Emergence of hydrodynamics from kinetic theory

Hydrodynamics and related observables in heavy ion collisions

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Based on work done in collaboration with Li Yan (see arXiv: 2106.10508 and earlier papers) and more recently with Sunil Jaiswal et al (2208.02750)

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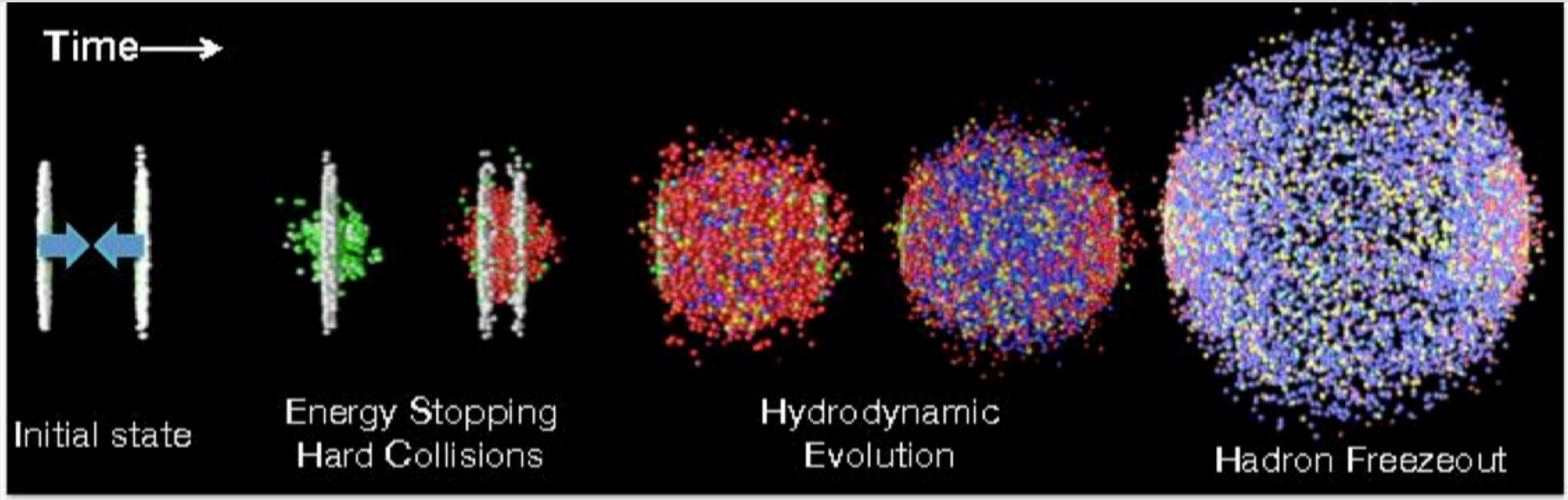


The hydrodynamic description of matter produced in heavy ion collisions works amazingly well !...

WHY IS THAT SO ?

IS THAT SO ?

Main message of the present talk:

The apparent success of Israel-Stewart second order hydrodynamics in allowing early time (out of equilibrium) description of matter expansion has nothing to do with "hydrodynamics" proper. It results from a subtle property of IS equations that mimic the early time, collisionless, regime.

What is hydrodynamics

Traditional view

Fluid behavior requires (some degree of) local equilibration (='thermalization').

Usual picture:

- **microscopic degrees of freedom relax quickly towards *local equilibrium***
- **long wavelength modes, associated to conservation laws, relax on longer time scales ⟶ *hydrodynamic modes***

Modern perspective

Effective theory for long wavelength modes (gradient expansions, etc

In either case the *long time behaviour* is controlled by hydrodynamics, irrespective of initial conditions

From kinetics to hydrodynamics (1)

System of freely moving particles

$$\rho(t, \boldsymbol{x}) = \sum_n \mathrm{e}_n \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_n(t)) \equiv J^0(t, \boldsymbol{x})$$

$$\boldsymbol{J}(t, \boldsymbol{x}) = \sum_n \mathrm{e}_n \frac{\mathrm{d}\boldsymbol{x}_n(t)}{\mathrm{d}t} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_n(t)) \equiv \boldsymbol{J}(t, \boldsymbol{x})$$

Local conservation law

$$\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0$$

In covariant notation

$$J^\mu(x) = \int \mathrm{d}\tau \sum_n \mathrm{e}_n \frac{\mathrm{d}x_n^\mu(\tau)}{\mathrm{d}\tau} \delta^{(4)}(x - x_n(\tau))$$

$$\partial_\mu J^\mu = 0$$

Energy-momentum tensor

$$T^{\mu\nu} = \int \mathrm{d}\tau \sum_n p_n^\mu \frac{\mathrm{d}x_n^\nu(\tau)}{\mathrm{d}\tau} \delta^{(4)}(x - x_n(\tau))$$

$$\partial_\mu T^{\mu\nu} = 0$$

The conservation laws remain valid for *average quantities*

over *initial conditions*

over phase space (*coarse graining*)

$$\partial_\mu \langle J^\mu \rangle = 0 \qquad \partial_\mu \langle T^{\mu\nu} \rangle = 0$$

From kinetics to hydrodynamics (2)

