{\nabla} \frac{\delta F}{\delta \phi} + \mathcal{O}(\Delta t^2)$$

$$\langle \vec{q}^{\,2} \rangle = 2 \Gamma\, T\, \Delta t + \mathcal{O}(\Delta t^2)$$

- Probability of a new configuration,

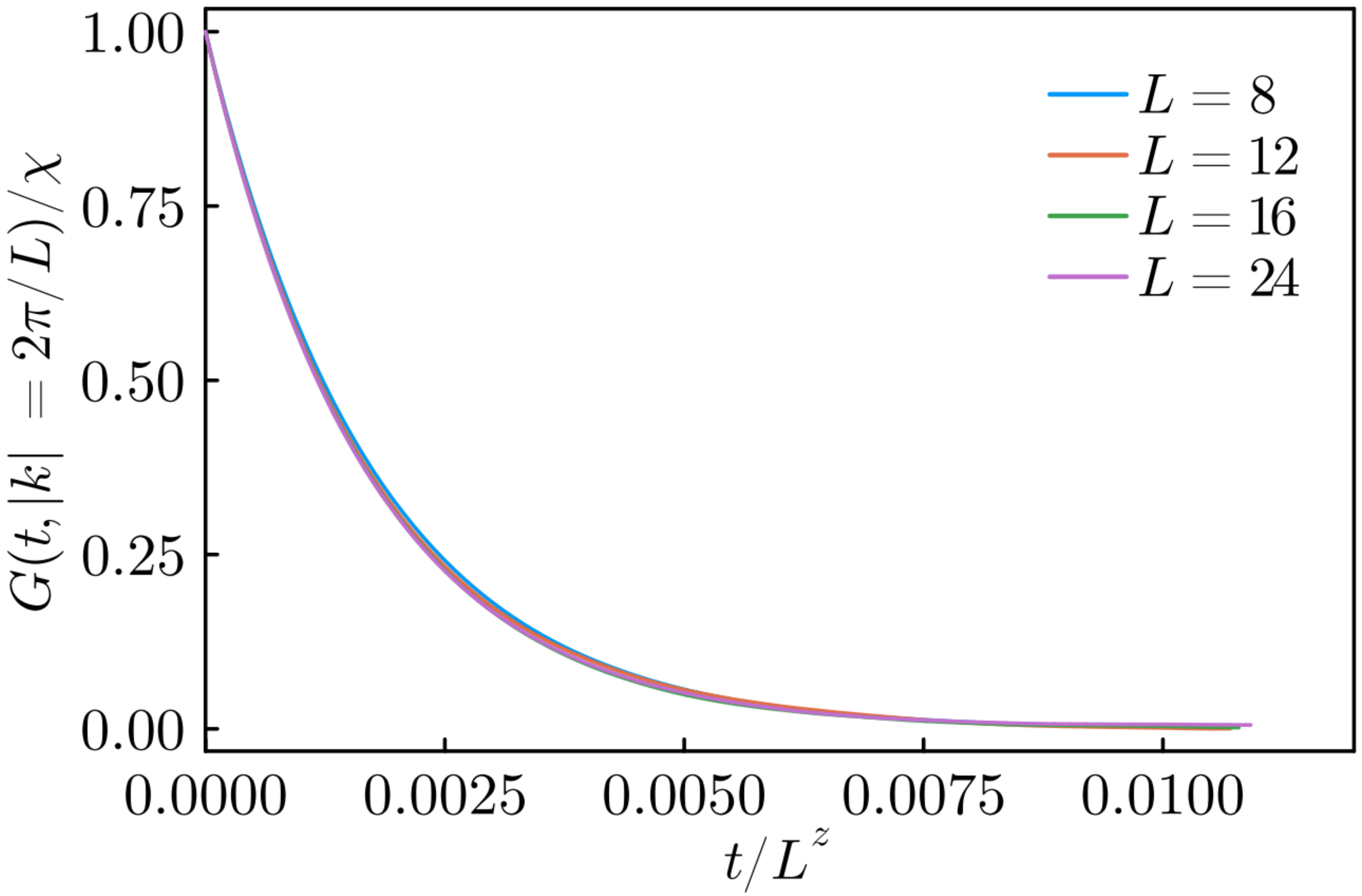
$$P\left(\phi(t,\vec{x}) \to \phi^{new}(t,\vec{x})\right) \sim \exp\left[-\left(F[\phi^{new}] - F[\phi]\right)\right]$$

irrespective of order of updates.

- The equilibrium distribution $\exp(-F[\phi]/T)$ is sampled even if Δt is not small.

- If Δt is not small, the diffusion eq. is approximately realized.

Results: Dynamic scaling



Data collapse occurs for $z \approx 3.97$. Theoretical expectation $z = 4 - \eta, \eta \approx 0.03$

- Scaling Hypothesis: Near a critical point the dynamic correlator, $\langle \phi(0,k)\,\phi(t,-k)\rangle$

$$G(t,k) = \tilde{G}(t/\xi^z, k\xi)$$

$\tilde{G}$ is a universal function.

- At the critical point $\xi \sim L$, thus $G(t,k)$ obtained in different volumes should collapse

$$G(t, k = 2\pi/L) \to \tilde{G}\left(\frac{t}{L^z}, 2\pi\right)$$

if time is scaled by L^z.

- z is the dynamic scaling exponent

- Part II: Coupling of the conserved order parameter to hydrodynamic modes (Model H)

Based on C.C., J. Ott, T. Schaefer, V. Skokov
PRL 133 (2024) 032301

Coupling to a fluid (Model H)

- Couple the order parameter ϕ to a fluid's momentum density $\vec{\pi}$

$$\frac{\partial \phi}{\partial t} = \Gamma \nabla^2 \frac{\delta H}{\delta \phi} - \left(\nabla \phi \cdot \frac{\delta H}{\delta \vec{\pi}_T} \right) + \zeta$$

diffusion advection noise

- Stochastic evolution equation of the momentum density

$$\frac{\partial \vec{\pi}_T}{\partial t} = \eta \nabla^2 \frac{\delta H}{\delta \vec{\pi}_T} + \left(\vec{\nabla} \phi \right) \cdot \frac{\delta H}{\delta \phi} - \left(\frac{\delta H}{\delta \vec{\pi}_T} \cdot \vec{\nabla} \right) \vec{\pi}_T + \vec{\xi}$$

diffusion Stress energy of ϕ advection noise

- The Hamiltonian $H = \int d^3x \left[\frac{\vec{\pi}_T^2}{2\rho} + \frac{1}{2} \left(\vec{\nabla} \phi \right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4 \right]$

Coupling to a fluid (Model H)

- Couple the order parameter to a fluid's momentum density $\vec{\pi}$

$$\frac{\partial \phi}{\partial t} = \Gamma \nabla^2 \frac{\delta H}{\delta \phi} - \left(\nabla \phi \cdot \frac{\delta H}{\delta \pi_T} \right) + \zeta$$

- Evolution equation of the momentum density

$$\frac{\partial \vec{\pi}_T}{\partial t} = \eta \nabla^2 \frac{\delta H}{\delta \vec{\pi}_T} + \left(\vec{\nabla} \phi \right) \cdot \frac{\delta H}{\delta \phi} - \left(\frac{\delta H}{\delta \vec{\pi}_T} \cdot \vec{\nabla} \right) \vec{\pi}_T + \vec{\xi}$$

- The Hamiltonian

$$H = \int d^3x \left[\frac{\vec{\pi}_T^2}{2\rho} + \frac{1}{2} \left(\vec{\nabla} \phi \right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4 \right]$$

For purposes of determining z It suffices to choose

- Non-relativistic fluid
- The momentum density is transverse $\vec{\nabla} \cdot \vec{\pi} = 0$

There are shear waves but no sound. No coupling to energy density or pressure.

Model H (deterministic part)

- Let's consider only the non-dissipative part of the equations

$$\frac{\partial \phi}{\partial t} + \frac{\vec{\pi}_T}{\rho} \cdot \vec{\nabla} \phi = 0, \qquad \frac{\partial \vec{\pi}_T}{\partial t} + \frac{\vec{\pi}_T}{\rho} \cdot \vec{\nabla} \vec{\pi}_T = \vec{\nabla} \phi \, \vec{\nabla}^2 \phi \longleftarrow$$

Third-order term, goes beyond usual Navier-Stokes

The third-order term is necessary for conserving energy

$$\frac{dH}{dt} = \int d^3x \left[\dot{\vec{\pi}}_T \cdot \frac{\vec{\pi}_T}{\rho} - \dot{\phi} \nabla^2 \phi + V'(\phi)\, \dot{\phi} \right] = 0$$

where the equations of motion have been used along with standard continuum manipulations

$$\int_x V'(\phi) \frac{\vec{\pi}_T}{\rho} \cdot \nabla \phi = \int_x \vec{\nabla} \cdot \left(\frac{\vec{\pi}_T}{\rho} V(\phi) \right) = 0 \qquad \frac{\pi_i^T}{\rho} \left(\frac{\pi_i^T}{\rho} \nabla_j \right) \pi_i^T = \nabla_i \left(\frac{\pi_i^T}{\rho} \frac{\pi_T^2}{2\rho} \right)$$

- These continuum manipulations are not necessarily allowed in the discretized theory.

Model H numerics (deterministic part)

- The equations in manifestly conserving form $\dot{\phi} = \vec{\nabla} \cdot \left(\frac{\vec{\pi}_T}{\rho} \phi \right)$ $\quad \dot{\pi}_i^T = -P_{ij}^T \nabla_k \left(\frac{1}{\rho} \pi_T^k \pi_T^j + \nabla_k \nabla_j \phi \right)$

- Use a skew symmetric derivative for the non-linear term

$$\nabla_\mu \left(\frac{1}{\rho} \pi_\mu^T \pi_\nu^T \right) \Bigg|_{skew} \equiv \frac{1}{2} \nabla_\mu \left(\frac{1}{\rho} \pi_\mu^T \pi_\nu^T \right) + \frac{1}{2} \frac{\pi_\mu^T}{\rho} \nabla_\mu \pi_\nu^T$$

Morinishi, Lund, Vasilyev, Moin, Journal of computational physics (143, 90 (1998)

along with a centred difference $\nabla_\mu^c \psi = (\psi(x + \hat{\mu}) - \psi(x - \hat{\mu})/2$

- The discretized evolution equations:

$$\dot{\phi} = -\frac{1}{\rho} \pi_T^\mu \nabla_\mu^c \phi, \qquad \dot{\pi}_T^\mu = -\left[\nabla_\mu \left(\frac{1}{\rho} \pi_\mu^T \pi_\nu^T \right) \Bigg|_{skew} + \left(\nabla_\mu^c \phi \right) \left(\nabla_\nu^c \nabla_\nu^c \phi \right) \right]$$

Model H numerics (deterministic part)

- The discretized eqs.

$$\dot{\phi} = -\frac{1}{\rho}\pi_T^\mu \nabla_\mu^c \phi \qquad \dot{\pi}_T^\mu = -\left[\nabla_\mu \left(\frac{1}{\rho} \pi_\mu^T \pi_\nu^T \right) \Big|_{skew} + \left(\nabla_\mu^c \phi \right) \left(\nabla_\nu^c \nabla_\nu^c \phi \right) \right]$$

conserves the kinetic energy of the system exactly:

$$\frac{dT}{dt} = \frac{d}{dt} \int d^3x \left[\frac{\pi_T^2}{2\rho} + \frac{(\nabla\phi)^2}{2} \right] = 0$$

- The equations are integrated in time using a Runge-Kutta scheme. After each step, project onto transverse part in Fourier space

$$\pi_\mu^T = P_{\mu\nu}^T \pi_\nu \qquad P_{\mu\nu}^T = \delta_{\mu\nu} + \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2}$$

- Total energy conservation in the deterministic step is found to hold to very good accuracy.

