Bulk evolution of linearized fluctuations

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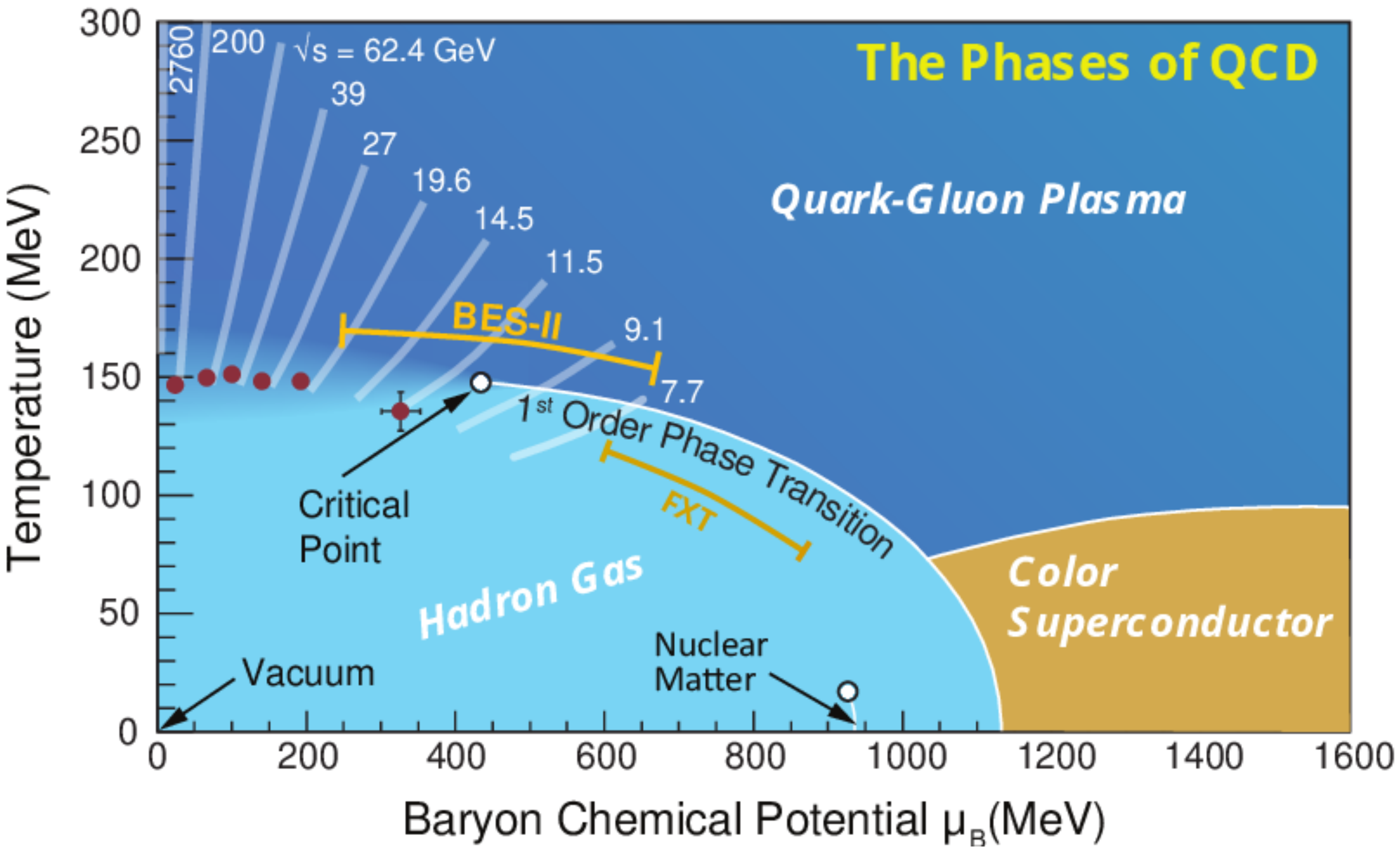
Introduction