Consider the (non-relativistic) Boltzmann equation

distribution function

$$\frac{\partial}{\partial t} f + \frac{\vec{p}}{m} \cdot \vec{\nabla}_r f - \vec{\nabla}_r U \cdot \vec{\nabla}_p f = C[f] \qquad f = f(\vec{r}, \vec{p}, t)$$

free motion force field collisions

Introduce density, velocity field, and pressure

$$n(\vec{r}, t) = \int \frac{d^3 p}{(2\pi)^3} f(\vec{r}, \vec{p}, t) \qquad n\vec{v}(\vec{r}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{\vec{p}}{m} f(\vec{r}, \vec{p}, t) \qquad P_{ij}(\vec{r}, t) = \int \frac{d^3 p}{(2\pi)^3} p_i \frac{p_j}{m} f(\vec{r}, \vec{p}, t)$$

When collisions dominate, the pressure becomes isotropic $P_{ij}(\vec{r}, t) = \delta_{ij} P(\vec{r}, t)$

and the kinetic equation reduces formally to hydrodynamics

$$\partial_t n + \vec{\nabla}(n\vec{v}) = 0$$

$$m\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} + \vec{\nabla} U + \frac{1}{n} \vec{\nabla} P = 0$$

Conservation laws for particle number and momentum

Thermalization

(relaxation towards local equilibrium)

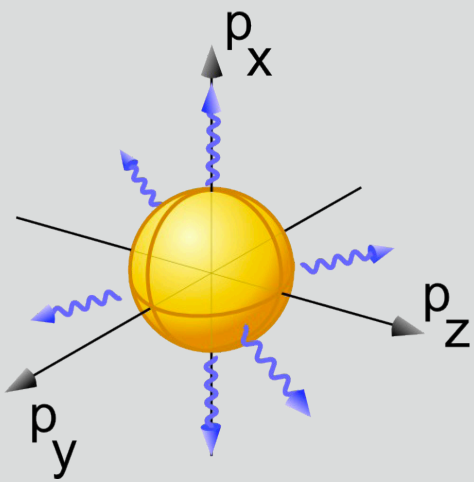
Two main issues

i) relative populations of different momentum modes

ii) isotropy of momentum distribution

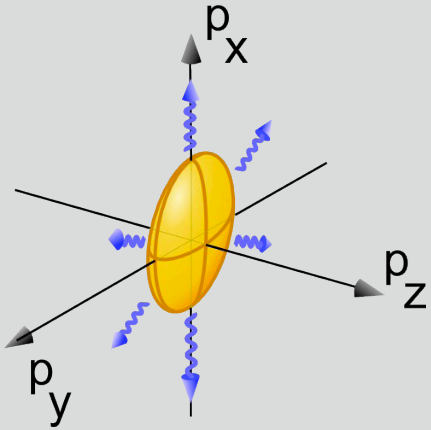
Main topic for the rest of this talk

"Isotropization"



Longitudinal expansion hinders isotropization

The fast expansion of the matter along the collision axis drives the momentum distribution to a very flat distribution



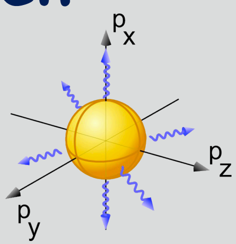
Translates into the existence of two different pressures

$$P_L = \int_p \frac{p_z^2}{p_0} f \qquad P_T = \int_p \frac{p_T^2}{p_0} f$$

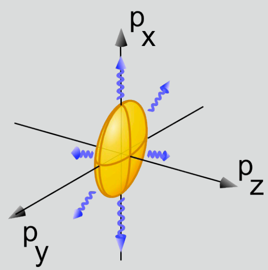
(longitudinal) (transverse)

Competition between

Collisions →



Expansion →



Anisotropy $(\sim P_L - P_T)$ relaxes slowly, like a 'collective' variable associated to a conservation law

Simple kinetic equation
(Bjorken flow)

- 1+1 dimensional expansion, in relaxation time approximation

$$\left[\partial_\tau - \frac{p_z}{\tau}\partial_{p_z}\right] f(\boldsymbol{p}/T) = -\frac{f(\boldsymbol{p},\tau) - f_{\rm eq}(p,\tau)}{\tau_R}$$

expansion collisions

- Describes the transition from the collisionless regime $(\tau \ll \tau_R)$ to the regime dominated by collisions, leading eventually to hydrodynamics $(\tau \gg \tau_R)$

- Can be solved straightforwardly by standard numerical techniques. We shall follow a less direct, but more insightful (semi) analytic approach.

Special moments of the momentum distribution

(JPB, Li Yan, 2017, 18, 19)

Special moments $p_z = p\cos\theta$

$$\mathcal{L}_n \equiv \int_p p^2 P_{2n}(\cos\theta) f(\boldsymbol{p})$$

(Legendre polynomial)

$$P_0(z) = 1 \qquad P_2(z) = \frac{1}{2}(3z^2 - 1)$$

Why these moments ?