Model H numerics (stochastic /dissipative part)

- For the fluctuating/dissipative part:

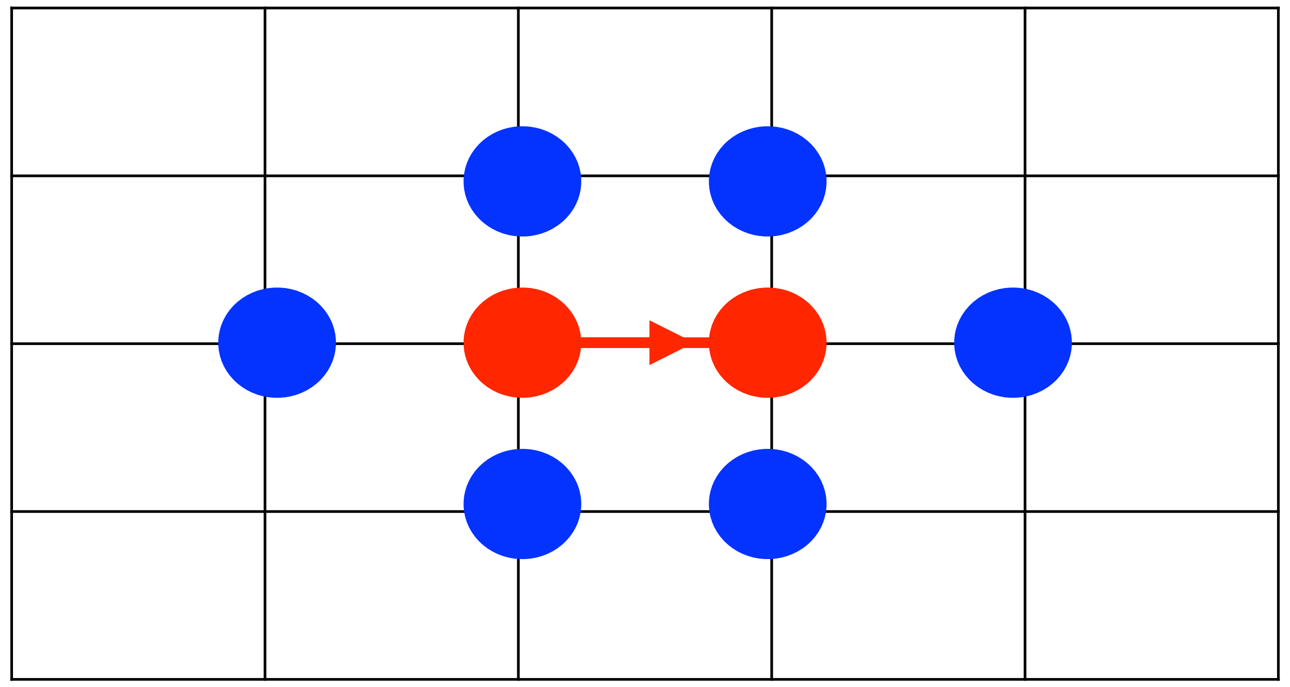
$$\phi^{\text{trial}}(t+\Delta t, \vec{x}) = \phi(t, \vec{x}) + q^{\mu}$$

$$\phi^{\text{trial}}(t+\Delta t, \vec{x}+\hat{\mu}) = \phi(t, \vec{x}+\hat{\mu}) - q^{\mu}$$

$$q^{\mu} = \sqrt{2\,T\,\Gamma\,\Delta t}\,\zeta_{\mu}$$

$$\langle \zeta^{\mu}\zeta^{\nu}\rangle = \delta^{\mu\nu}$$

Same as Model B update



- Similarly for the momentum densities:

$$\pi_{\mu}^{\text{trial}}(t+\Delta t, \vec{x}) = \pi_{\mu}(t, \vec{x}) + r_{\mu}^{(\nu)}$$

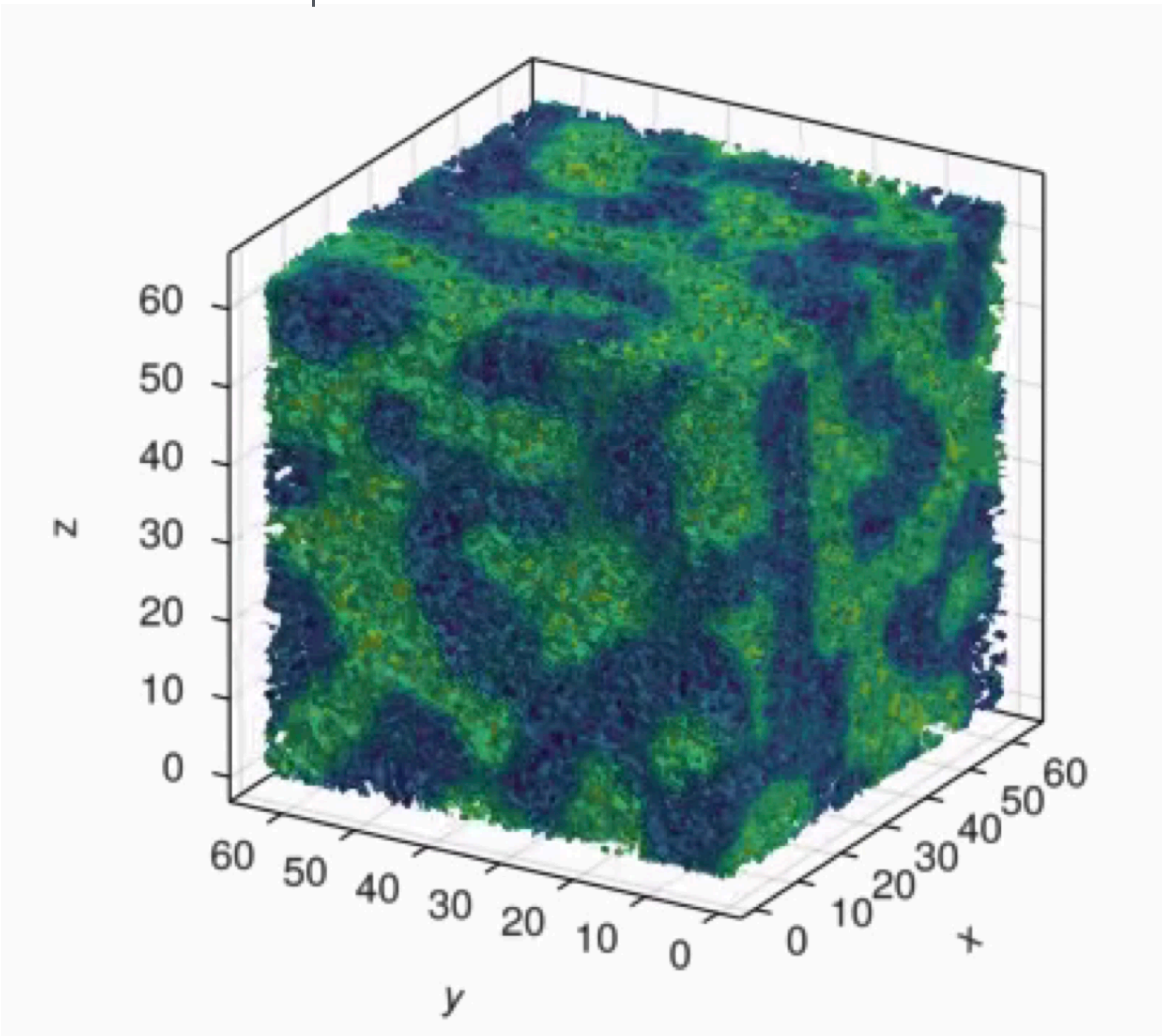
$$\pi_{\mu}^{\text{trial}}(t+\Delta t, \vec{x}+\hat{\nu}) = \pi_{\mu}(t, \vec{x}+\hat{\nu}) - r_{\mu}^{(\nu)}$$

$$r_{\mu}^{(\nu)} = \sqrt{2\eta T\,\Delta t}\,\zeta_{\mu}^{(\nu)}$$

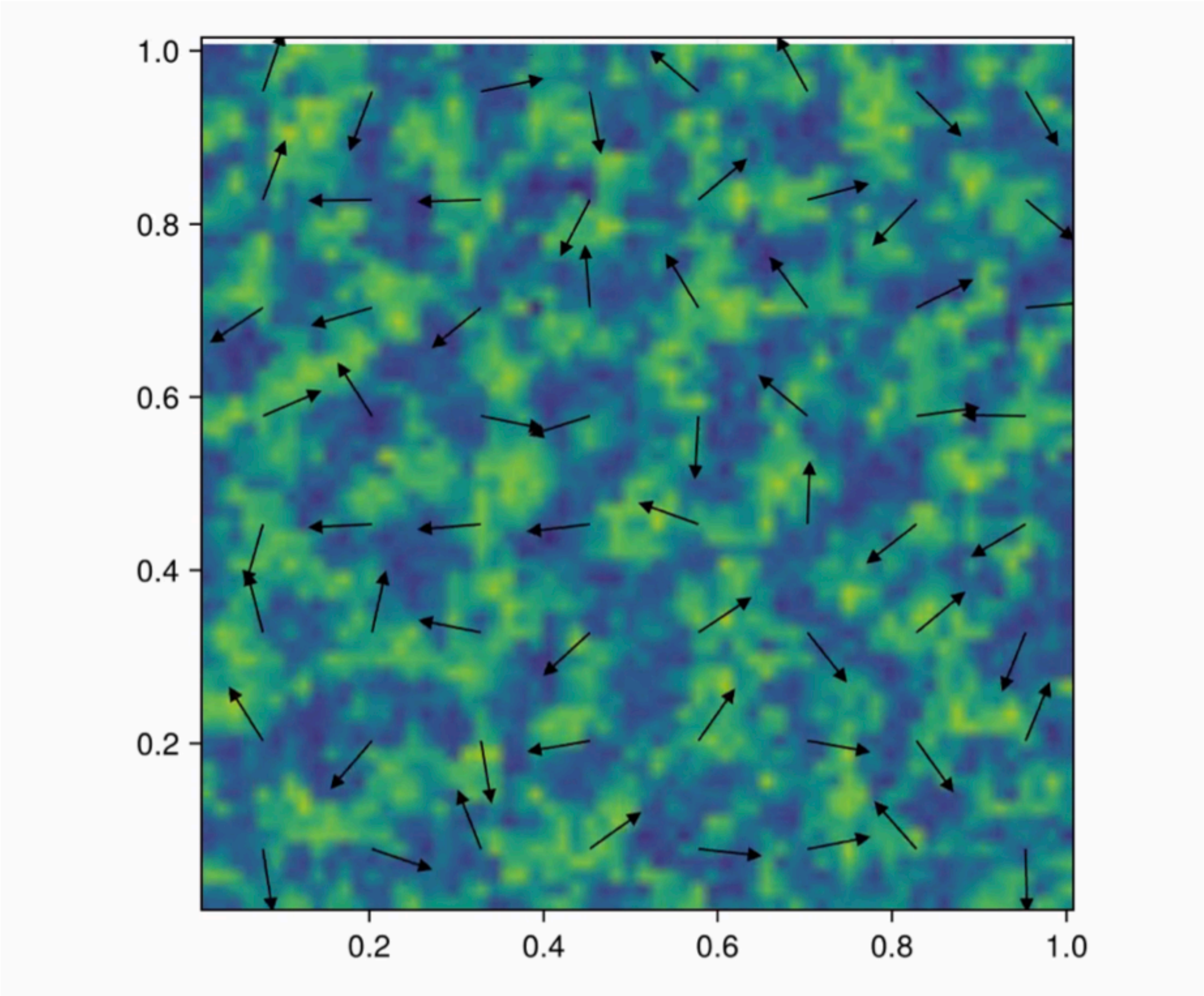
- Calculate change in energy. Accept/reject with $P = \min(1, \exp(-\Delta H/T))$