Fluctuations in general

- In viscous relativistic fluid dynamics
- Quantum fluctuations
    - Initial fluctuations
- Thermal fluctuations
    - related to susceptibilities and EoS -> phase structure of QCD
    - fluctuation-dissipation relations

Dynamical fluctuations

- Stochastic
    - discretized noise sampled event-by-event
    - observables calculated after statistical averaging
- Hydro-kinetics
    - deterministic kinetic equations for the two-point functions of fluid dynamical fields
    - linearization of stochastic fluid dynamics
- Critical fluctuations - inclusion of non-linearities

Stochastic fluctuations

$$\partial_\mu T^{\mu\nu} = 0, \qquad T^{\mu\nu} = T^{\mu\nu}_{ideal} + T^{\mu\nu}_{viscous} + \Xi^{\mu\nu}$$

$$\partial_\mu J^\mu = 0, \qquad J^\mu = J^\mu_{ideal} + J^\mu_{viscous} + I^\mu_{noise}$$

$$< u^\gamma \partial_{;\gamma} \pi^{\mu\nu} > = -\frac{\pi^{\mu\nu} - \pi^{\mu\nu}_{NS}}{\tau_\pi} - \frac{4}{3}\pi^{\mu\nu}\partial_{;\gamma}u^\gamma$$

- From fluctuation-dissipation relation

$$\partial_t \Xi^{ij} = -\frac{1}{\tau_\pi}\left(\Xi^{ij} - \xi^{ij}\right)$$

- Correlator of the noise

$$\langle \xi^{\mu\nu}(x)\xi^{\alpha\beta}(x')\rangle = \left[2\eta T(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}) + 2\left(\zeta - \frac{2}{3}\eta\right)T\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\delta^4(x - x')$$

- Discretization of delta function leads to
  - Lattice spacing dependence
  - Large noise contributions can locally lead to negative densities

Linearized equations

- Introducing a perturbation to hydro equations

$$\partial_\nu \left(T_0^{\mu\nu} + \delta T^{\mu\nu} \right) = 0$$

- Rewriting the equations as

$$\partial_0 Q^\mu + \partial_i \overrightarrow{F}^\mu = 0,$$ where $Q^\mu = T^{0\mu}$ and $F^{i\mu} = T^{i\mu}$

- Decoupling for background and perturbations

$$\partial_0 Q_0^\mu + \partial_i \overrightarrow{F}_0^\mu = 0$$

$$\partial_0 \, \delta Q^\mu + \partial_i \, \delta \overrightarrow{F}^\mu = 0$$

- perturbations have zero mean over the ensemble average

Linearized equations

- Introducing the perturbation to primitive variables

$$\varepsilon = \varepsilon_0 + \delta\varepsilon$$

$$p = p_0 + \delta p \quad \text{and} \quad \pi^{\mu\nu} = \pi_0^{\mu\nu} + \delta\pi^{\mu\nu}$$

$$u^\mu = u_0^\mu + \delta u^\mu$$

- We arrive at the set of equations

$$Q_0^\mu = (\varepsilon_0 + p_0)u_0^0 u_0^\mu - p_0 g^{\mu 0} + \pi_0^{\mu 0}$$

$$F_0^{\mu i} = (\varepsilon_0 + p_0)u_0^\mu u_0^i - p_0 g^{\mu i} + \pi_0^{\mu i}$$

$$\delta Q^\mu = (\varepsilon_0 + p_0)(u_0^\mu \delta u^0 + u_0^0 \delta u^\mu) + (\delta\varepsilon + \delta p)u_0^0 u_0^\mu - \delta p g^{\mu 0} + \delta\pi^{\mu 0}$$

$$\delta F^{\mu i} = (\varepsilon_0 + p_0)(u_0^\mu \delta u^i + u_0^i \delta u^\mu) + (\delta\varepsilon + \delta p)u_0^\mu u_0^i - \delta p g^{\mu i} + \delta\pi^{\mu i}$$