- **There is too much information in the distribution function**
- **We want to focus on the angular degrees of freedom**

The energy momentum tensor is described by first two moments

$$T^{\mu\nu} = \int_p f(\boldsymbol{p}) p^\mu p^\nu \qquad \mathcal{L}_0 = \varepsilon \qquad \mathcal{L}_1 = \mathcal{P}_L - \mathcal{P}_T$$

We are looking for an effective theory for these two moments

Coupled equations for the moments

$$\frac{\partial \mathcal{L}_n}{\partial \tau} = -\frac{1}{\tau}\left[a_n \mathcal{L}_n + b_n \mathcal{L}_{n-1} + c_n \mathcal{L}_{n+1}\right] - \frac{\mathcal{L}_n}{\tau_R} \quad (n \geq 1)$$

(Free streaming)

(collisions)

$$\frac{\partial \mathcal{L}_0}{\partial \tau} = -\frac{1}{\tau}\left[a_0 \mathcal{L}_0 + c_0 \mathcal{L}_1\right]$$

- **The coefficients a_n, b_n, c_n are pure numbers ($a_0 = 4/3$, $c_0 = 2/3$)**
- **Interesting system of coupled linear equations, with nearest neighbour couplings**
- **Exact solution provides exact values for the energy density and pressures, but does not allow the complete reconstruction of the distribution function**
- **The competition between expansion and collisions is made obvious. Note the absence of collisional damping for the energy density.**

Effective theory obtained by 'eliminating' moments $\mathcal{L}_{n>1}$

Two-moment truncation

(effective theory)

$$\frac{\partial}{\partial \tau}\begin{pmatrix}\mathcal{L}_0\\ \mathcal{L}_1\end{pmatrix} = -\frac{1}{\tau}\begin{pmatrix}a_0 & c_0\\ b_1 & a_1+\frac{\tau}{\tau_R}\end{pmatrix}\begin{pmatrix}\mathcal{L}_0\\ \mathcal{L}_1\end{pmatrix}$$

- Contains second order viscous hydrodynamics à la Israel-Stewart
- Views hydrodynamics as a coupled mode problem
- Amenable to analytic solution, bringing insight into the notions of attractors, general features of the gradient expansion, and its resummations in terms of trans-series, etc. [not discussed here]

[for analytic solution see JPB and L. Yan, PLB 820:136478 (2021)]

- Captures most important features of more sophisticated approaches, and can be made quantitatively accurate with a simple renormalization of a second order transport coefficient (a1) [see later].

Free streaming fixed points

One can transform the coupled linear equations into a single non linear differential equation for the quantity

$$\boxed{g_0(\tau) = \frac{\tau}{\mathcal{L}_0}\frac{\partial \mathcal{L}_0}{\partial \tau}} \qquad \left(\frac{P_L - P_T}{\varepsilon} = -\frac{1}{c_0}(a_0 + g_0)\right)$$

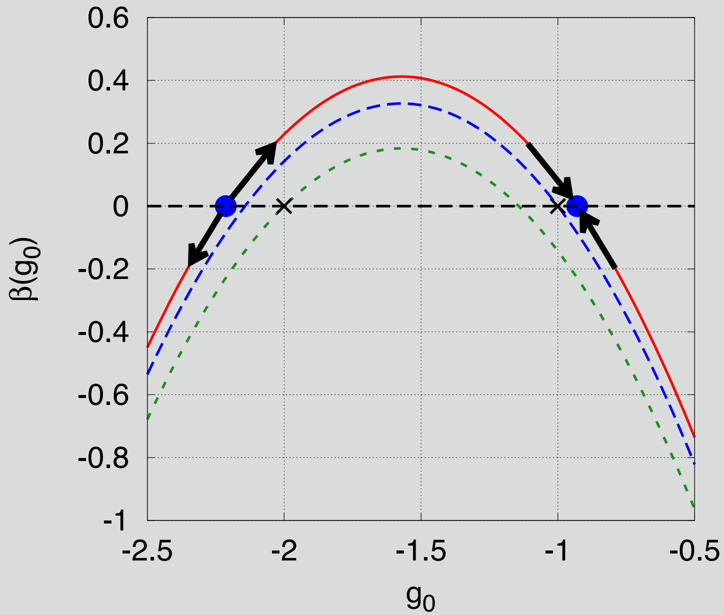
$$\tau\frac{\mathrm{d}g_0}{\mathrm{d}\tau} + g_0^2 + (a_0 + a_1)\,g_0 + a_1 a_0 - c_0 b_1 - \left[c_0 c_1 \frac{\mathcal{L}_2}{\mathcal{L}_0}\right] = 0$$

Write this as

$$\tau\frac{dg_0}{d\tau} = \beta(g_0) \qquad \beta(g_0) = -g_0^2 - (a_0 + a_1)g_0 - a_0 a_1 + c_0 b_1$$

$$g_0^* = -\lambda_1 = -2.21$$

(unstable)



$$g_0^* = -\lambda_0 = -0.929$$

(stable)

exact fixed point (-1) can be recovered by adjusting L2 to its exact value, known near a fixed point, e.g. at the stable fixed point,

$$\mathcal{L}_2/\mathcal{L}_0 = 3/8$$

- **This fixed point structure is only moderately affected by higher moments**
- **This structure is approximately captured by Israel-Stewart hydrodynamics**

Including collisions

$$w\frac{dg_0}{dw} = \beta(g_0, w) \qquad\qquad w \equiv \tau/\tau_R$$

$$\beta(g_0) = -g_0^2 - (a_0 + a_1 + w)\, g_0 - a_1 a_0 + c_0 b_1 - a_0 w + \left[c_0 c_1 \frac{\mathcal{L}_2}{\mathcal{L}_0}\right]$$

This non linear equation is formally identical to that resulting from Israel-Stewart formulation of second order viscous hydrodynamics [Heller, Spalinski , 2015]

