Relativistic Stochastic Hydrodynamics from Metropolis Updates

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I.The Density Frame

Viscous hydro issues, problems, and things to discuss

Hydrodynamics: Conservation laws, equilibrium & symmetry

$$T^{\mu\nu} = \underbrace{eu^\mu u^\nu + p\Delta^{\mu\nu}}_{ideal} - \underbrace{\eta\sigma^{\mu\nu}}_{dissipation} + \underbrace{\xi^{\mu\nu}}_{noise}$$

$$\sigma^{\mu\nu} = \Delta^{\mu\rho}\Delta^{\nu\sigma}(\partial_\rho u_\sigma + \partial_\sigma u_\rho - \frac{2}{3}\delta_{\rho\sigma}\partial_\gamma u^\gamma)$$

1. Run away numerical solutions as EOM are 2^{nd}-order in time, which are regulated by introducing non-hydro modes
2. Choice of hydro frame (u^μ) makes the solutions non-unique
   - Landau frame, Eckart frame $J^\mu = nu^\mu$, and Density frame

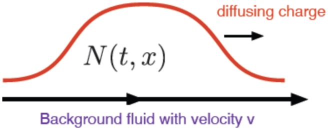
Density frame: $T^{0i} \propto v^i$

3. Stochastic hydrodynamics is natural in density frame

A prototype for hydro: the advection-diffusion equation

Consider a dilute conserved charge density $N(t,x) \equiv J^0$ in a moving fluid

$$J^\mu_{LF} = nu^\mu + j^\mu_D$$
$$j^\mu_D = -D\Delta^{\mu\nu}\partial_\nu n$$



In an arbitrary Lorentz frame

$$\partial_\mu J^\mu = 0$$

contains two time derivatives and hence has run away solutions

Relativistic advection-diffusion without boost symmetry

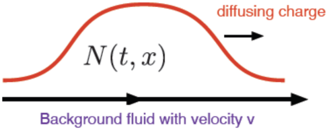
Hydro without boosts:

I. Novak, J. Sonner, and B. Withers (2020); J. de Boer et. al (2020) and **J. Armas & A. Jain (2021)**

Consider a dilute conserved charge density $N(t,x) = Q/V$ in a moving fluid

$$\partial_t N + \partial_i J^i = 0$$

$$J^i = \underbrace{Nv^i}_{\text{Advection}} + \underbrace{J^i_D}_{\text{Diffusion}}$$



J^i_D is Taylor expanded in gradients of the charge density

$$J^i_D = -\underbrace{D_{\parallel}(v)\hat{v}^i\hat{v}^j\partial_j N}_{\text{diffusion} \parallel \text{ to v}} - \underbrace{D_{\perp}(v)(\delta^{ij} - \hat{v}^i\hat{v}^j)\partial_j N}_{\text{Diffusion} \perp \text{ to v}}$$

Restoring boost symmetry and back to Landau frame

The current in the Landau frame

$$J^\mu = n u^\mu + j_D^\mu \quad \text{with} \quad j_D^\mu = -D \Delta^{\mu\nu} \partial_\nu n$$

Identify $N(t, x) \equiv J^0$, and use lowest order EOM

$$\partial_t n + v^i \partial_i n \approx 0$$

to replace the time derivatives with spatial ones to obtain the current, in the Density frame

$$J^i = N v^i - \underbrace{\frac{D}{\gamma^3} \hat{v}^i \hat{v}^j \partial_j N}_{\text{Diffusion } \parallel \text{ v}} - \underbrace{\frac{D}{\gamma} (\delta^{ij} - \hat{v}^i \hat{v}^j) \partial_j N}_{\text{Diffusion } \perp \text{ to v}}$$

with,

$$D_\parallel(v) = \frac{D}{\gamma^3} \quad \text{and} \quad D_\perp(v) = \frac{D}{\gamma}$$

Summary

The advection-diffusion equation is not covariant in density frame

$$\partial_t N + \partial_i(N v^i) = \partial_i(D^{ij}\partial_j N) \quad \text{where} \quad D^{ij} = \frac{D}{\gamma}(\delta^{ij} - v^i v^j)$$

The equation are strictly first order in time and thus stable solutions do exist. Each Lorentz observer has their own hydrodynamic frame where