Linearized equations

- Linearizing the transport coefficients

$$\eta = \eta_0 + \delta\eta$$
$$\zeta = \zeta_0 + \delta\zeta$$
$$\tau_\pi = \tau_{\pi 0} + \delta\tau_\pi$$
$$\tau_\Pi = \tau_{\Pi 0} + \delta\tau_\Pi$$

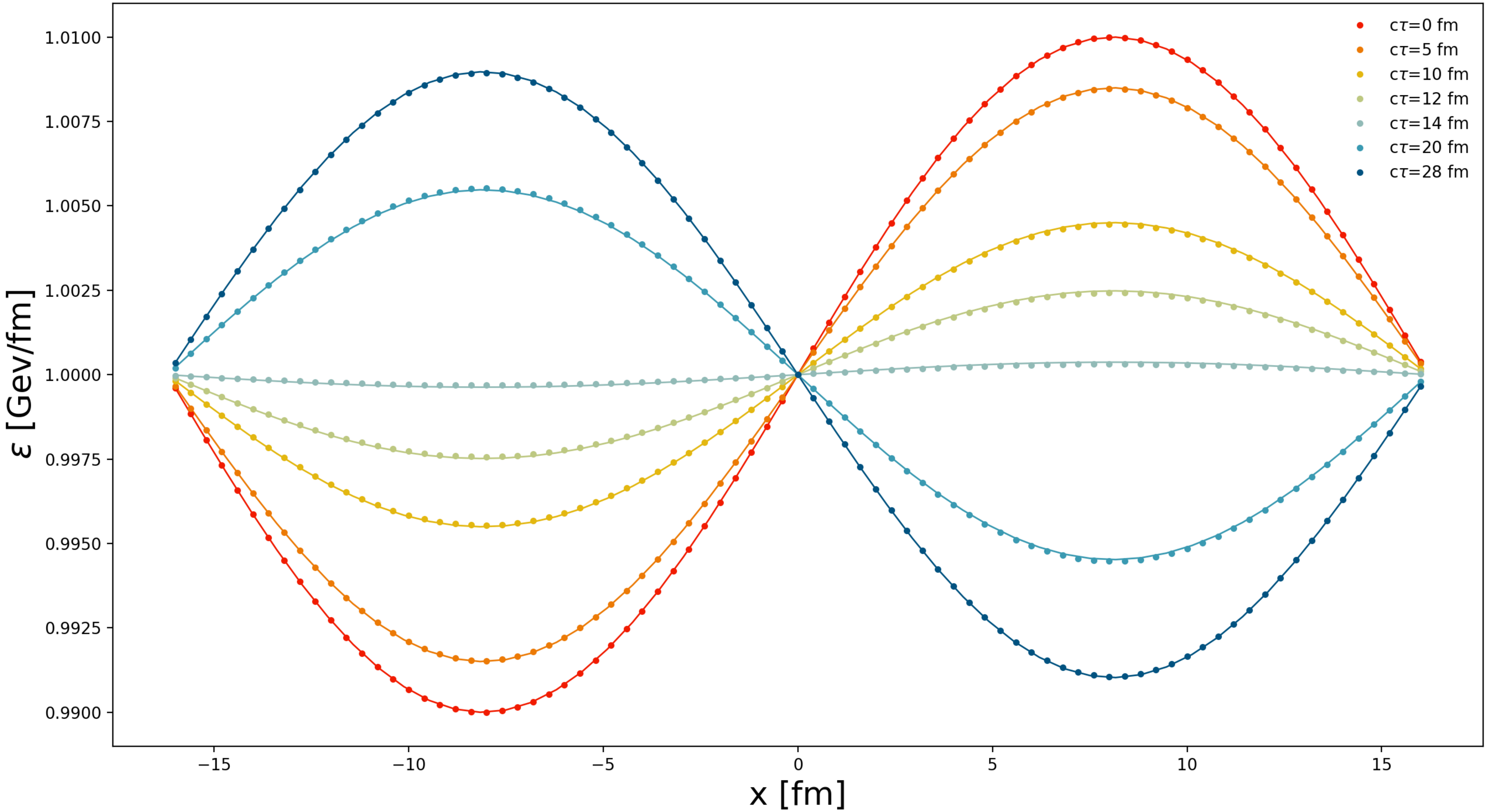
$$< > = \frac{1}{2}\Delta^\mu_\alpha \Delta^\nu_\beta + \frac{1}{2}\Delta^\nu_\alpha \Delta^\mu_\beta - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta} = < >_0 + < >_\delta$$