$w \ll 1$ $(\tau \ll \tau_R)$ one recovers the two free streaming fixed points

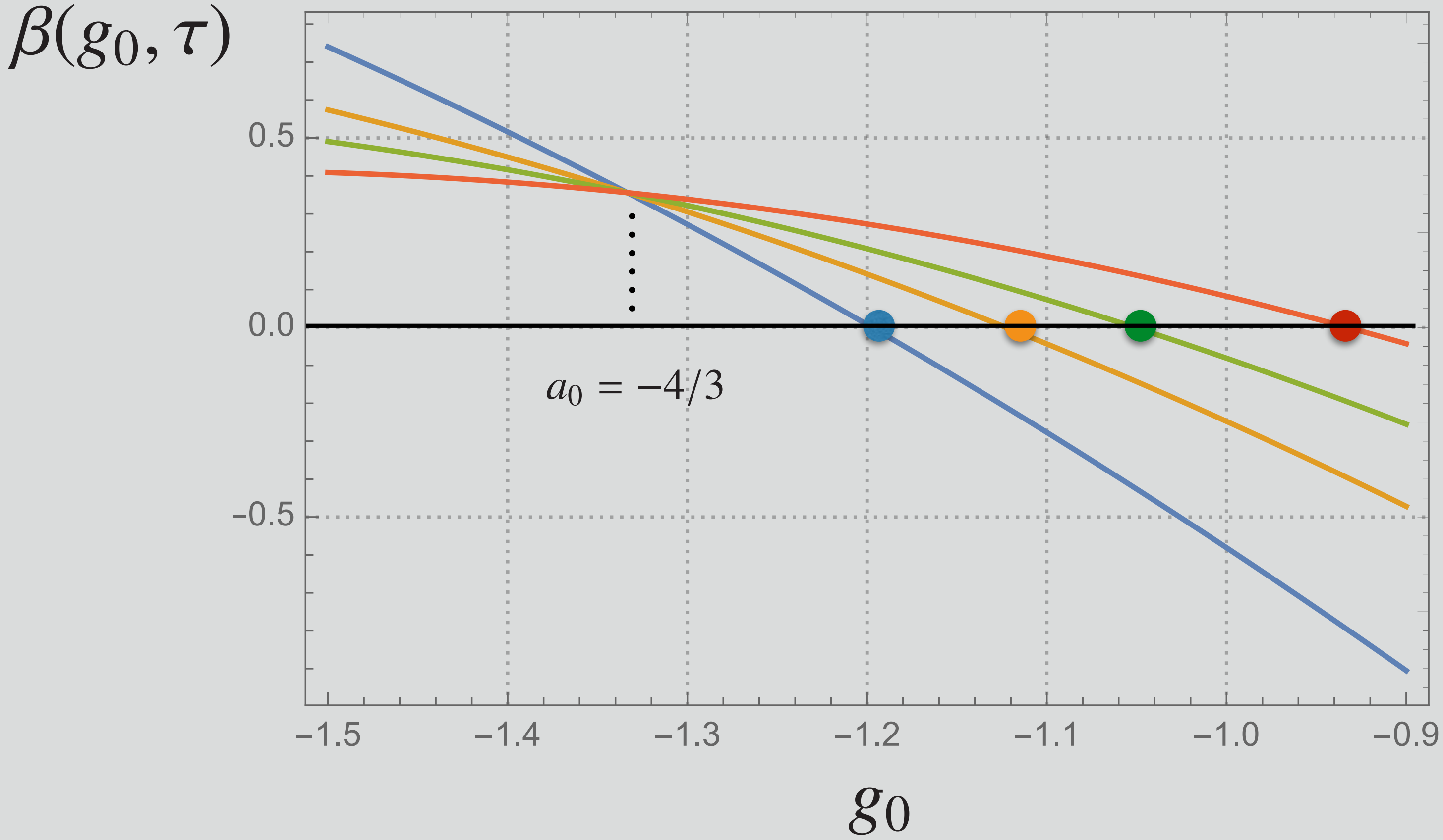
$w \gg 1$ $(\tau \gg \tau_R)$ $g_0 + a_0 = 0, \quad g_0 = -4/3$ hydrodynamic fixed point

The attractor solution is the particular solution that starts from the stable collisionless fixed point at small time and evolves "slowly" to the hydrodynamic fixed point at late time.

All solutions converge, soon or later depending on the initial conditions, towards the attractor (hence to hydrodynamics) at late time.

Attractor

Under the effect of collisions, the stable collisionless fixed point evolves "slowly" into the hydrodynamic fixed point



"Pseudo fixed points" move "slowly" towards -4/3 as time increases

The "attractor" is the solution $g_0(\tau)$ that joins the (stable) collisionless fixed point at early time to the hydrodynamic fixed point at late time.