Model H simulations

Order parameter field in 3d



Order parameter + velocity field in 2d



Simulations by Josh Ott

Results: Dynamics of momentum density

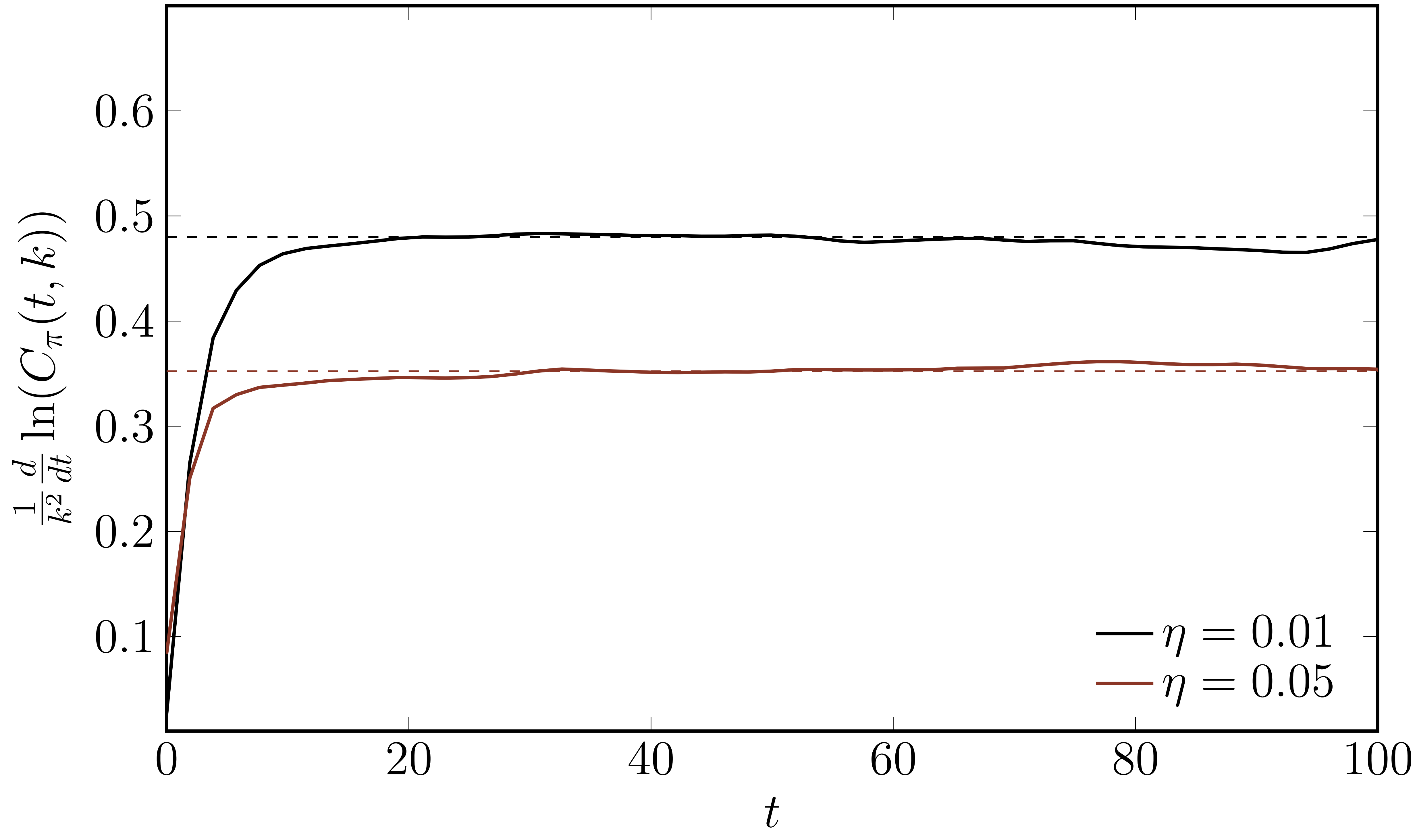
- Consider the time-dependent correlation function of the momentum density

$$\langle \pi_i^T(0,\vec{k})\, \pi_j^T(0,-\vec{k})\rangle \equiv C_{ij}(t,\vec{k}), \quad \text{where} \quad C_{ij}(t,\vec{k}) = \left(\delta_{ij} - \hat{k}_i\hat{k}_j\right) C_\pi(t,k)$$

- In linearized hydrodynamics $C_\pi(t,k) = \rho\, T \exp\left(-\frac{\eta}{\rho}k^2\, t\right)$

- Compute $C_\pi(t,k)$ in Model H to extract effective η

- Thermal fluctuations and non-linear effects modify linear hydro result (even away from T_c)



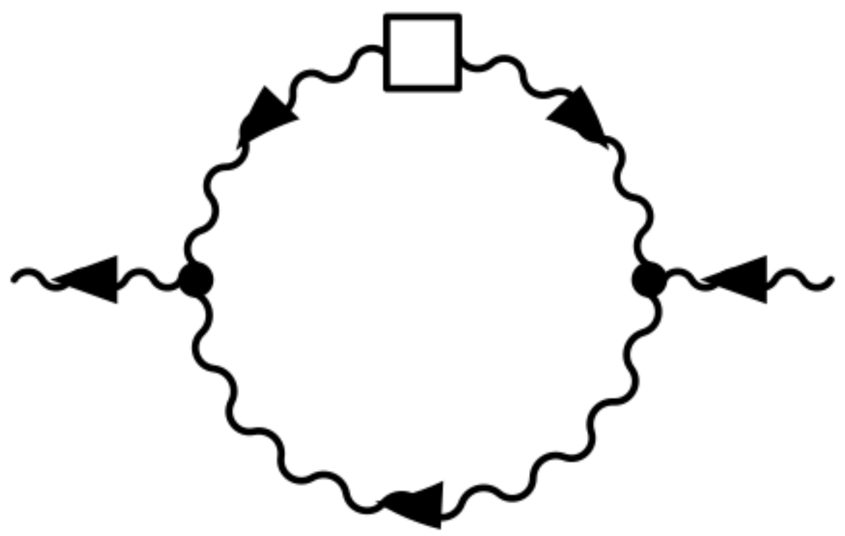
Dynamics: Loop corrections

Non-linear interactions between modes $\vec{\pi}_T, \phi$ can be represented diagrammatically

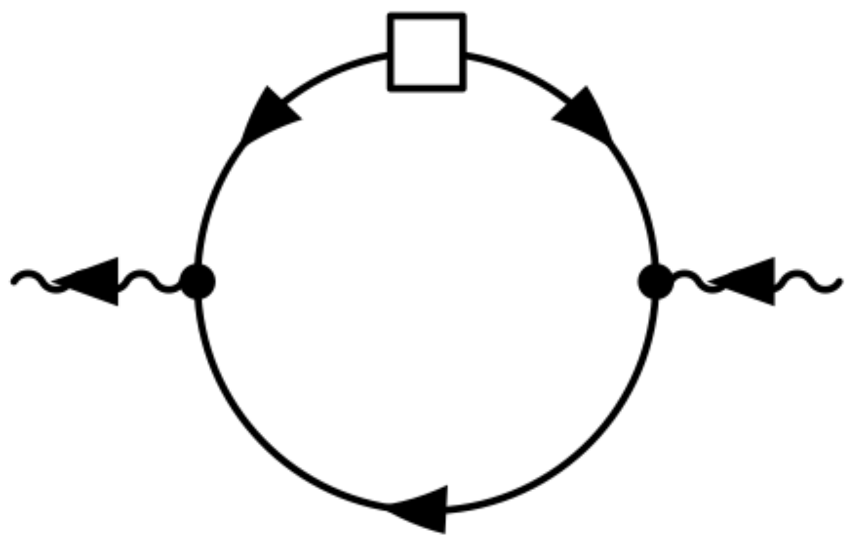
Green's functions for π_T 

Green's functions for ϕ

Corrections to momentum corr. function

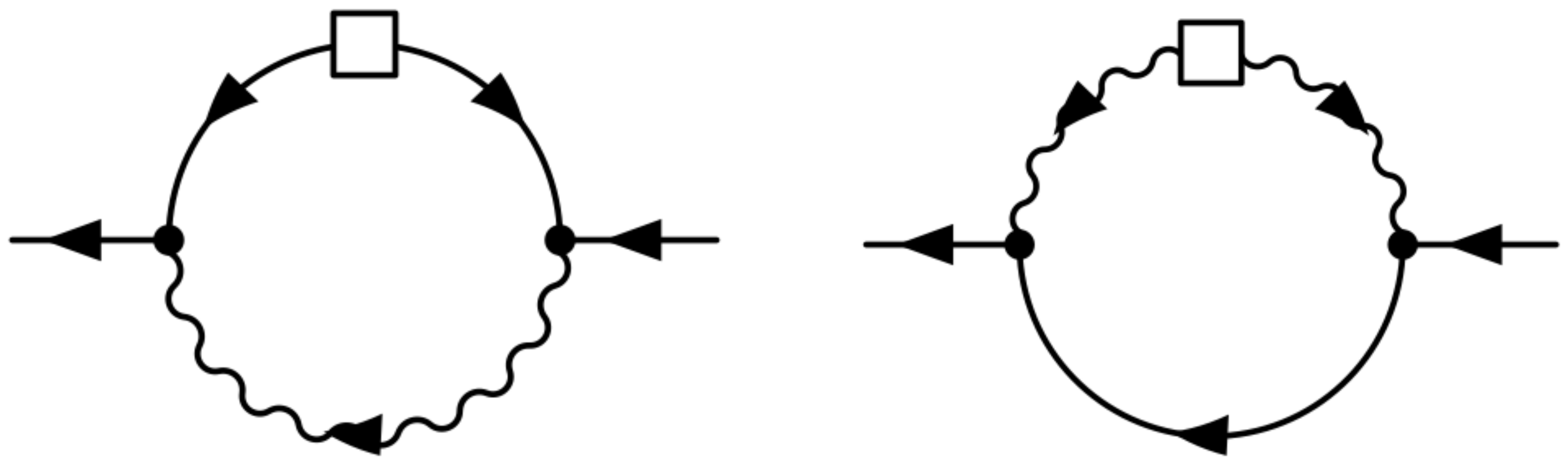


Self-advection of π_T



Coupling of π_T to ϕ

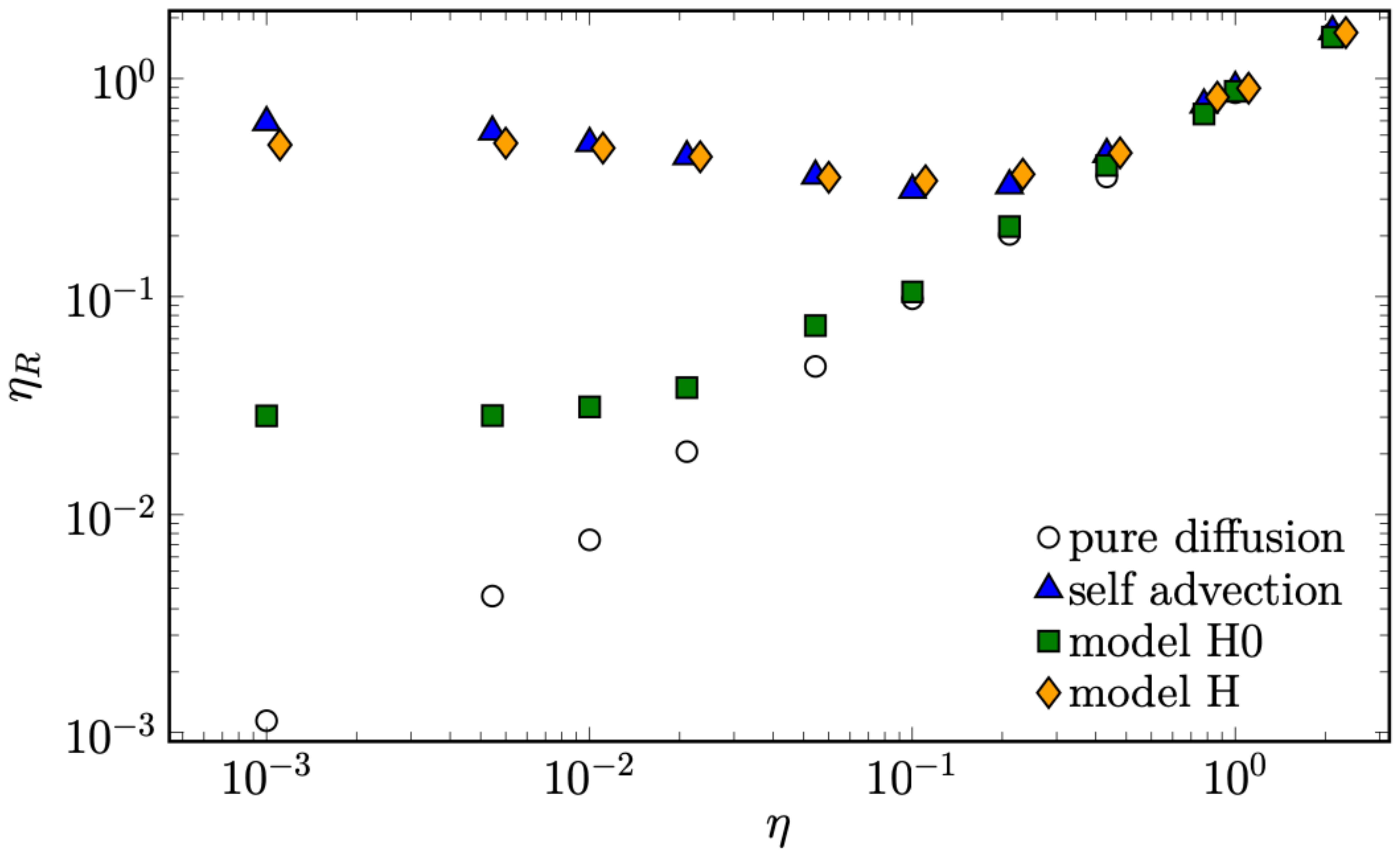
Corrections to corr. function of ϕ



Advection of ϕ by π_T

Renormalized viscosity

- For pure diffusion, the eq. is linear
- Effective viscosity becomes as small as the bare one