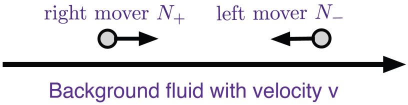
$$dS^0 = \frac{\mu}{T} dJ^0$$

The chemical potential is defined to all orders by the charge, in the density frame in contrast to the the Landau frame where it's defined by the 4-current

$$\mu_{DF} = \frac{J^0}{\chi u^0} \quad \text{vs} \quad \mu_{LF} = \frac{-u_\mu J^\mu}{\chi}$$

II. Analytically tractable problem

Test: 1+1D boosted random walk of massless particles



The particles experience Poissonian random kicks with transition rates given by

$$\Gamma_+ = \frac{1}{2\tau_R}\sqrt{\frac{1-v}{1+v}} \quad \text{and} \quad \Gamma_- = \frac{1}{2\tau_R}\sqrt{\frac{1+v}{1-v}}$$

The right and left movers obey

$$\partial_t N_+ + c\partial_x N_+ = -\Gamma_+ N_+ + \Gamma_- N_-$$
$$\partial_t N_- + c\partial_x N_- = -\Gamma_- N_- + \Gamma_+ N_+$$

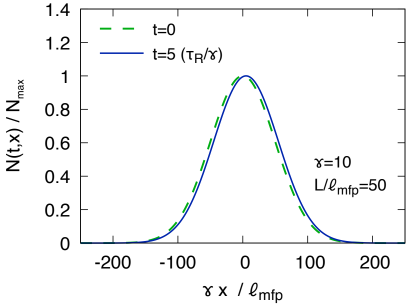
With charge density and current density given by

$$J = (N_+ - N_-)c \quad N = N_+ + N_-$$

Initialization and analysis using the density frame

At $t = 0$ we drop a Gaussian charge density $N(x)$ with $J_D = 0$ in the lab frame. The DF predicts that a current will develop in time

$$J = Nv + J_D \quad \textit{where} \quad J_D = \frac{D}{\gamma^3}\partial_x N$$



Density frame analysis is approached on a timescale of τ_R/γ

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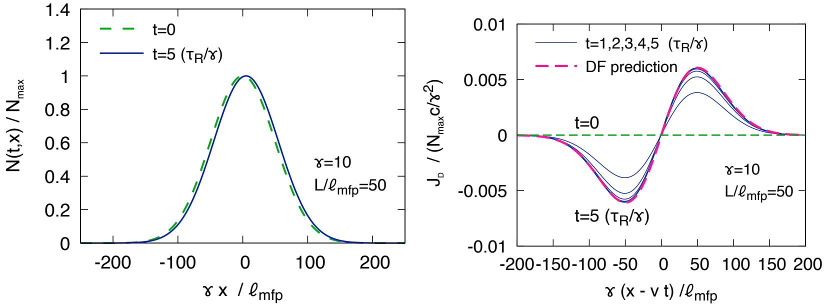
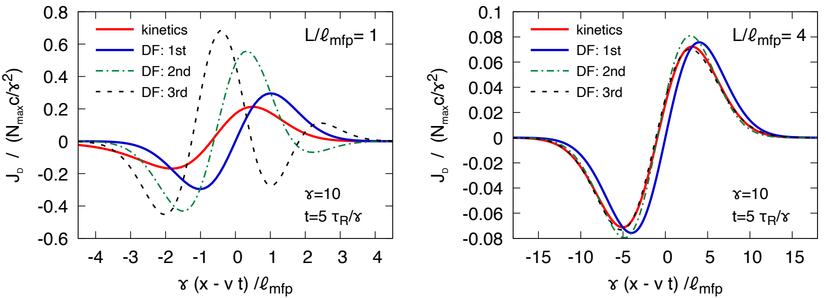


Density frame analysis is approached on a timescale of τ_R/γ

Convergence of gradient expansion

$$-\gamma^2 J_D = \underbrace{c_1(v)\left(\frac{\tau_R}{\gamma}\right)\partial_x N}_{\text{first}} + \underbrace{c_2(v)\left(\frac{\tau_R}{\gamma}\right)^2 \partial_x^2 N}_{\text{second}} + \underbrace{c_3(v)\left(\frac{\tau_R}{\gamma}\right)^3 \partial_x^3 N}_{\text{third}}$$