- We get Israel-Stewart equations

$$< u_0^\gamma \partial_{\delta;\gamma}\delta\pi^{\mu\nu} >_0 + < \delta u^\gamma \partial_{0;\gamma}\pi_0^{\mu\nu} >_0 + < u_0^\gamma \partial_{0;\gamma}\pi_0^{\mu\nu} >_\delta =$$
$$= -\frac{\delta\pi^{\mu\nu} - \delta\pi^{\mu\nu}_{NS}}{\tau_{\pi 0}} - \frac{\pi_0^{\mu\nu} - \pi^{\mu\nu}_{0_{NS}}}{\tau_{\pi 0}^2}\delta\tau_\pi - \frac{4}{3}(\pi_0^{\mu\nu}\partial_{\delta;\gamma}\delta u^\gamma + \delta\pi^{\mu\nu}\partial_{0;\gamma}u_0^\gamma)$$

Test of linearized equations in box mode

- vHLLE - periodic boundaries, Cartesian coordinates, static background, NS limit
- Perturbation of sinus wave $\varepsilon = 0.01 \sin\left(\frac{2\pi}{L}x\right)$



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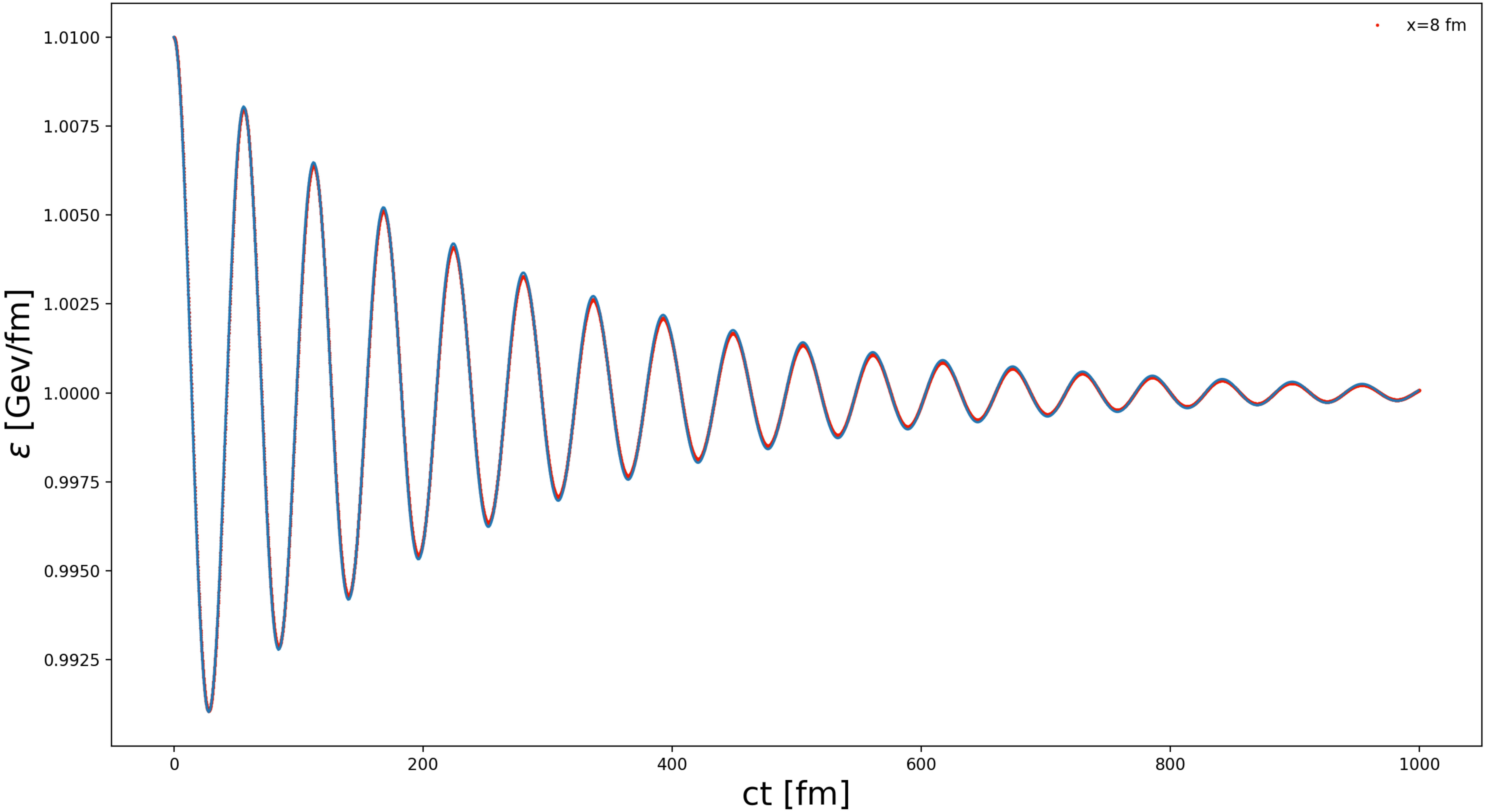