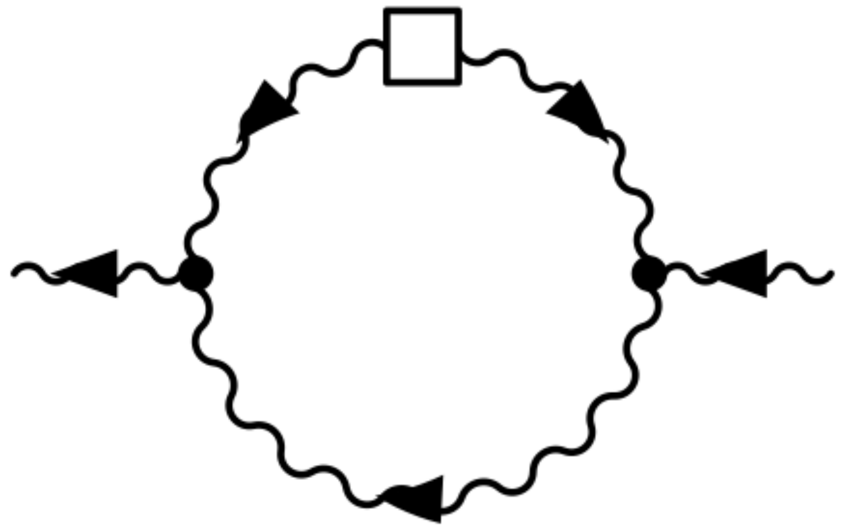


Pure-diffusion

$$\frac{\partial \vec{\pi}_T}{\partial t} + \cancel{\frac{\vec{\pi}_T}{\rho} \vec{\nabla}} \vec{\pi}_T = \frac{\eta}{\rho} \nabla^2 \vec{\pi}_T + \cancel{\vec{\nabla} \phi \nabla^2 \phi} + \vec{\xi}$$

Renormalized viscosity

The "stickiness of shear" Schaefer & Chafin

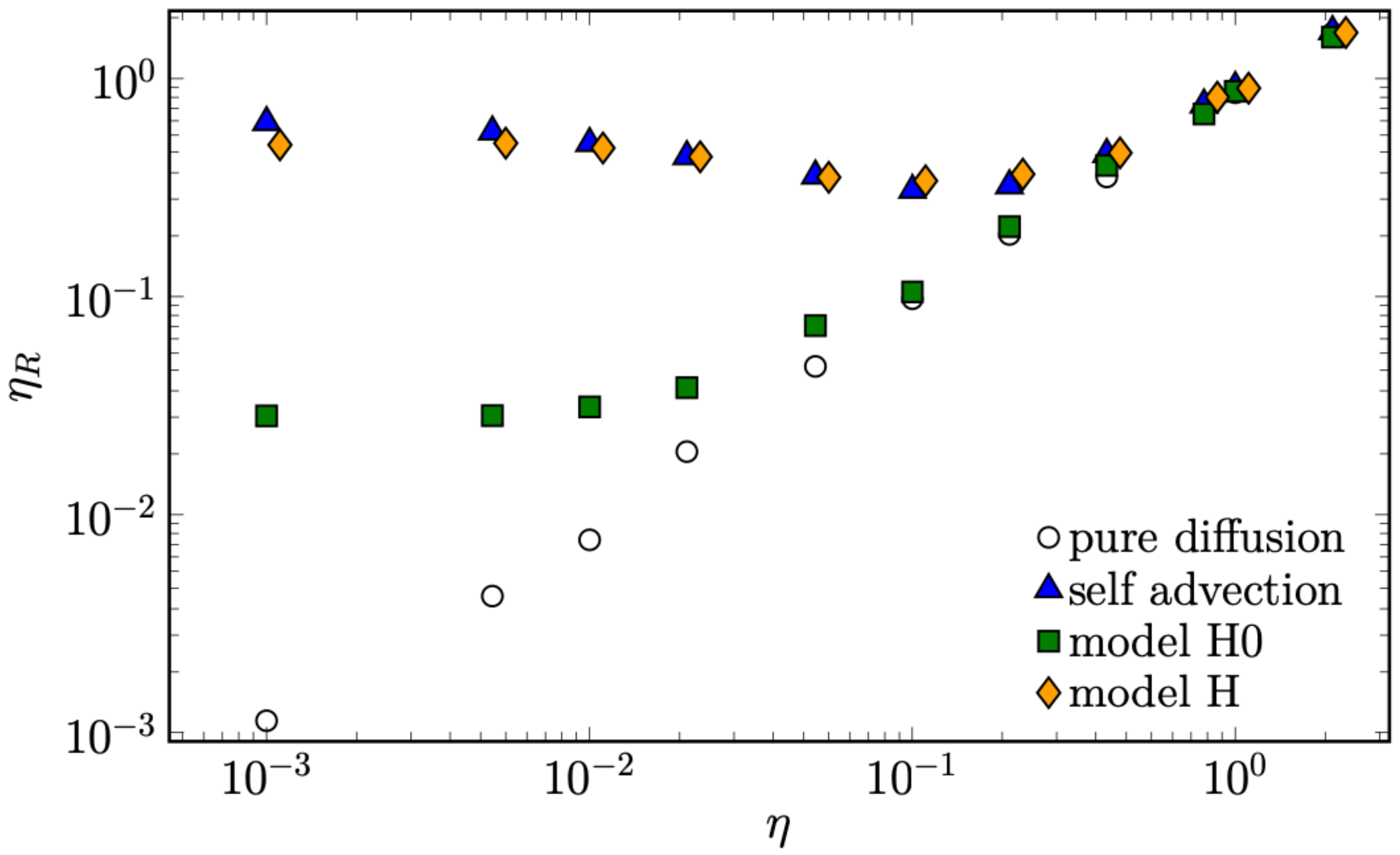


$$\eta_R = \eta + \frac{7}{60\pi^2}\frac{\rho T\Lambda}{\eta}$$

Effective viscosity levels off, then increases.

Thermal fluctuations + Non-linearity of hydro

$\Longrightarrow$ shear viscosity has a minimum



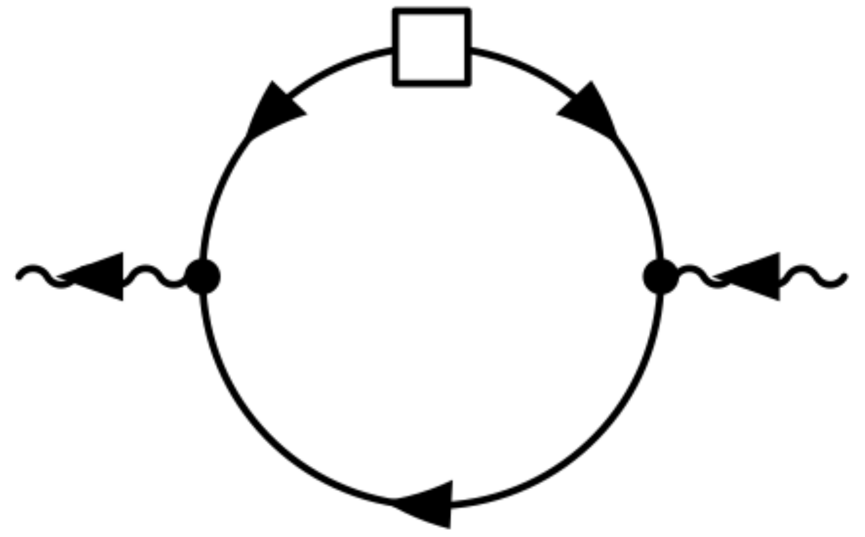
Self-advection

$$\frac{\partial \vec{\pi}_T}{\partial t} + \frac{\vec{\pi}_T}{\rho}\cdot\vec{\nabla}\vec{\pi}_T = \frac{\eta}{\rho}\nabla^2\vec{\pi}_T + \cancel{\vec{\nabla}\phi\nabla^2\phi} + \vec{\xi}$$