Gaussian charge density $N(x)$ with different system sizes



III. Adding noise

Brownian motion: A prototype for all dissipative stochastic processes

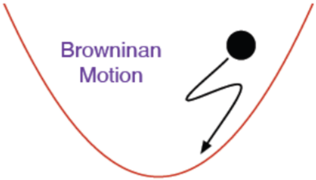
Consider the Free energy of the system

$$\mathcal{H} = \frac{p^2}{2m} + V(q)$$

Stochastic equations of motion

$$\partial_t q + \{q, \mathcal{H}\} = 0$$

$$\partial_t p + \{p, \mathcal{H}\} = -\eta \underbrace{\left(\frac{\partial \mathcal{H}}{\partial p}\right)}_{\text{velocity}} + \underbrace{\xi}_{\text{noise}}$$



The system relaxes to the distribution $\mathcal{P}_{\mathbf{eq}} \propto e^{-\beta\mathcal{H}}$

Dissipative dynamics from metropolis updates

Consider the drag & stochastic pieces

$$\partial_t p = -\eta \left(\frac{\partial \mathcal{H}}{\partial p} \right) + \xi \quad \text{and} \quad \langle \xi(t)\xi(t') \rangle = 2T\eta\delta_{tt'}$$

In Metropolis updates one proposes a random momentum transfer

$$p \to p + \Delta p \quad \text{with} \quad \langle \Delta p^2 \rangle = 2T\eta\Delta t$$

Calculate the change in free energy

$$\Delta\mathcal{H} = \mathcal{H}(p + \Delta p) - \mathcal{H}(p) \approx \left(\frac{\partial \mathcal{H}}{\partial p} \right) \Delta p$$

If $\Delta\mathcal{H} < 0$ accept the proposal, $\Delta\mathcal{H} > 0$ accept with probability

$$\mathcal{P}_{up} \propto e^{-\beta\Delta\mathcal{H}}$$

This reproduces the correct mean dissipation term

$$\langle \Delta p \rangle = -\eta \left(\frac{\partial \mathcal{H}}{\partial p} \right) \Delta t$$

1+2D Stochastic relativistic advection-diffusion equation

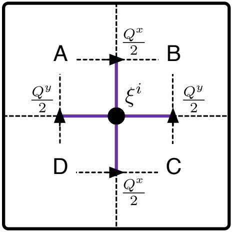
The equation is

$$\partial_t N + \partial_i(N v^i) = \partial_i(D^{ij}\partial_j N + \xi^i)$$

with dissipation matrix and noise

$$D^{ij} = \frac{D}{\gamma}(\delta^{ij} - v^i v^j)$$

$$\langle \xi^i(x)\xi^j(x')\rangle = 2T\chi D^{ij}\delta(x - x')$$



Implement the Metropolis algorithm

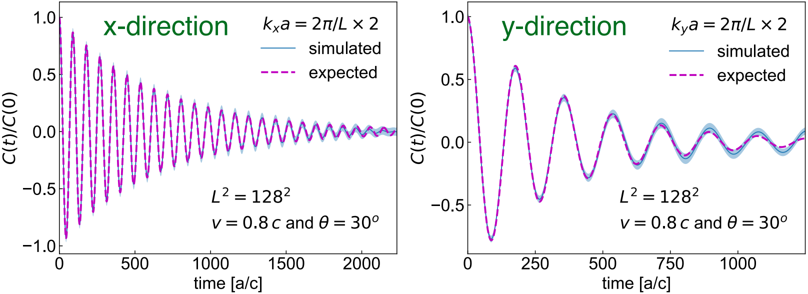
- Perform an ideal advection step
- Propose a charge transfer with 'appropriate' transverse and longitudinal variances
- Accept/Reject proposal based on the $\pm\Delta S$

This yields the correct mean diffusive current $\langle J^i_D\rangle = -D^{ij}\partial_j N$

Results for the charge density correlation functions

Equilibrium corr function with $v = 0.8c$ moving at angle $\theta = 30^o$

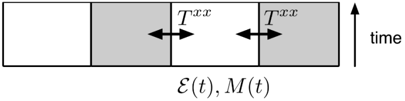
$$C(t) = \langle N(t,\boldsymbol{k}) N(0,-\boldsymbol{k})\rangle = T\chi \underbrace{cos(\boldsymbol{v}\cdot\boldsymbol{k}\,t)}_{\text{advection}} \times \underbrace{e^{-D^{ij}k_i k_j t}}_{\text{diffusion}}$$