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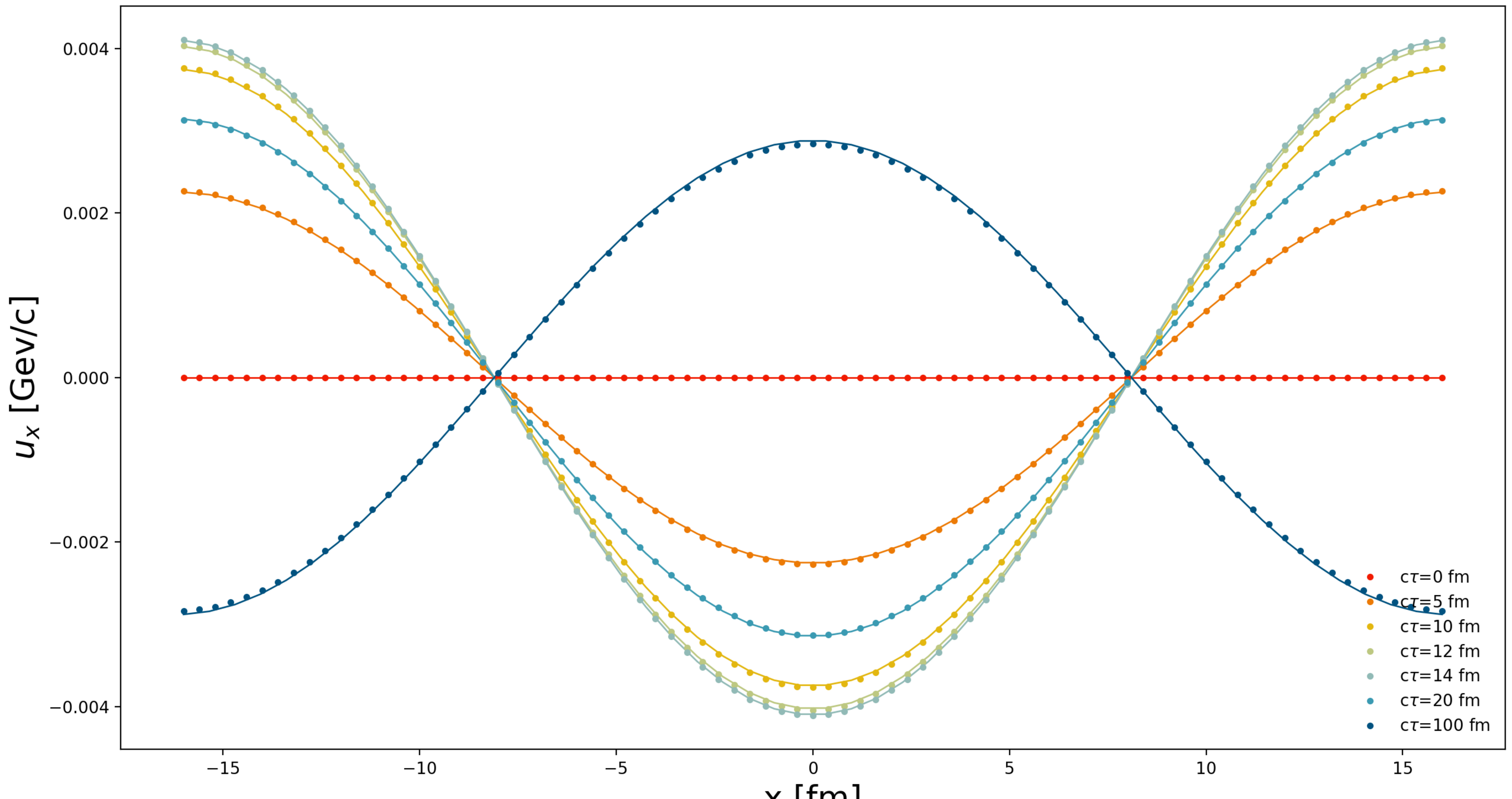


