Emergence of hydrodynamics from kinetic theory

Hydrodynamics and related observables in heavy ion collisions Subatech, Nantes

October 28, 2024

Based on work done in collaboration with Li Yan (see arXiv: 2106.10508 and earlier papers) and more recently with Sunil Jaiswal et al (2208.02750)

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The hydrodynamic description of matter produced in heavy ion collisions works amazingly well !…

WHY IS THAT SO ?

IS THAT SO ?

Main message of the present talk:

The apparent success of Israel-Stewart second order hydrodynamics in allowing early time (out of equilibrium) description of matter expansion has nothing to do with "hydrodynamics" proper. It results from a subtle property of IS equations that mimic the early time, collisionless, regime.

What is hydrodynamics

Traditional view

Fluid behavior requires (some degree of) local equilibration (='thermalization').

Usual picture:

- **• microscopic degrees of freedom relax quickly towards** *local equilibrium*
- **• long wavelength modes, associated to conservation laws, relax on longer time scales** *hydrodynamic modes*

Modern perspective

Effective theory for long wavelength modes (gradient expansions, etc

In either case the *long time behaviour* **is controlled by hydrodynamics, irrespective of initial conditions**

From kinetics to hydrodynamics (1)

Consider a system of particles moving along trajectories that are given by x(C). We system or rreely moving p \blacksquare **System of freely moving particles**

 $\rho(t, x) = \sum_{n} e_n \delta^{(3)}(x - x_n(t)) \equiv J^0(t, x)$

 $c=$

 $x_n(t)$ \equiv

 $e_n \delta^{(3)}$ (

 $\mathrm{d}x_n(t)$

 dt

Local conservation law

$$
\rho(t, x) = \sum_{n} e_n o^{\omega}(x - x_n(t)) \equiv J^{\omega}(t, x)
$$

$$
J(t, x) = \sum_{n} e_n \frac{dx_n(t)}{dt} \delta^{(3)}(x - x_n(t)) \equiv J(t, x)
$$

$$
\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0
$$

where the conserved charge carried carried by the particle, and the sum over \mathbb{R} all the particles of the system. Let us verify that the charge is conserved. We have **In covariant notation** covariant form

 \boldsymbol{n}

 $\overline{ }$

 \boldsymbol{n}

 e_n

 $\rho(t, x) = \sum$

 $J(t, x) = \sum$

$$
J^{\mu}(x) = \int d\tau \sum_{n} e_n \frac{dx_n^{\mu}(\tau)}{d\tau} \delta^{(4)}(x - x_n(\tau)) \qquad \partial_{\mu} J^{\mu} = 0
$$

 $J^0(t,x)$

om $\overline{\mathbf{X}}$ $\overline{\mathsf{h}}$ \mathbf{u} ⇥ ⇣(3) (^x ^x=(C))⇤ ^dx=(C) **Energy-momentum tensor**

$$
T^{\mu\nu} = \int d\tau \sum_{n} p_n^{\mu} \frac{dx_n^{\nu}(\tau)}{d\tau} \delta^{(4)}(x - x_n(\tau)) \qquad \partial_{\mu} T^{\mu\nu} = 0
$$

ier initial cond waaion rawo n <mark>ii</mark> inne $\overline{}$ $\overline{}$?⇡ oarse:
C ^d^C *.* (2.39) **The conservation laws remain valid for** *average quantities* **over** *initial conditions* $\partial_{\mu}\langle J^{\mu}\rangle = 0$ $\partial_{\mu}\langle T^{\mu\nu}\rangle = 0$ Given Eq. (2.39) it is clear that)⇠⇡ =)⇡⇠, that it)⇠⇡ is a symmetric tensor. The **conservation laws remain valid for a I also is the energy density.**

From kinetics to hydrodynamics (2)

free motion force field collisions

Consider the (non-relativistic) Boltzmann equation

 $\frac{r}{m} \cdot V$

 $\frac{\partial}{\partial t} f + \frac{\vec{p}}{m}$

distribution function

$$
\vec{\nabla}_r f - \vec{\nabla}_r U \cdot \vec{\nabla}_p f = C[f] \qquad f = f(\vec{r}, \vec{p}, t)
$$

Introduce density, velocity field, and pressure

$$
n(\vec{r},t) = \int \frac{d^3 p}{(2\pi)^3} f(\vec{r},\vec{p},t) \qquad n\vec{v}(\vec{r},t) = \int \frac{d^3 p}{(2\pi)^3} \frac{\vec{p}}{m} f(\vec{r},\vec{p},t) \qquad P_{ij}(\vec{r},t) = \int \frac{d^3 p}{(2\pi)^3} p_i \frac{p_j}{m} f(\vec{r},\vec{p},t)
$$

When collisions dominate, the pressure becomes isotropic \vec{r}, t = $\delta_{ij}P(\vec{r}, t)$ **and the kinetic equation reduces formally to hydrodynamics**

$$
\partial_t n + \vec{\nabla}(n\vec{v}) = 0
$$

$$
m\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} + \vec{\nabla}U + \frac{1}{n}\vec{\nabla}P = 0
$$