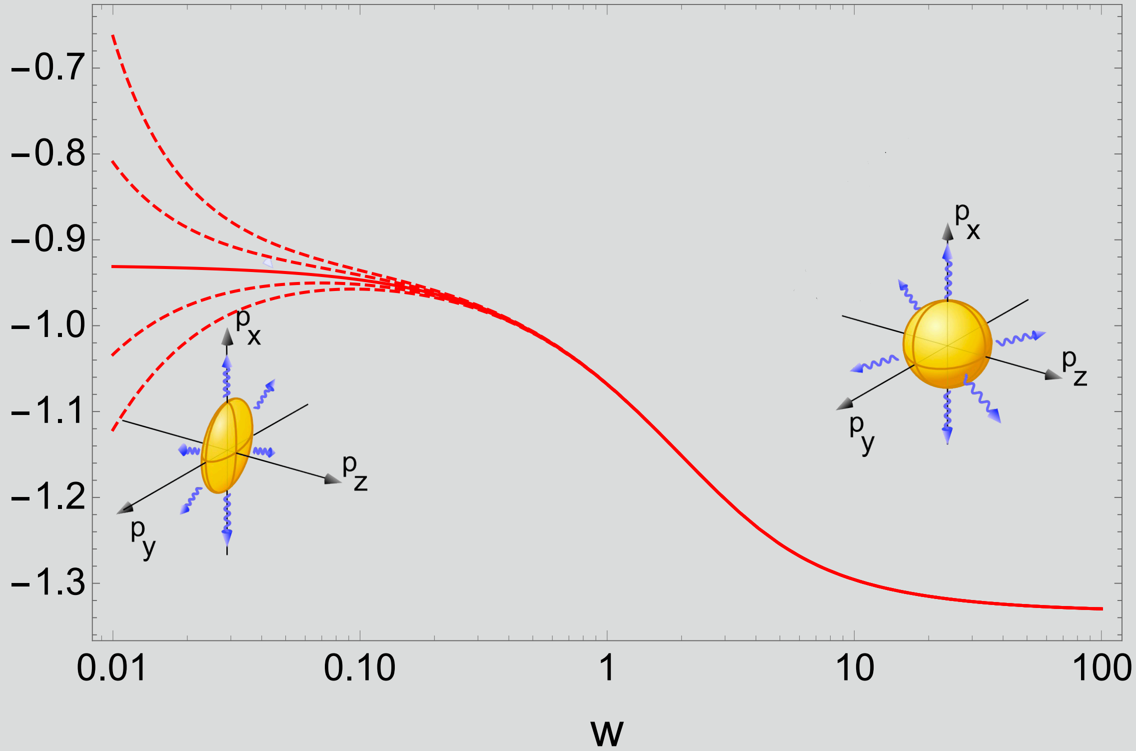
The transition from free streaming to hydrodynamics

(Attractor solution)

Early and late times are controlled by the free streaming and the hydrodynamic fixed points, respectively

$g_0 \approx -0.93$

Collisionless fixed point



$(w \equiv \tau/\tau_R)$

Hydro fixed point

(Universal!)

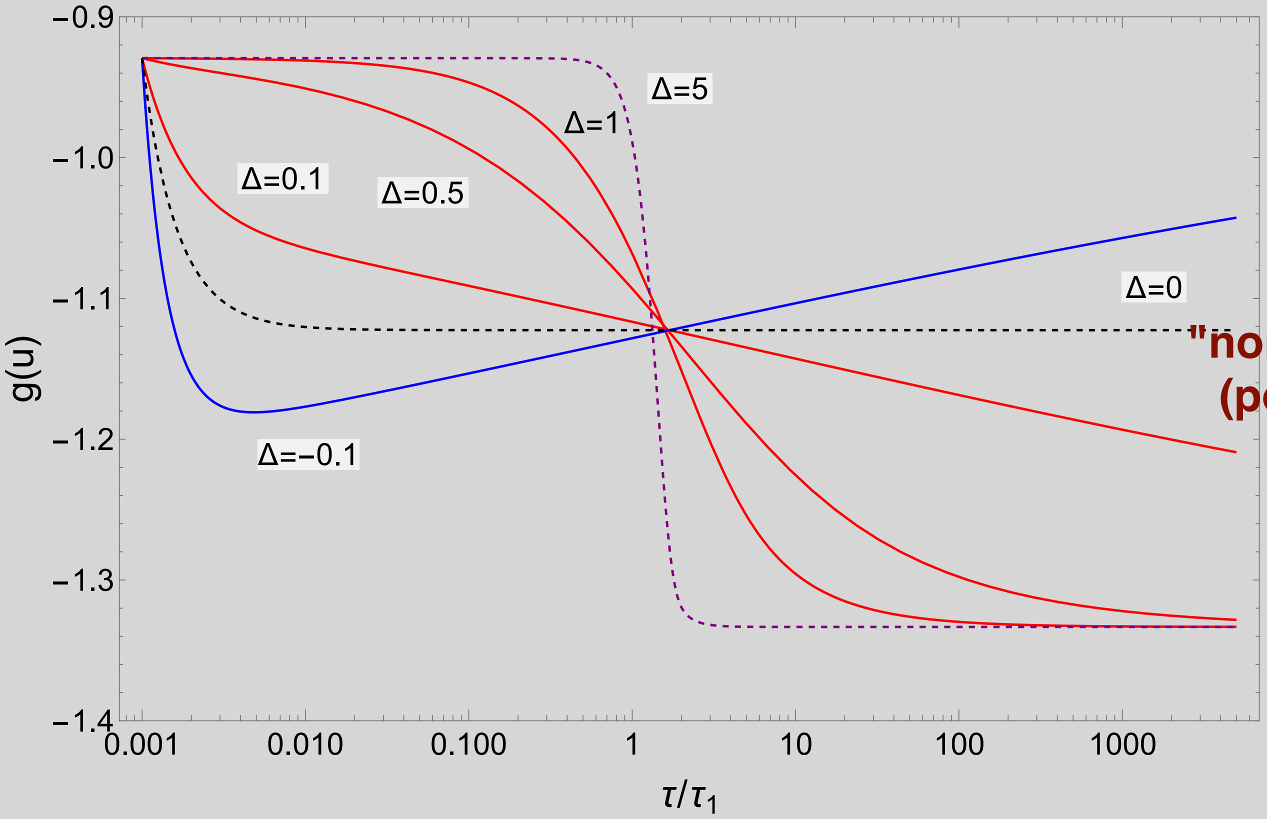
$g_0 = -\frac{4}{3}$

The transition region occurs when the collision rate is comparable to the expansion rate $(\tau \sim \tau_R)$