Introducing of stochastic noise to linearized equations

- Given

$$\partial_t \Xi^{ij} = -\frac{1}{\tau_\pi}\left(\Xi^{ij} - \xi^{ij}\right)$$

$$\langle \xi^{\mu\nu}(x)\xi^{\alpha\beta}(x')\rangle = \left[2\eta T_0(\Delta_0^{\mu\alpha}\Delta_0^{\nu\beta} + \Delta_0^{\mu\beta}\Delta_0^{\nu\alpha}) + 2\left(\zeta - \frac{2}{3}\eta\right)T_0\Delta_0^{\mu\nu}\Delta_0^{\alpha\beta}\right]\delta^4(x-x')$$

- $\xi^{\mu\nu}$ has the same structure as $\pi^{\mu\nu}$

$$T^{\mu\nu} = T^{\mu\nu}_{id} + T^{\mu\nu}_{visc} + \Xi^{\mu\nu} = T^{\mu\nu}_{id} + T^{'\mu\nu}_{visc}$$ and $$\delta\pi^{'\mu\nu} = \delta\pi^{\mu\nu} + \xi^{\mu\nu}$$

$$< u_0^\gamma \partial_{\delta;\gamma}\delta\pi^{'\mu\nu} >_0 + < \delta u^\gamma \partial_{0;\gamma}\pi_0^{\mu\nu} >_0 + < u_0^\gamma \partial_{0;\gamma}\pi_0^{\mu\nu} >_\delta =$$

$$= -\frac{\delta\pi^{'\mu\nu} - \delta\pi^{\mu\nu}_{NS} - \xi^{\mu\nu}}{\tau_{\pi 0}} - \frac{\pi_0^{\mu\nu} - \pi^{\mu\nu}_{0_{NS}}}{\tau^2_{\pi 0}}\delta\tau_\pi - \frac{4}{3}(\pi_0^{\mu\nu}\partial_{\delta;\gamma}\delta u^\gamma + \delta\pi^{'\mu\nu}\partial_{0;\gamma}u_0^\gamma)$$