In analogy to "stickiness of sound" Kovtun, Moore & Romatschke

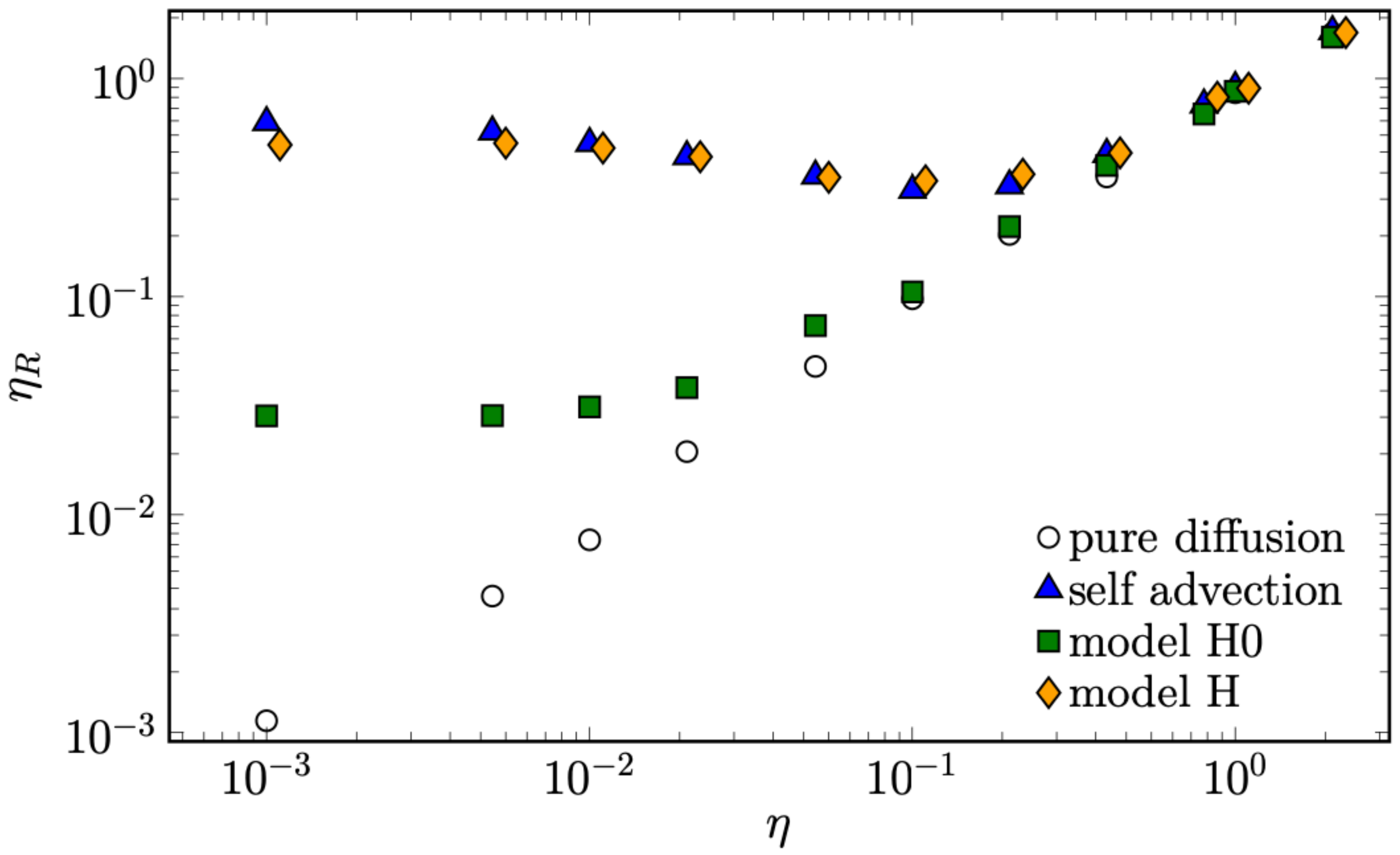
Renormalized viscosity

The renormalization of η due to coupling to the order parameter



$$\eta_R = \eta + \frac{1}{160\pi}\frac{T\xi_0}{\Gamma}$$

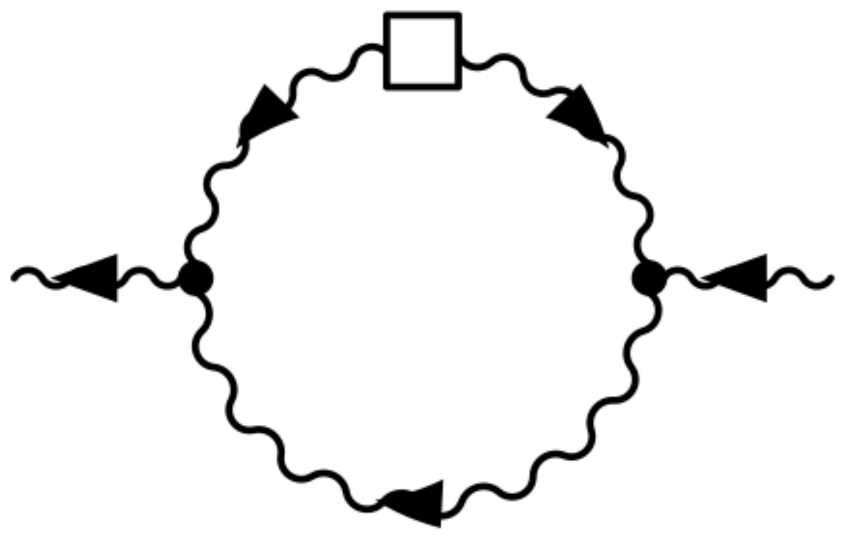
Much smaller effect than self-advection



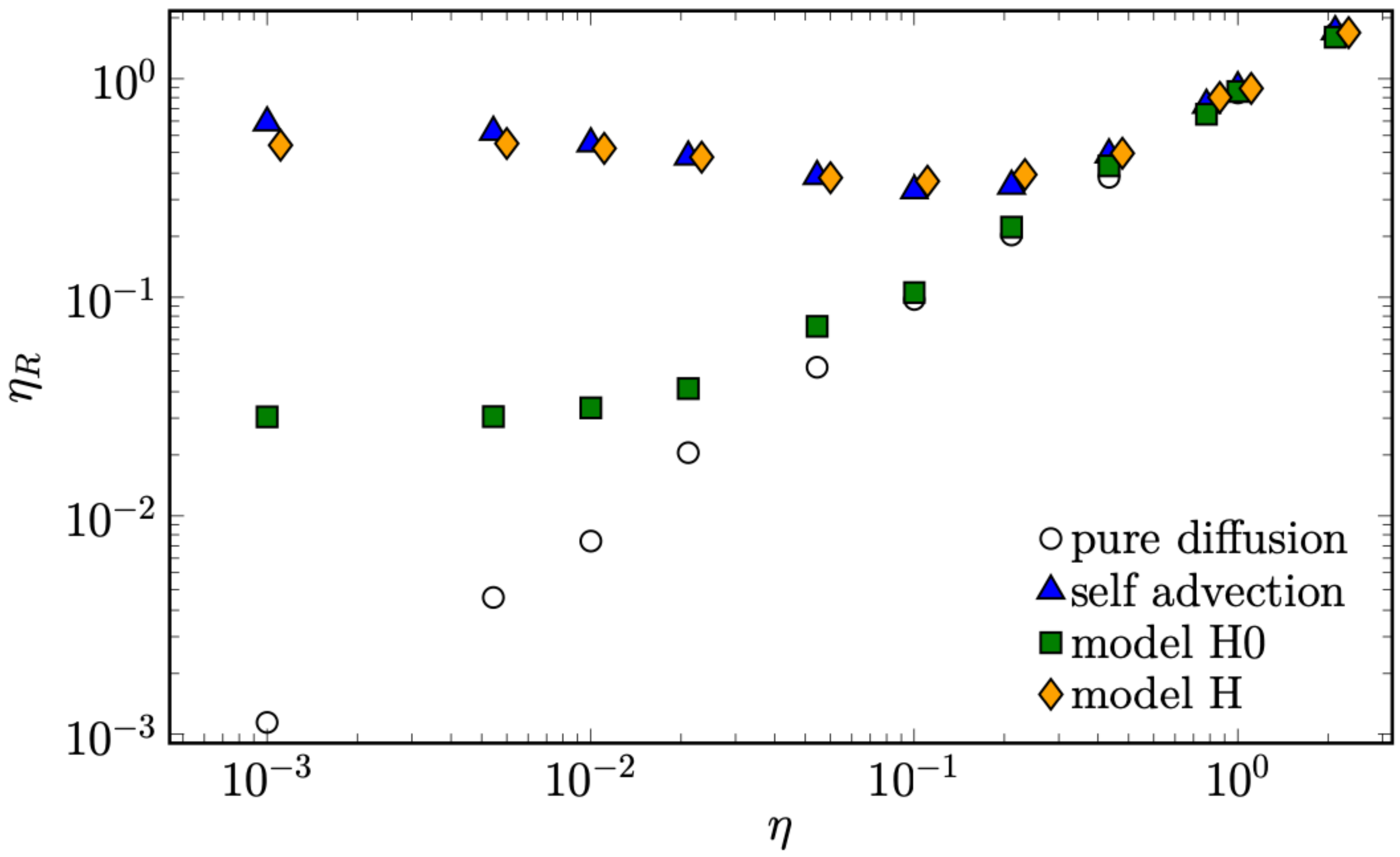
Model H0 $$\frac{\partial \vec{\pi}_T}{\partial t} + \cancel{\frac{\vec{\pi}_T}{\rho}\vec{\nabla}\,\vec{\pi}_T} = \frac{\eta}{\rho}\nabla^2\vec{\pi}_T + \vec{\nabla}\phi\,\nabla^2\phi + \vec{\xi}$$

Renormalized viscosity

Model H effective viscosity dominated by self-advection of π_T



$$\eta_R = \eta + \frac{7}{60\pi^2}\frac{\rho T \Lambda}{\eta}$$



Model H

$$\frac{\partial \vec{\pi}_T}{\partial t} + \frac{\vec{\pi}_T}{\rho} \cdot \vec{\nabla}\vec{\pi}_T = \frac{\eta}{\rho}\nabla^2\vec{\pi}_T + \vec{\nabla}\phi\,\nabla^2\phi + \vec{\xi}$$

Dynamics: Order parameter

- Using the time dependent correlation function of the order parameter

$$C(t,\vec{k}) = \langle \phi(0,\vec{k})\, \phi(t,-\vec{k}) \rangle$$

a wave-number dependent relaxation rate is defined $C(t,\vec{k}) \sim \exp(-\Gamma_k t)$

- A model for Γ_k was proposed by Kawasaki:

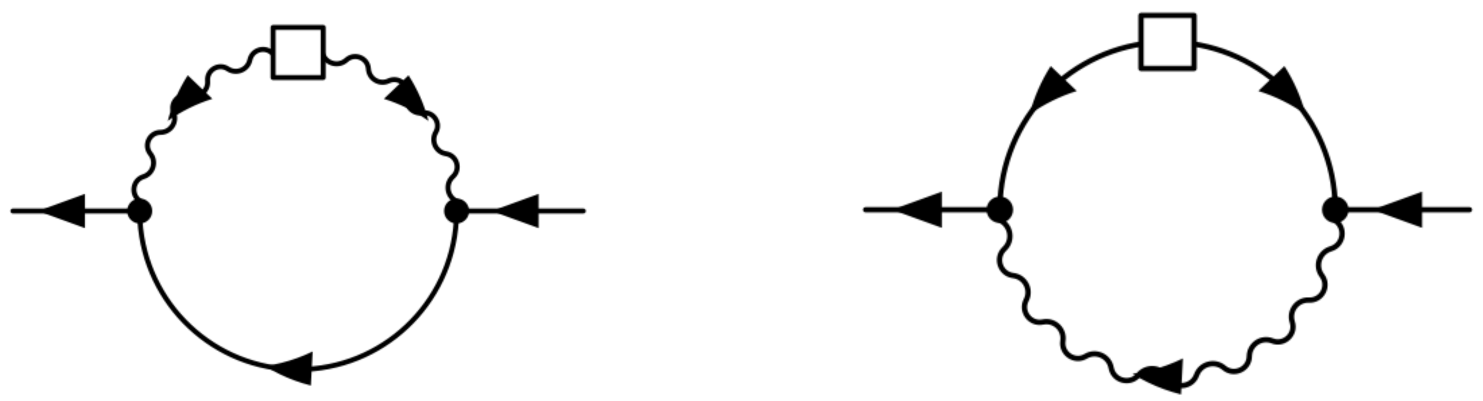
Kawasaki function

$$\Gamma_k = \frac{\Gamma}{\xi^4}\,(k\xi)^2\left(1+(k\xi)^2\right) + \frac{T}{6\pi\eta_R\xi^3}K(k\xi)$$

Pure Model B prediction
using mean field approx.

Arises from coupling
between ϕ and π_T

Diagrams computed with
certain approximations

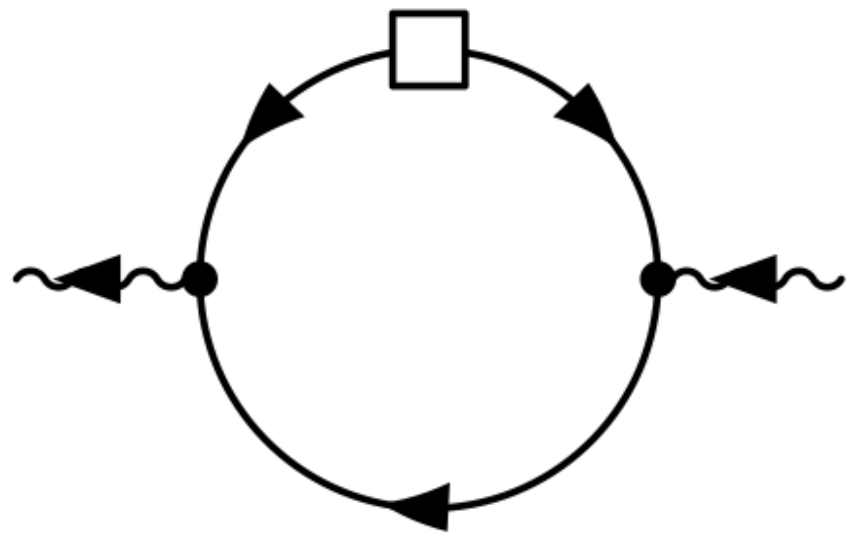


Dynamics: Kawasaki approximation

- The Kawasaki approximation: $$\Gamma_k = \frac{\Gamma}{\xi^4}\left(k\xi\right)^2\left(1+\left(k\xi\right)^2\right) + \frac{T}{6\pi\eta_R\xi^3}K(k\xi)$$

- Near critical point, relaxation-rate for wavenumbers $k = k_* \sim 1/\xi$ should cross over from $z = 4$ (pure diffusive dynamics) to $z = 3$ (pure Model H behavior).