The Metropolis steps reproduces the diffusive tensor

IV. Hydrodynamics

Ideal hydrodynamics



First step find the temperature and flow velocity, $\beta^\mu(\mathcal{E}, M) = u^\mu/T$, given the charges

$$T^{00} \equiv \mathcal{E} = (e(T) + p(T))u^0u^0 - p(T)$$
$$T^{0x} \equiv M = (e(T) + p(T))u^0u^x$$

Equations of motion

$$\partial_t\mathcal{E} + \partial_x M = 0 \quad \text{and} \quad \partial_t M + \partial_x \mathcal{T}^{xx}(\beta) = 0$$

$$\mathcal{T}^{xx}(\beta) = (e(T) + p(T))u^xu^x + p(T) + \textit{corrections}$$

$\beta^\mu(\mathcal{E}, M)$ **do not get corrections, Only the spatial stress must be specified and corrected by gradients** $\partial\beta$

Viscous hydrodynamics in Density frame

$$T^{xx} = \mathcal{T}^{xx}(\beta) + \Pi^{xx}_{DF}(\partial\beta)$$

where

$$\Pi^{xx}_{DF} = -\kappa^{xxxx}(v)\partial_x\beta_x \quad \text{and} \quad \kappa^{xxxx}(v) = T\left(\tfrac{4}{3}\eta + \zeta\right)\left(\frac{1}{\gamma^4}\frac{1}{1 - c_s^2 v^2}\right)$$

Viscous hydrodynamics in Density frame

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$$\Pi^{xx}_{DF} = -\kappa^{xxxx}(v)\partial_x\beta_x \quad \text{and} \quad \kappa^{xxxx}(v) = T\left(\tfrac{4}{3}\eta + \zeta\right)\left(\frac{1}{\gamma^4}\frac{1}{1-c_s^2v^2}\right)$$

The DF stress tensor follows from the one in LF, $\beta^\mu_{LF} \equiv \beta^\mu + \delta\beta^\mu$

$$\underbrace{T^{0\mu}}_{\text{frame indep object}} = \underbrace{\mathcal{T}^{0\mu}(\beta+\delta\beta) + \Pi^{0\mu}_{LF}}_{\text{Landau frame}} = \underbrace{\mathcal{T}^{0\mu}(\beta)}_{\text{Density frame}}$$

But the shifts $\delta\beta^\mu$ propagate into spatial components,

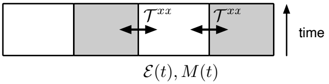
$$T^{xx} = \mathcal{T}^{xx}(\beta+\delta\beta) + \Pi^{xx}_{LF} \quad \text{or} \quad \Pi^{xx}_{DF} = \Pi^{xx}_{LF} + \frac{\partial\mathcal{T}^{xx}(\beta)}{\partial\beta_\mu}\delta\beta_\mu$$

Using ideal EOM to eliminate time derivative gives κ^{xxxx}

Stochastic viscous hydrodynamics

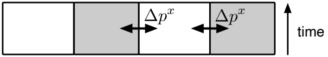
Take an ideal hydro step:

$$\partial_t \mathcal{E} + \partial_x M = 0$$
$$\partial_t M + \partial_x \mathcal{T}^{xx}(\beta) = 0$$



Take a viscous step using the Metropolis algorithm for the equations:

$$\partial_t \mathcal{E} = 0$$
$$\partial_t M + \partial_x \Pi^{xx} = 0$$



Propose a momentum transfer through the walls of cells:

$$\Delta p^x = \Delta t \xi^{xx} \quad \text{with} \quad \langle \xi^{xx} \xi^{xx} \rangle = \kappa^{xxxx} / (\Delta t \Delta x)$$

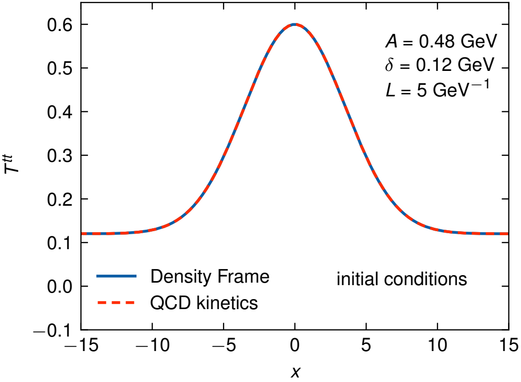
This produces the correct mean stress: $\langle \Pi^{xx} \rangle = -T\kappa^{xxxx} \partial_x \beta_x$

V. Comparative tests

Comparison of QCD kinetics with DF hydro: Setup

We thermally initialize our system of gluons with a Gaussian dist.