Discretization and sampling of noise

- Discretizing the delta function

$$\langle \xi^{\mu\nu}(x)\xi^{\alpha\beta}(x')\rangle = \left[2\eta_0 T_0(\Delta_0^{\mu\alpha}\Delta_0^{\nu\beta} + \Delta_0^{\mu\beta}\Delta_0^{\nu\alpha}) + 2\left(\zeta_0 - \frac{2}{3}\eta_0\right)T_0\Delta_0^{\mu\nu}\Delta_0^{\alpha\beta}\right]\boxed{\frac{1}{\Delta t \Delta V}}$$

- Sampling from Gaussian with covariance

$$\boxed{\langle \xi^{\mu\nu}(x)\xi^{\alpha\beta}(x')\rangle = 2\eta_0 T_0(\Delta_0^{\mu\alpha}\Delta_0^{\nu\beta} + \Delta_0^{\mu\beta}\Delta_0^{\nu\alpha})\frac{1}{\Delta t \Delta V}}$$

- Symmetric tensor

- Subtracting 1/3 of trace from spatial elements

[C. Young, *Phys.Rev.C* 89 (2014) 2]

Structure factor

- Static constant background
- Structure factor - correlation of fields - power spectrum

$$S(\omega, \vec{k}) = A \cdot \langle \delta \hat{U}(\omega, \vec{k}) \delta \hat{U}(\omega', -\vec{k}) \rangle$$

- where *A* is normalization
- Related to susceptibilities via fluctuation-dissipation relation
- Equal time correlation - static structure factor

$$S(\vec{k}) = A \cdot \langle \delta \hat{U}(\vec{k}) \delta \hat{U}(-\vec{k}) \rangle$$

Normalization of structure factor

$$\partial_t U = LU + KW$$

- in NS limit in 1D [A. Donev et al., CAMCOS *(2009)*]

$$\begin{bmatrix} \partial_t \delta\varepsilon \\ \partial_t \delta u^x \end{bmatrix} = -\partial_x \begin{bmatrix} (\varepsilon_0 + p_0)\delta u^x \\ \dfrac{c_s^2}{\varepsilon_0 + p_0}\delta\varepsilon \end{bmatrix} + \partial_x \begin{bmatrix} 0 \\ \dfrac{4}{3}\dfrac{\eta}{\varepsilon_0 + p_0}\partial_x \delta u^x \end{bmatrix} + \frac{1}{\varepsilon_0 + p_0}\partial_x \begin{bmatrix} 0 \\ \xi \end{bmatrix}$$

- dentifiing L and K matrices

$$\hat{L} = -ik \begin{pmatrix} 0 & \varepsilon_0 + p_0 \\ \dfrac{c_s^2}{\varepsilon_0 + p_0} & -ik\dfrac{4}{3}\dfrac{\eta}{\varepsilon_0 + p_0} \end{pmatrix}, \hat{K} = ik \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{\varepsilon_0 + p_0} \end{pmatrix} = -\hat{K}^* \text{ and } C_W = \begin{pmatrix} 0 & 0 \\ 0 & \xi \end{pmatrix} \text{ where } \xi = \frac{8}{3}\eta_0 T_0$$