- Digression: Using Γ_k one can re-recompute the renormalization of η due to coupling of π_T to ϕ:



$$\eta_R = \eta\left[1 + \frac{8}{15\pi^2}\log\left(\frac{\xi}{\xi_0}\right)\right]$$

Near critical point, viscosity diverges, but only weakly

$$\eta_R \sim \xi^{x_\eta} \quad \text{with } x_\eta \approx 0.05$$

Extraction of dynamic critical exponent (numerics)

- Compute time dependent correlator of the order parameter

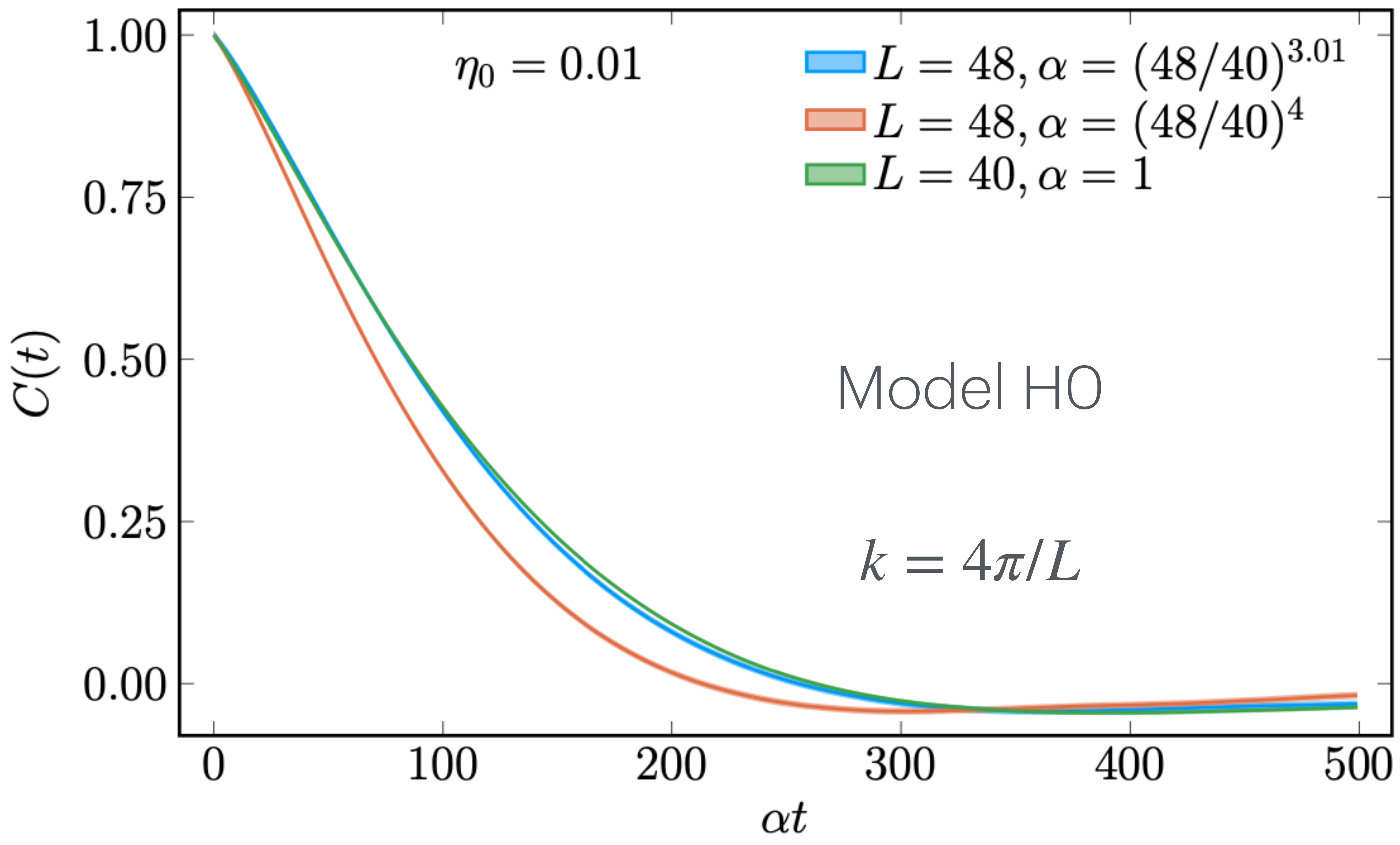
$$C(t,\vec{k}) = \langle \phi(0,\vec{k})\,\phi(t,-\vec{k})\rangle$$

at the critical point.

- Dynamic scaling at critical point :

$$C(t,k) = \tilde{C}\left(t/L^z, k\,L\right)$$

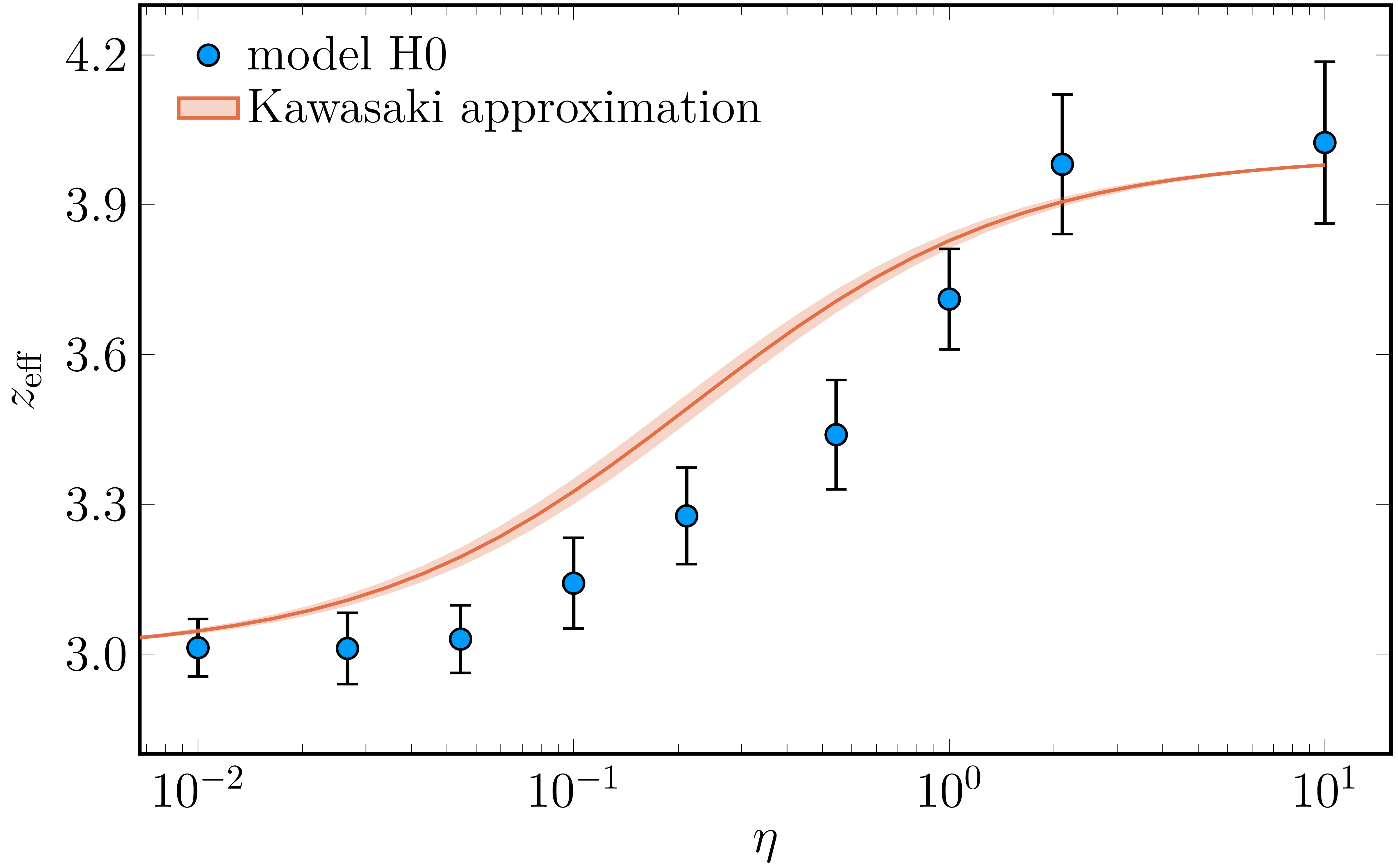
- Hold kL fixed, vary lattice size. Extract z by looking for data collapse.



$$z\,(\eta = 0.01) = 3.01$$

Cross-over of z

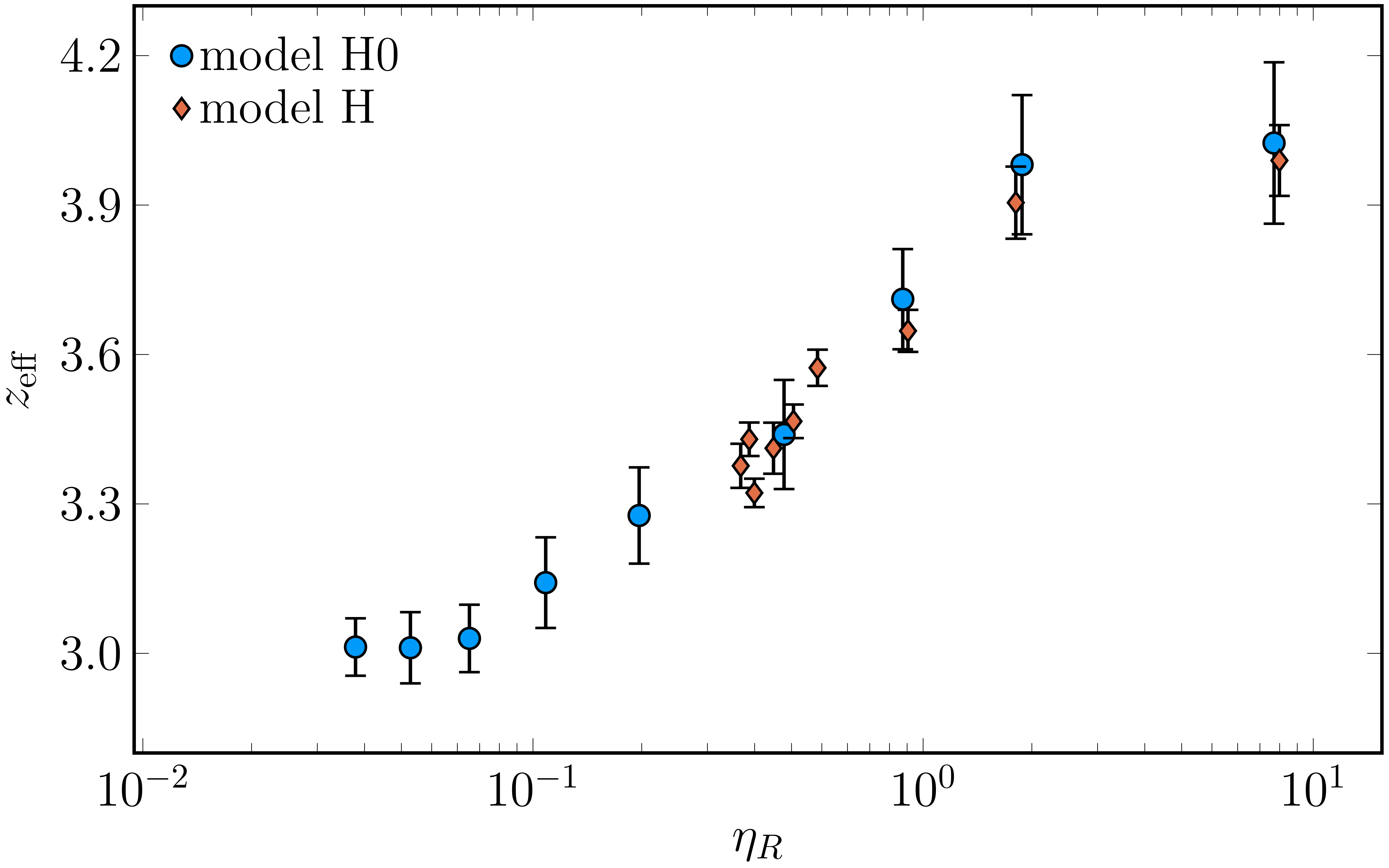
- Extract z for various η
- In Model H0, η_R can become quite small.
  - Dynamic exponent crosses over from $z = 4$ (pure diffusion) to $z = 3$ (Model H expectation)