$$e = Ae^{-x^2/L^2} + \delta$$

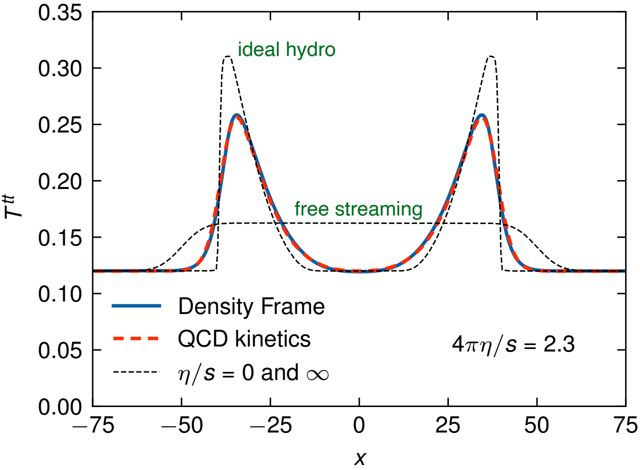


$$\frac{\ell_{mfp}}{L} \equiv \frac{\eta}{(e+p)c_s L} = \left(\frac{4\pi\eta/s}{20.0}\right)$$

Comparison of QCD kinetics with DF hydro

QCD kinetics in 1+1D by Fabian Zhou, Aleksas Mazeliauskas

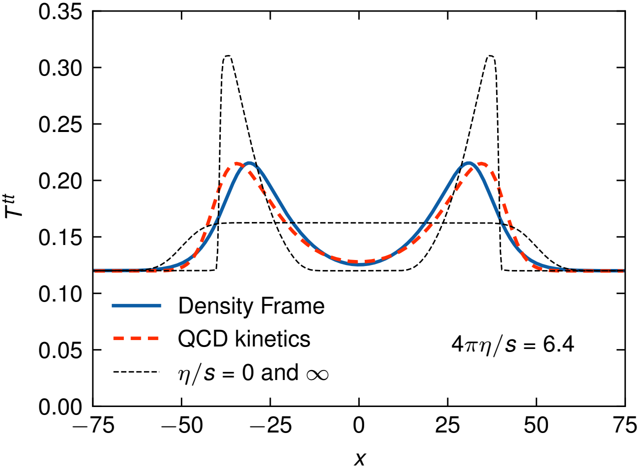
$$\frac{\ell_{mfp}}{L} = 0.12$$



Comparison of QCD kinetics with DF hydro

QCD kinetics in 1+1D by Fabian Zhou, Aleksas Mazeliauskas

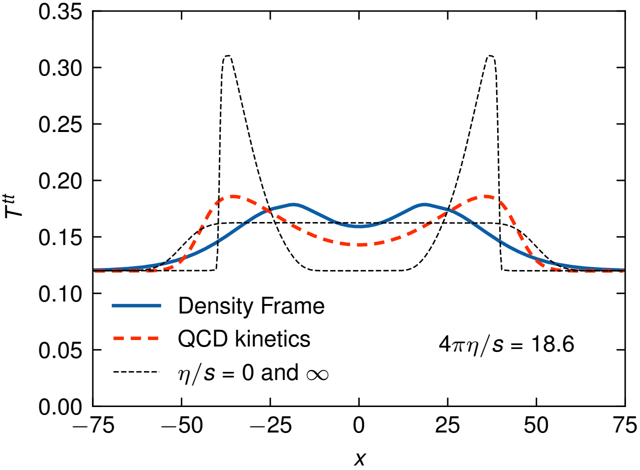
$$\frac{\ell_{mfp}}{L} = 0.32$$



Comparison of QCD kinetics with DF hydro

QCD kinetics in 1+1D by Fabian Zhou, Aleksas Mazeliauskas

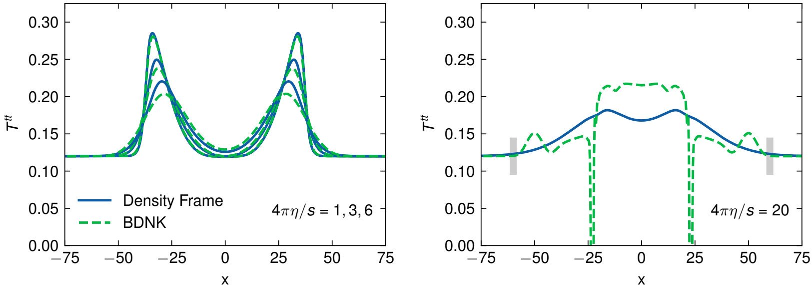
$$\frac{\ell_{mfp}}{L} = 0.93$$



Comparison with BDNK: Bemfica, Disconzi, Noronha; Kovtun

Initialize the system with a Gaussian energy distribution

$$e = Ae^{-x^2/L^2} + \delta$$



- The BDNK and Density frames agree for low shear viscosity
  - BDNK code from Pandya & Pretorius, PRD arXiv:2104.00804
  - The mean free path is