- Using the equation

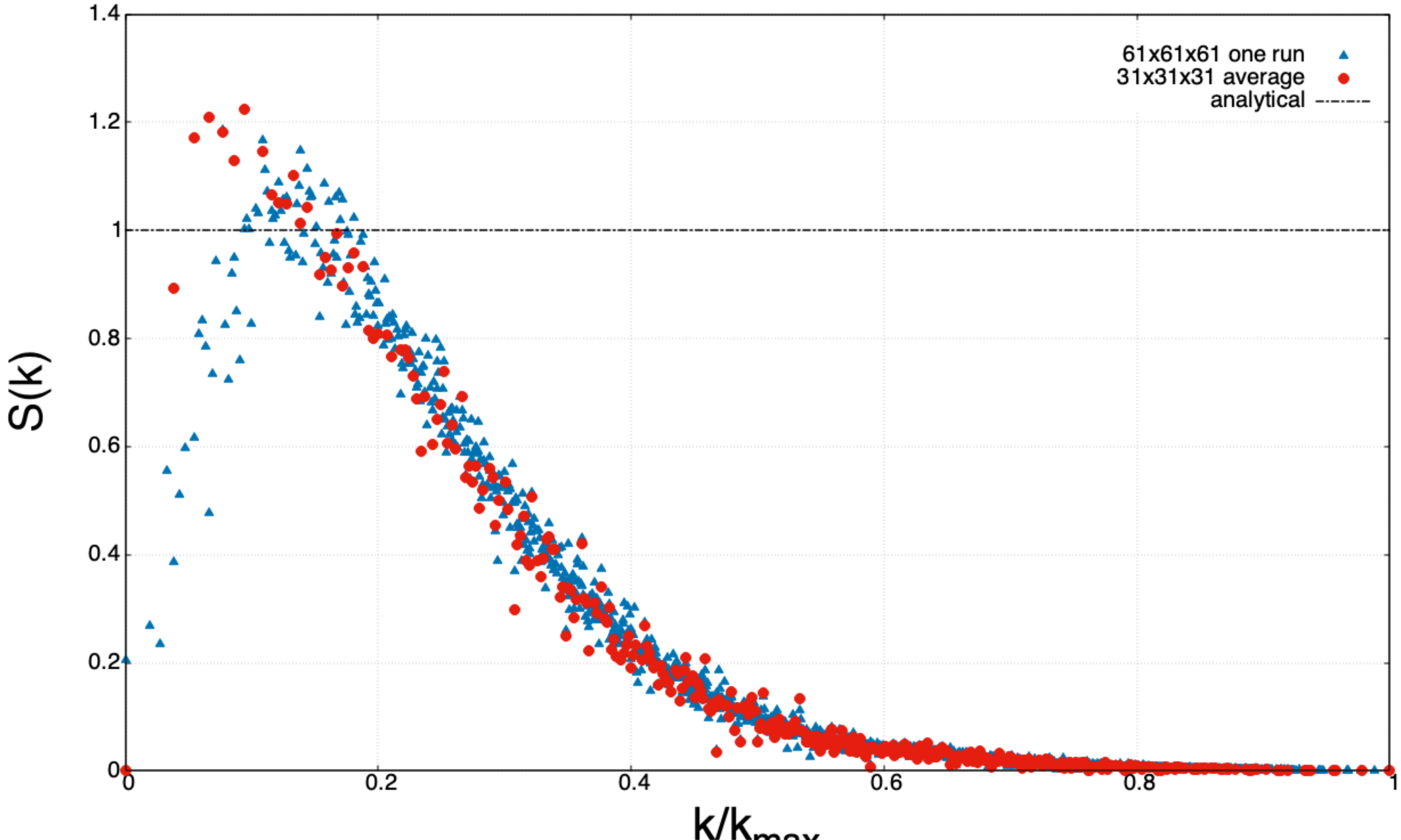
$$\hat{L}S + S\hat{L}^* = -\hat{K}C_W\hat{K}^*$$

The structure factor matrix $S(k) = \begin{pmatrix} c_s^{-2}(\varepsilon_0 + p_0)T_0 & 0 \\ 0 & (\varepsilon_0 + p_0)^{-1}T_0 \end{pmatrix}$ independent of k

Current status

- Using KISS FFT to transform fields to Fourier space

$$S(k) = \frac{V}{c_s^2(\varepsilon_0 + p_0)T_0}\langle \delta\varepsilon(k)\delta\varepsilon(-k)\rangle$$



Conclusion and further steps

- Thermal fluctuations should be included - Fluctuation dissipation theorem
- Fluctuations provide good basis for studying phase diagram
    - Critical fluctuation for studying critical point
- Stochastic fluid dynamics can include non-linear terms
    - But it has some difficulties - fluctuation larger than background, discretization dependence
- Further steps
    - Calculating static structure factor for finer grid
    - Dynamic structure factor
    - Renormalization of the grid dependence