The Kawasaki approximation: $$\Gamma_k = \frac{\Gamma}{\xi^4}\left(k\xi\right)^2\left(1+(k\xi)^2\right) + \frac{T}{6\pi\eta_R\xi^3}K(k\xi)$$

Cross-over of z

- Dynamic scaling exponent as a function of renormalized viscosity.
- z for full Model H coincides with Model H0
- In full Model H, η_R cannot become too small $\Longrightarrow$ min(z) ≈ 3.3



Model H $\quad \dfrac{\partial \vec{\pi}_T}{\partial t} + \dfrac{\vec{\pi}_T}{\rho} \cdot \vec{\nabla}\, \vec{\pi}_T = \dfrac{\eta}{\rho} \nabla^2 \vec{\pi}_T + \cdots$

Model H0 $\quad \dfrac{\partial \vec{\pi}_T}{\partial t} + \cancel{\dfrac{\vec{\pi}_T}{\rho} \cdot \vec{\nabla}\, \vec{\pi}_T} = \dfrac{\eta}{\rho} \nabla^2 \vec{\pi}_T + \cdots$

Evolution of higher moments

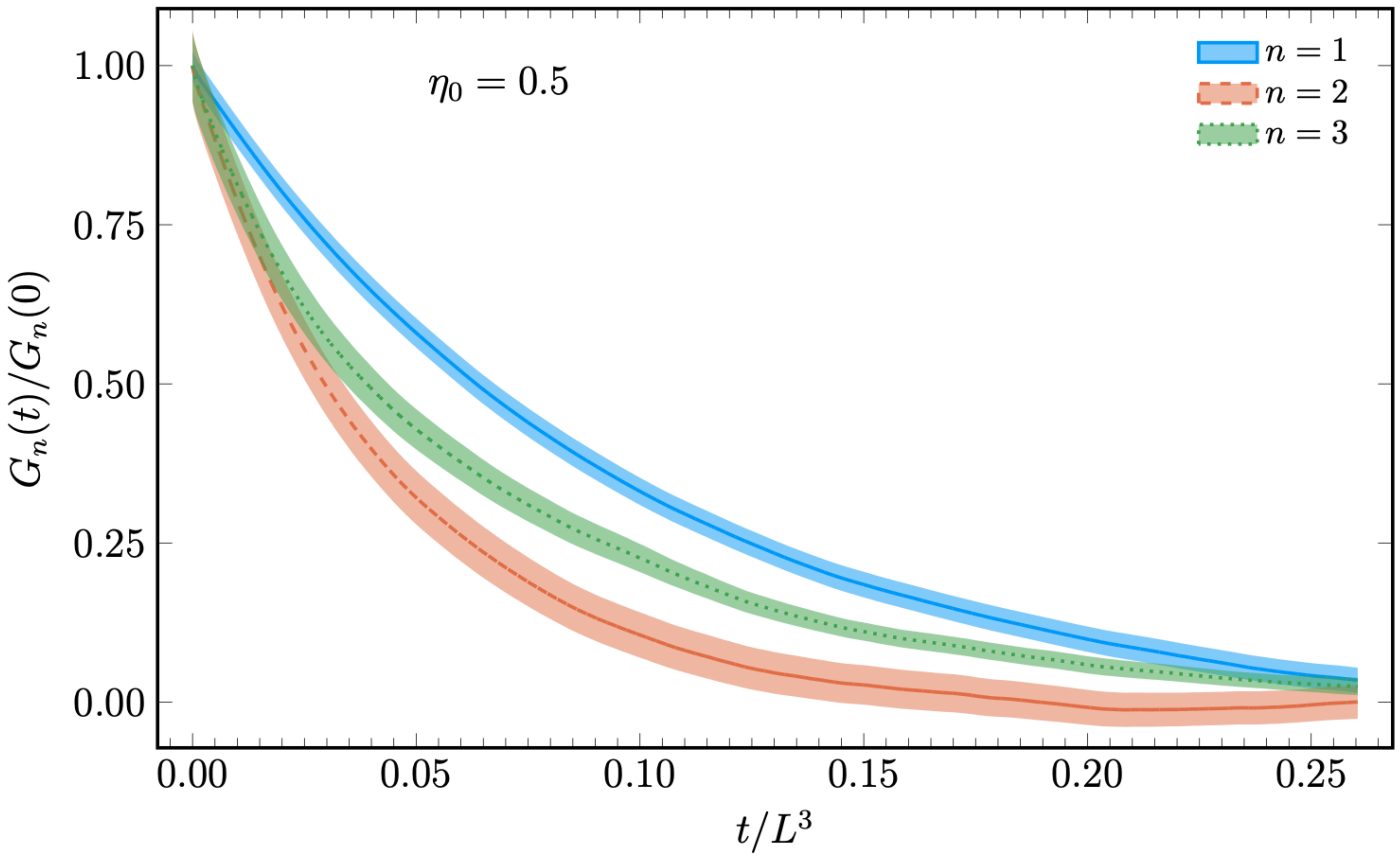
- Consider higher-point correlations

$$G_n(t) = \langle M^n(t) M^n(0) \rangle$$

$$M(t) = \int_V d^3x \, \phi(t, \vec{x})$$

- Correlation functions satisfy dynamical scaling

- Relaxation rate depends on 'n'. Not compatible with mean field expectations



Summary & Outlook

- Performed numerical simulations of stochastic fluid dynamics near a critical point. Observed renormalization of shear viscosity and dynamical scaling.

  - Self-coupling of momentum density is important in limiting the smallness of effective viscosity.

  - Dynamic scaling exponent depends sensitively on value of correlation length and effective shear viscosity.

  - Pure Model H behavior $z \approx 3$ requires both large ξ and small η_R .

To generalize this to relativistic fluids with non-trivial expansions and cooling, inclusion of sound modes and critical equation of state.

Thank you!

Backup: determination of m_c^2 in Model A

- At a critical point, susceptibilities $\langle M^2 \rangle$ diverge (infinite vol). In finite volume there are peaks. Possible strategy: Thermalize Model B configurations, compute $\langle M^2 \rangle$ at different m^2 and look for peaks.

- Mean-field estimates that Model B configurations take $\tau_{\text{therm}} \sim L^z$ with z ~ 4 to thermalize. Computationally demanding.

- Use a model in the same static universality class but with smaller $z \Longrightarrow$ Model A, relaxational dynamics of an order-parameter (z = 2).

$$\frac{\partial \phi}{\partial t} = -\Gamma \frac{\delta F}{\delta \phi} + \zeta \qquad F[\phi] = \int d^3x \left[\frac{1}{2}\left(\nabla \phi\right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4\right]$$

$$\langle \zeta(t, \vec{x})\, \zeta(t', \vec{x}') \rangle = 2\Gamma\, T\, \delta(\vec{x} - \vec{x}')\, \delta(t - t')$$

Backup: The Metropolis scheme in Model A

- Take a trial update $\phi(t+\Delta t, x)_{\text{trial}} = \phi(t, x) + \sqrt{2\Gamma\, T\, \Delta t}\, \theta, \quad \langle \theta^2 \rangle = 1$

- The change in free energy due to this update

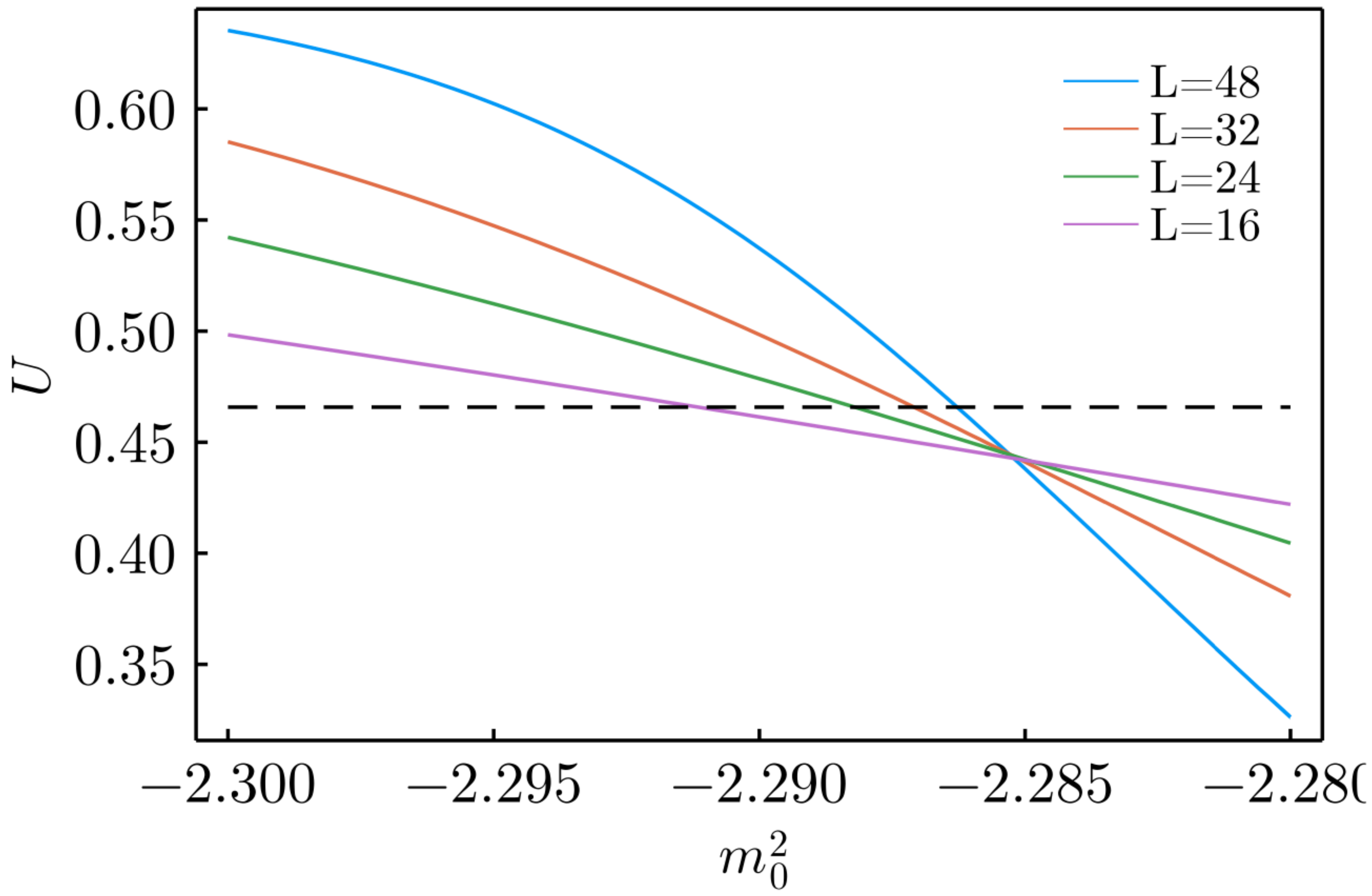
$$\Delta F = \left(d + \frac{m^2}{2}\right)\left(\phi^2_{\text{trial}}(x) - \phi^2(x)\right) + \frac{\lambda}{4}\left(\phi^4_{trial}(x) - \phi^4(x)\right)$$
$$-\left(\phi_{\text{trial}}(x) - \phi(x)\right)\sum_{\hat{\mu}=1}^{d}\left(\phi(x+\hat{\mu}) - \phi(x-\hat{\mu})\right)$$

- Accept the update with probability

$$p = \min(1, \exp(-\Delta F/T))$$

Backup: m_c^2 using Binder cumulants

- Fluctuation observables like $\langle M^2 \rangle$ and $\langle M^4 \rangle$ shows peaks at m_c^2.
- The location of these peaks differs from infinite volume limit.