$$\frac{\ell_{mfp}}{L} = \left(\frac{4\pi\eta/s}{20}\right)$$

Comparison with BDNK: Bemfica, Disconzi, Noronha; Kovtun

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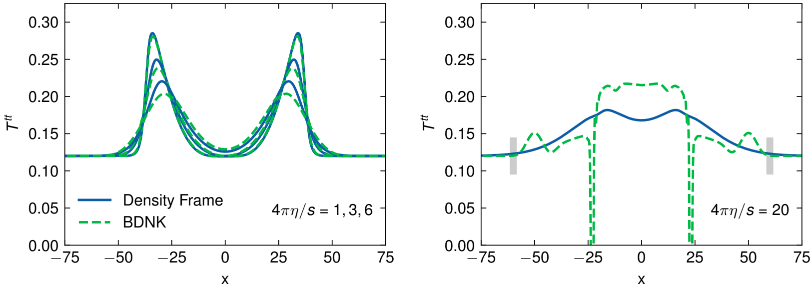


- For large shear viscosity the Density frame's current becomes increasingly diffusive without non-hydro modes
- In BDNK the perturbations must strictly vanish outside the causality edge, but this causes Gibb's oscillations

Conclusions

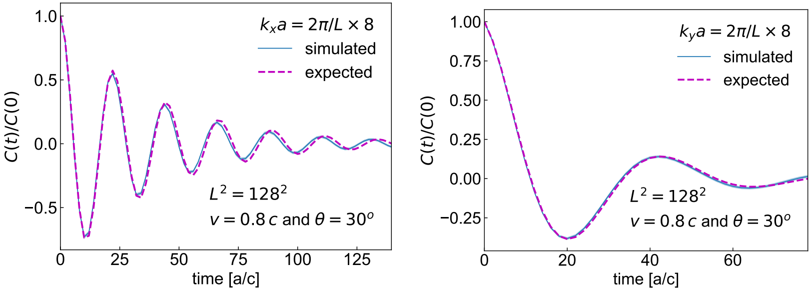
- **Robust results in good agreement with a variety of tests:**
  - Stable 1^{st} order equations and no non-hydro modes
  - Noise comes first and dissipation is an afterthought
- **General coordinates in 1+3D has been worked out:**
  - Take an ideal step
  - The momentum proposal is parallel transported from the cell-face to the cell centers for the accept/reject step

The parallel transport reproduces the covariant derivatives in the dissipative strain and corrections to the energy equation

Thank You

Results for the charge density correlation functions

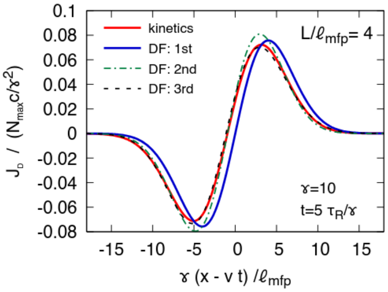
$$C(t) = \langle N(t,\boldsymbol{k})N(0,-\boldsymbol{k})\rangle = T\chi\,\underbrace{cos(\boldsymbol{v}\cdot\boldsymbol{k}t)}_{\text{advection}}\times\underbrace{e^{-D^{ij}k_ik_jt}}_{\text{diffusion}}$$

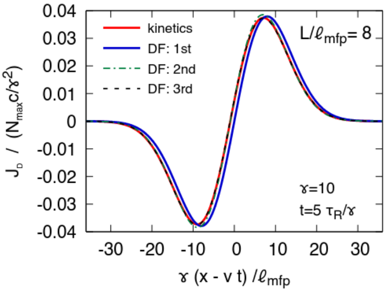


Convergence of gradient expansion

$$-\gamma^2 J_D = \underbrace{c_1(v)\left(\frac{\tau_R}{\gamma}\right)\partial_x N}_{\text{first}} + \underbrace{c_2(v)\left(\frac{\tau_R}{\gamma}\right)^2 \partial_x^2 N}_{\text{second}} + \underbrace{c_3(v)\left(\frac{\tau_R}{\gamma}\right)^3 \partial_x^3 N}_{\text{third}}$$