Time dependent relaxation time

$$\tau_R \sim \tau^{1-\Delta}$$

Δ **controls the "speed" of the transition**



"non thermal fixed point"
(power law behaviors)

Müller-Israel-Stewart hydrodynamics in the context of Bjorken flow

$$T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu - P g^{\mu\nu} + \pi^{\mu\nu}$$

(viscous pressure)

Gradient expansion

$$\pi = \frac{4\eta}{3}\frac{1}{\tau} \longrightarrow$$

NS eqn. $$\partial_\tau \varepsilon = -\frac{a_0}{\tau}\left(\varepsilon - \frac{\eta}{\tau}\right)$$

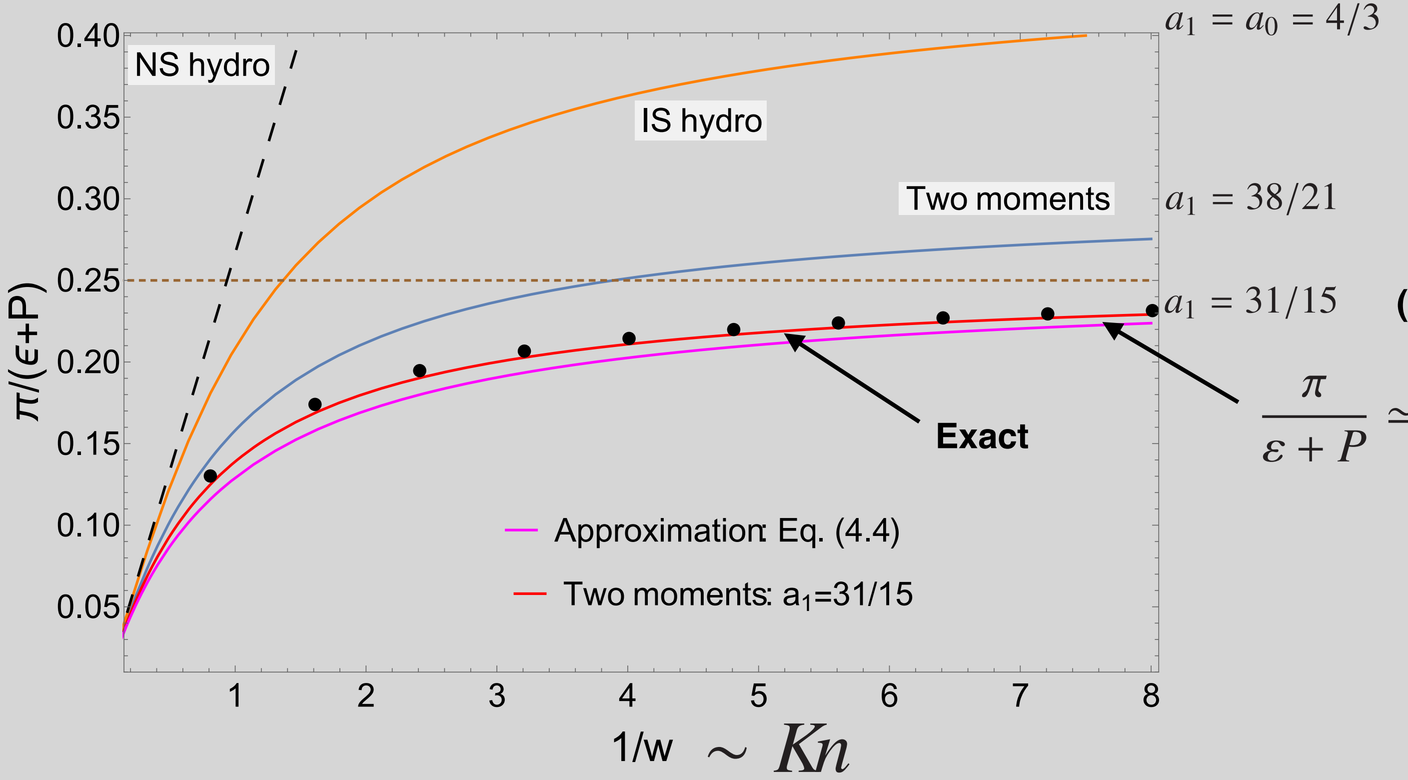
MIS hydro

$$\partial_\tau \pi + \frac{a_1^{IS}}{\tau}\pi = -\frac{1}{\tau_\pi}\left(\pi - \frac{4\eta}{3}\frac{1}{\tau}\right)$$

	a_1	b_1	τ_π
Two moments/DNMR hydro	38/21	8/15	τ_R
Navier-Stoke hydro	undetermined	$2\eta/(\varepsilon\tau_R)$	0^+
Isreal-Stewart hydro	a_0 or $a_0 + 21/10$	$2\eta/(\varepsilon\tau_R) = 8/15$	τ_π
BRSSS hydro	$a_0 + \frac{2C_{\lambda_1}}{3C_\tau}$	$2a_0\frac{C_\eta}{C_\tau}$	$\frac{C_\tau}{T}$
Kinetic hydro	31/15	8/15	τ_R