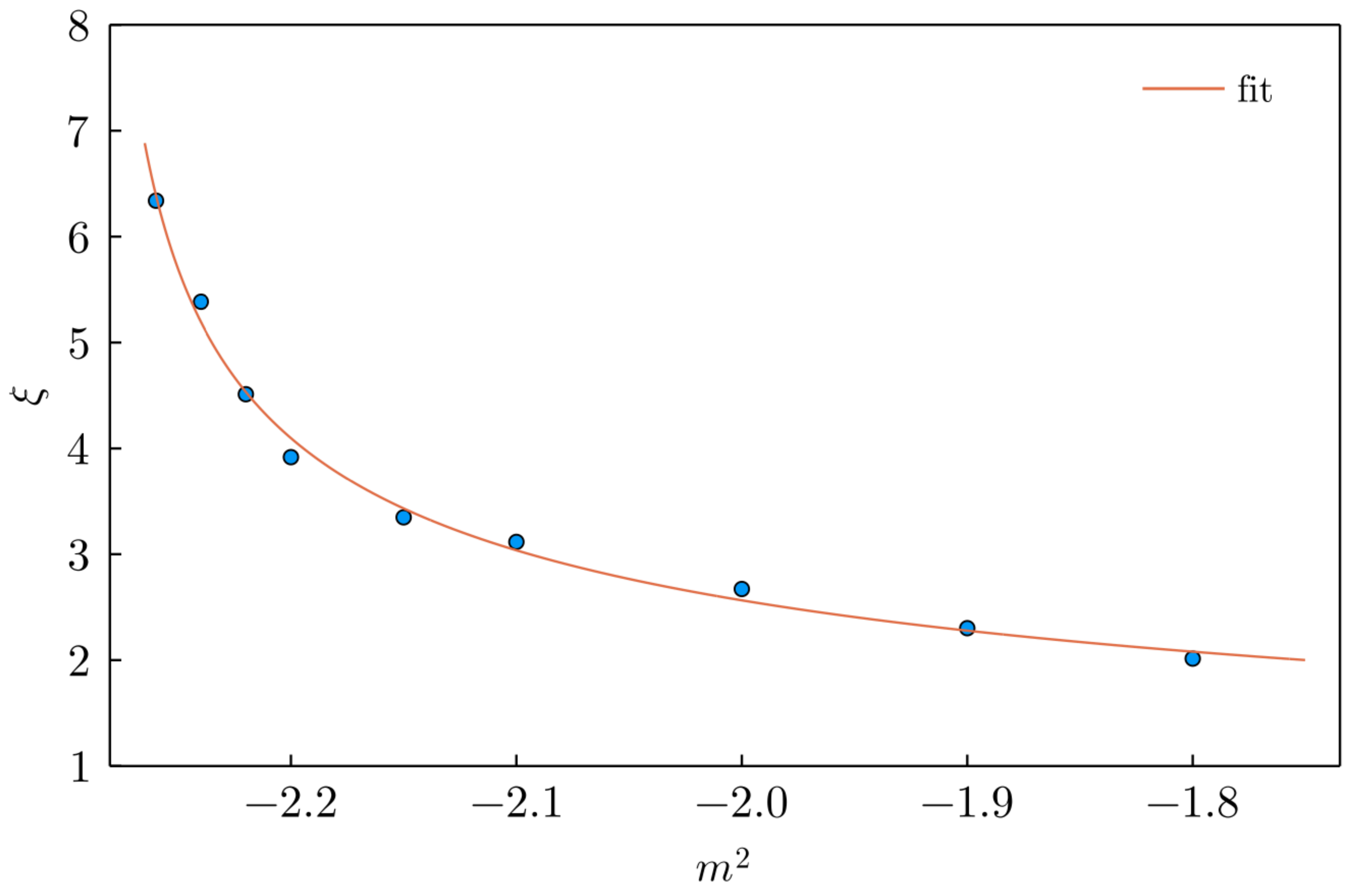
At the true critical point, finite volume effects on the Binder cumulant U cancels

$$U = 1 - \frac{\langle M^4 \rangle}{3(\langle M^2 \rangle)^2}$$

Strategy: Thermalize lattice using Metropolis update up to a long time, $t \sim L^2$

Compute $U(m^2)$ and estimate where the curves cross the infinite volume result

Backup: Correlation length in Model B



The static correlator in Fourier space

$$C(k) = \langle \phi(0,\vec{k})\, \phi(0, -\vec{k}) \rangle$$

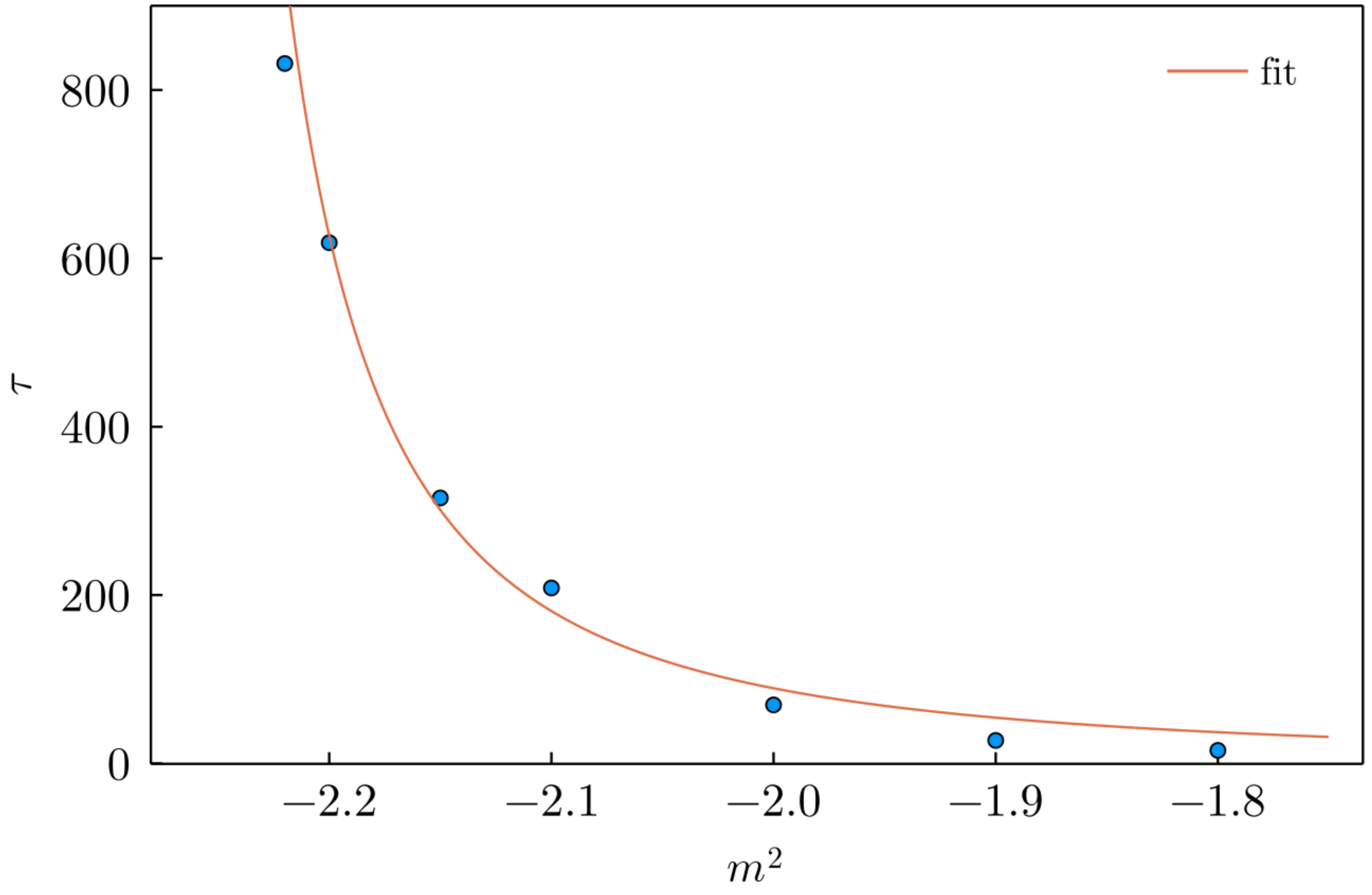
Extract correlation length by fitting with mean field expectation

$$C(k) \sim \frac{1}{k^2 + 1/\xi^2}$$

The correlation length grows as

$$\xi \sim \frac{1}{(m^2 - m_c^2)^\nu} \quad \text{with} \quad \nu \approx 0.54$$

Backup: Relaxation time in Model B



- Compute the dynamic correlator

$$C(t,k) = \langle \phi(0,\vec{k})\, \phi(t,-\vec{k}) \rangle$$

- The correlator damps in time

$$C(t,k) \sim \exp(-t/\tau_k)$$

Where τ_k is a momentum-dependent relaxation time.

- In figure, τ for a given m^2 (or ξ) is defined as τ_k at $k = 1/\xi$

- Relaxation time grows as $\tau \sim \xi^z$

Backup: The stickiness of sound

Kovtun, Moore & Romatschke

Linearized energy-momentum tensor in presence of noise

$$T_{00,\xi} = \delta e \qquad T_{0i,\xi} = -\left(e_0 + P_0\right)\delta u_i \qquad T_{ij,\xi} = \delta_{ij}\, c_s^2\, \delta e - \eta \left(\partial_i \delta u_j + \partial_j \delta u_i - \frac{2}{3}\delta_{ij} \vec{\nabla}\cdot \delta\vec{u}\right) + \xi_{ij}$$

Noise is Gaussian: $\langle \xi_{ij}(x)\xi_{kl}(y)\rangle = 4\,\eta\, T \Delta_{ijkl}\, \delta^4(x-y)$

Averages of any quantity is obtained by using a functional integral $\langle \mathcal{O}\rangle \equiv \int D\xi_{ij}\, e^{-S_\xi}\, \mathcal{O}$

$$S_\xi = \int d^3x\, \xi_{ij} \left(\frac{1}{8T\eta}\Delta^{ijkl}\right) \xi_{kl}$$

Can compute any correlation functions, for eg., $\langle T^{12}(x)\, T^{12}(y)\,\rangle \equiv G^{12,12}(x,y)$

Backup: The stickiness of sound

Beyond linearized regime, consider terms up to 2nd order in perturbation (also take low momentum limit) $T^{12}_{\xi} = \left(e_0 + P_0\right)\, \delta u^1\, \delta u^2\, + \xi^{12}$

The symmetric correlator $G^{12,12}_{\rm sym}(x,y) = \langle \xi^{12}(x)\xi^{12})(y)\rangle_\xi + (\epsilon_0 + P_0)^2 \langle \delta u^1(x)\delta u^2(x)\delta u^1(y)\delta u^2(y)\rangle_\xi.$

In Fourier space,
$$G^{12,12}_{\rm sym}(\omega, k\to 0) = 2T\eta + \int \frac{d\omega'}{2\pi}\frac{d^{d-1}k'}{(2\pi)^{d-1}} \left[G^{01,01}_{\rm sym}(\omega',\mathbf{k}')G^{02,02}_{\rm sym}(\omega-\omega',-\mathbf{k}') + G^{01,02}_{\rm sym}(\omega',\mathbf{k}')G^{02,01}_{\rm sym}(\omega-\omega',-\mathbf{k}')\right]$$

For example, $G^{01,01}_{\rm sym} = -\dfrac{2T}{\omega}\left(e_0 + \dfrac{k^2\eta}{i\omega - \gamma_\eta k^2}\right)$ $\quad\gamma_\eta = \eta/(e_0 + P_0)$

Finally, one obtains
$$G^{12,12}(\omega, k\to 0) = -i\omega\left(\eta + \frac{17T\Lambda_{UV}}{120\pi^2\gamma_\eta}\right) + (1+i)\omega^{3/2}\frac{\left(7 + \left(\frac{3}{2}\right)^{3/2}\right)T}{240\pi\gamma_\eta^{3/2}}$$

Renormalization of shear

Kovtun, Moore & Romatschke