Gaussian charge density $N(x)$ with different system sizes





1+2D Stochastic Viscous Hydrodynamics

The stress tensor is:

$$(T^{00}, T^{0i}) \equiv (\mathcal{E}, M^i)$$

$$T^{ij} = \mathcal{T}_0^{ij} + \Pi^{ij}$$

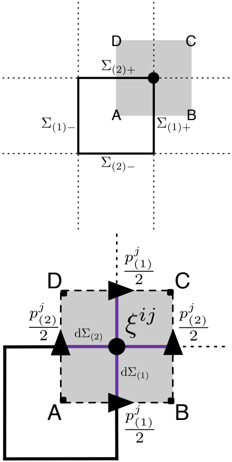
Take a viscous step using the Metropolis algorithm for the equations:

$$\partial_t \mathcal{E} = 0$$

$$\partial_t M^i + \partial_j \Pi^{ij} = 0$$

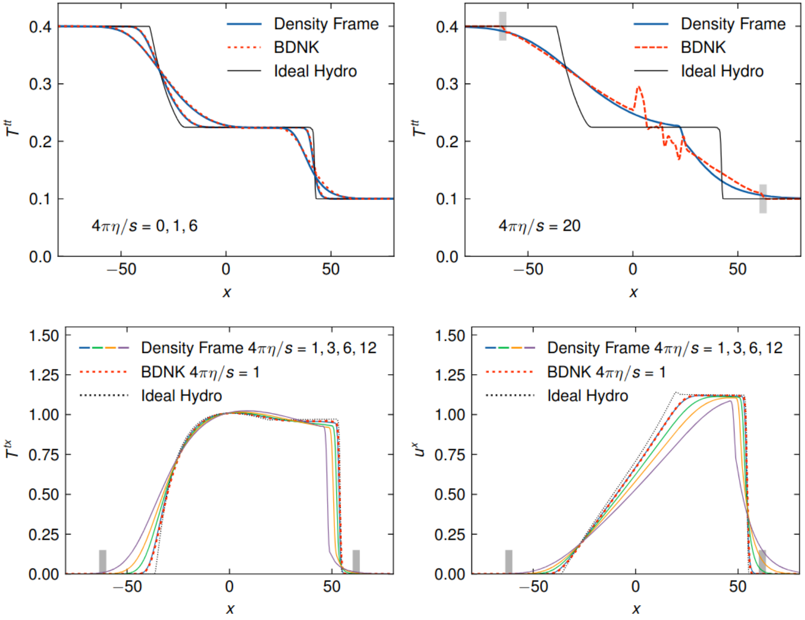
Propose a momentum transfer through the walls of cell A:

$$\delta p^i_{(x)} = \Delta t \, \mathrm{d}\Sigma_{(x)} \, \xi^{xi}$$



This produces the correct mean stress: $\Pi^{ij} = -T\kappa^{ijmn}\partial_{(m}\beta_{n)}$

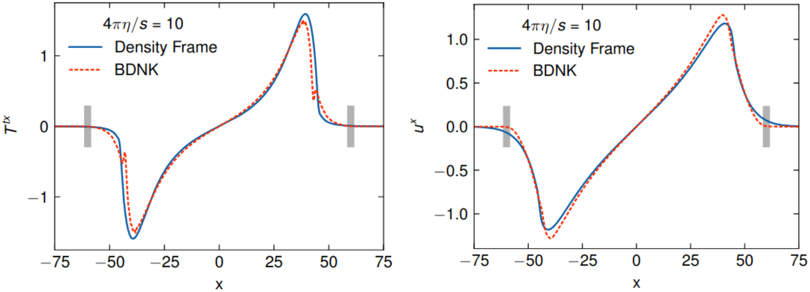
Shock tube test



Comparison with BDNK: Test 2

Initialize the system with a highly relativistic Gaussian charge distribution

$$e = Ae^{-x^2/L^2} + \delta$$



- The stress tensor again experiences an oscillatory behaviour and the code crashes
- u^μ is strictly vanishes outside the computational volume in BDNK, while for the density frame it has an exponentially supressed tail.