Time dependent relaxation time $(\tau_R \sim \tau)$

Constant cross sections $\tau_R \sim \dfrac{1}{\sigma n} \sim \tau \quad (n \sim 1/\tau)$ **[Denicol, Noronha , 2020]**



$$\frac{\pi}{\varepsilon + P} \simeq \frac{b_1}{2} \frac{Kn}{1 + 2b_1 Kn}$$

$$\left(\eta = \frac{b_1}{2}\varepsilon\tau_R\right)$$

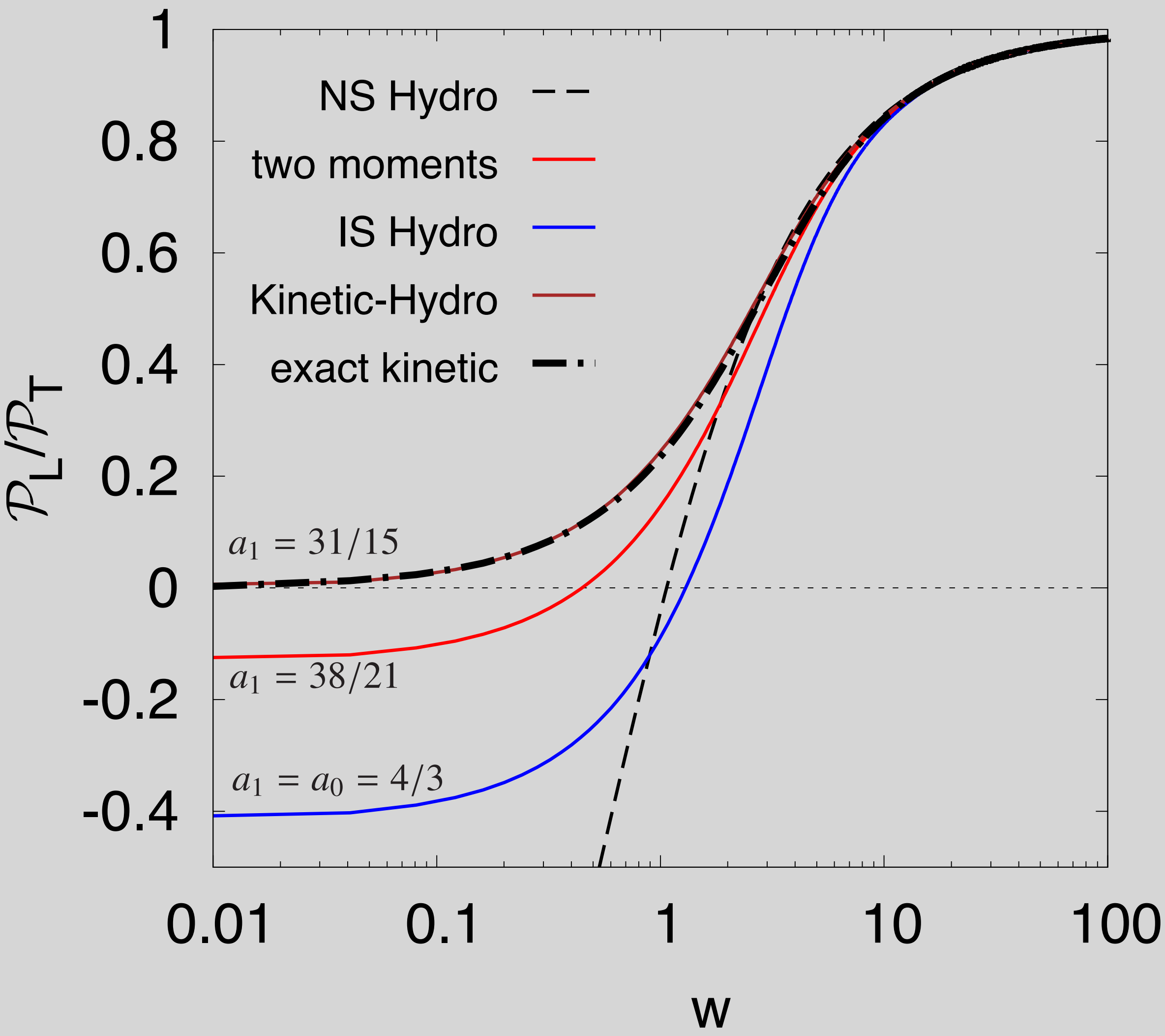
Changing a1 (a second order 'transport coefficient') does not "improve" hydrodynamics, but rather improves the location of the collisionless fixed point

Renormalization of a1

$$-c_0 c_1 \frac{\mathcal{L}_2}{\mathcal{L}_0} = -c_1 c_0 \frac{A_2}{A_1}\frac{\mathcal{L}_1}{\mathcal{L}_0} = c_1 \frac{A_2}{A_1}(g_0 + a_0) \quad \longrightarrow \quad a_1 \mapsto a_1' = a_1 + c_1 \frac{A_2}{A_1} = \frac{31}{15}$$

Renormalizing a1 cures unphysical features of two-moment truncation (and other Israel-Stewart calculations)

$$\tau \frac{\mathrm{d}g_0}{\mathrm{d}\tau} + g_0^2 + \left(a_0 + a_1 + \frac{\tau}{\tau_R} \right) g_0 + a_1 a_0 - c_0 b_1 + \frac{a_0 \tau}{\tau_R} = 0$$



Conclusions

The solution of a simple kinetic equation for Bjorken flow was analysed in terms of special moments of the distribution function.

The simplest two moment-truncation yields an 'effective' theory that captures the main qualitative features of the dynamics, in particular the transition from the collisionless regime to hydrodynamics. It encompasses all versions of second order (Israel-Stewart) hydrodynamics

The collisionless regime is characterized by two fixed points, one stable, the other unstable. The effect of the collisions is to move "slowly" the stable free streaming fixed point into the (universal) hydrodynamic fixed point.

Conclusions

The "attractor" emerges as the solution that joins the collisionless fixed point at t=0 to the hydrodynamic fixed point at large time. The vicinities of the two fixed points are easy to control (known ratios of moments in free streaming, Navier-Stokes in hydrodynamics). Large deviations from the hydrodynamic fixed point involves information about the collisionless fixed point.

Terminologies "hydrodynamic attractor", "early time attractor" are somewhat misleading. Vicinity of hydro fixed point is genuine hydro. Early time fixed point exists in collisionless regime of kinetic theory, not in holographic descriptions.

Hydrodynamic behavior emerges when it is supposed to do so, i.e. within kinetic theory when the collision rate is comparable to the expansion rate.

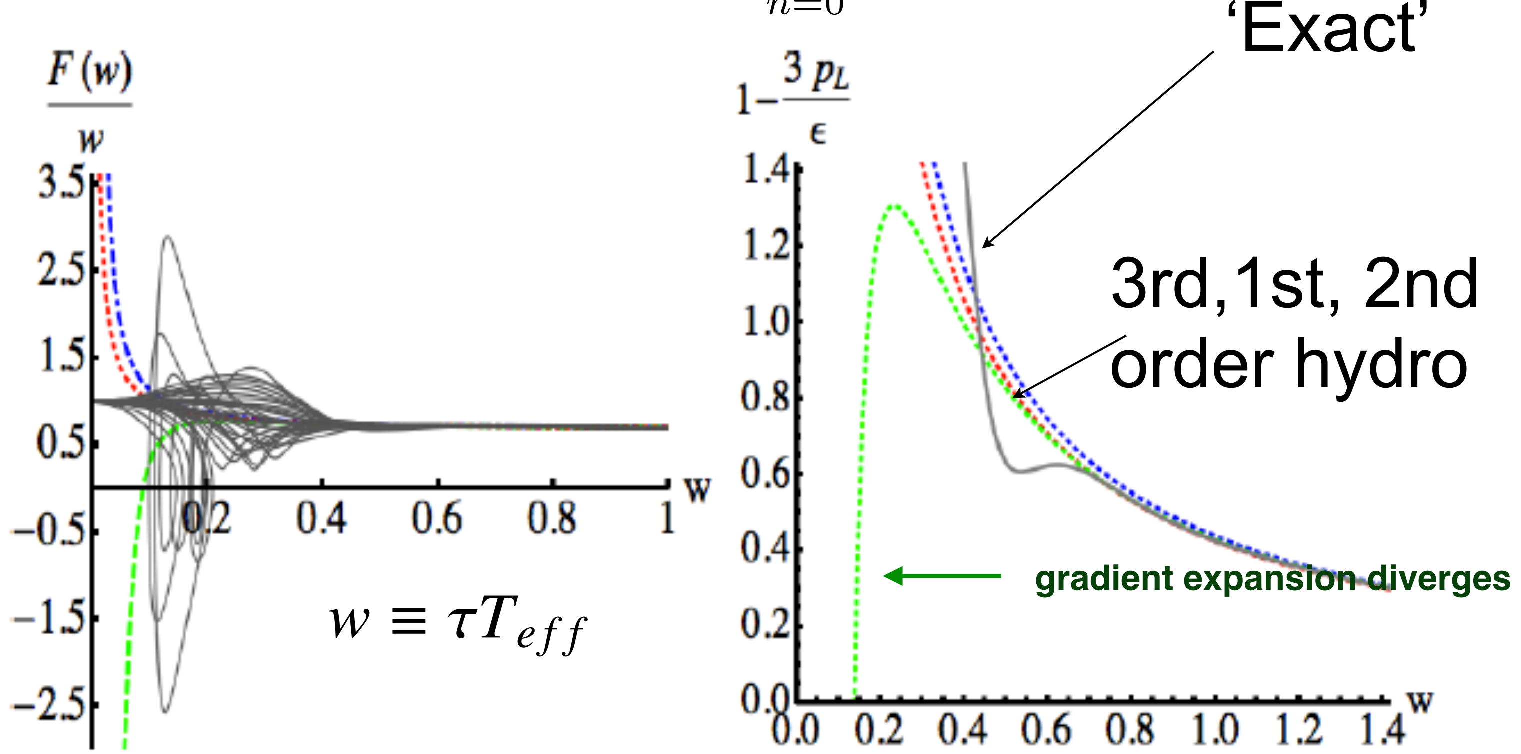
By 'improving' the transition region between the fixed points (i.e., adjusting the collisionless fixed point), one does not 'improve hydrodynamics'!

The present analysis extends with 'minor' modifications to the non-conformal case (2208.02750)

Holographic description of a boost invariant plasma

(Heller, Janik, Witaszczyk, [1103.3452])

Define $\mathcal{R} = \frac{\mathcal{P}_T - \mathcal{P}_L}{\epsilon}$ $\qquad f \equiv \frac{2}{3} + \frac{\mathcal{R}}{6} = \sum_{n=0}^{\infty} f_n w^{-n}$



$$1 - \frac{3P_L}{\varepsilon} \simeq 0.6 \leftrightarrow \frac{P_L}{P_T} \simeq 0.3$$

Viscous hydro can cope with "partial thermalization", and "large" differences between longitudinal and transverse pressures.