Conservation laws for particle number and momentum

Thermalization

(relaxation towards local equilibrium)

Two main issues

i) relative populations of different momentum modes

ii) isotropy of momentum distribution

Main topic for the rest of this talk

Longitudinal expansion hinders isotropization

The fast expansion of the matter along the collision axis drives the momentum distribution to a very flat distribution

Translates into the existence of two different pressures

Competition between

$$
P_L = \int_p \frac{p_z^2}{p_0} f
$$

$$
f \qquad P_T = \int_p \frac{p_T^2}{p_0} f
$$

 p_{7}

(longitudinal) (transverse)

Anisotropy $\left(\sim P_{L}-P_{T}\right)$ relaxes slowly, like a 'collective' **variable associated to a conservation law**

Simple kinetic equation (Bjorken flow)

 \mathbf{V} is the equation **• 1+1 dimensional expansion, in relaxation time approximation**

$$
\left[\partial_{\tau} - \frac{p_z}{\tau} \partial_{p_z}\right] f(\boldsymbol{p}/T) = -\frac{f(\boldsymbol{p}, \tau) - f_{\text{eq}}(p, \tau)}{\tau_R}
$$

 T this equation describes the competition between two e \mathbf{c} **expansion collisions**

- Describes the transition from the collisionless regime $(\tau \ll \tau_R)$ **to the regime dominated by collisions, leading eventually to** In this paper, we shall assume that $\frac{1}{2}$ is a constant, or that $\frac{1}{2}$ is a constant, in which case $\frac{1}{2}$ is a constant of $\frac{1}{$ • Describes the transition from the collisionless regime $(\tau \ll \tau_R)$ hydrodynamics $(\tau \gg \tau_R)$
- Can be solved straightforwardly by standard numerical techniques. **We shall follow a less direct, but more insightful (semi) analytic approach.**

Special moments of the momentum distribution (JPB, Li Yan, 2017, 18, 19) 017, 18, 19) A. The moments *Lⁿ* L MOM *Lⁿ* ⌘ \overline{L} *p p*₂*n*(*p*)[,] (2)^{*n*}/^{*p*}₂*n*(*p*)[,] (2)^{*n*}/^{*p*}₂*n*(*p*)^{*,*} (2)^{*n*}/^{*p*}₂*n*(*p*)^{*,*} (2)^{*n*}/^{*p*} where *P*2*ⁿ* is a Legendre polynomial of order 2*n*, and cos ✓ = *pz/p*. Recall that

Special moments *^P*0(*z*)=1*, P*2(*z*) = ¹

 $p_z = p \cos \theta$ (3*z*₂ 1)*.* (3*z*² 1). (3*z*² 1*3z*³ 1*3*</sub> 1*3*

$$
\mathcal{L}_n \equiv \int_p p^2 P_{2n}(\cos \theta) f(\mathbf{p}) \qquad P_0(z) = 1 \qquad P_2(z) = \frac{1}{2}(3z^2 - 1)
$$

why these moments? And cos *Departments* 2007 of the distribution function under parity (or under reflection with respect to the *z* = 0 plane, i.e. *p^z* ! *p^z* and se moments
. of the distribution function under parity (or under reflection with respect to the *z* = 0 plane, i.e. *p^z* ! *p^z* and **Why these moments ?**

 \mathbf{r}

- **P** There is too much information in the distribution function For an expanding system with Bjorken geometry, order moments vanish as a consequence of the invariance of the i
The invariance of the invariance of th ere is too much information in the distribution function **• There is too much information in the distribution function** *L*⁰ = "*, L*¹ = *P^L P^T .* (4)
- We want to focus on the angular degrees of freedom For an expanding system with Bjorken geometry, odd order moments vanish as a consequence of the invariance • We want to focus on the angular degrees of freedom

Solution to the angular degrees of f

The e Note that the energy-momentum tensor in kinetic theory is given by Z The energy momentum tensor is described by first two moments

$$
T^{\mu\nu} = \int_p f(\mathbf{p}) p^{\mu} p^{\nu} \qquad \mathcal{L}_0 = \varepsilon \qquad \mathcal{L}_1 = \mathcal{P}_L - \mathcal{P}_T
$$

Example 20 We are looking for an effective theory for these two moments and it is interested in the interest of the in
It is a booking for an effective theory for t **We are looking for an effective theory for these two moments**

Coupled equations for the moments (4*ⁿ* 1)(4*ⁿ* + 3) ' <u>ار</u> $\frac{1}{2}$ ⁶⁴*n*² ⁵ By using the recursion relations among the Legendre coupled equations for the mome finite) set of coupled equations of coupled equations of coupled equations of coupled equations of coupled equ
The coupled equations of coupled experiments of coupled experiments of coupled experiments of coupled and coupl here. Note that all the *Lⁿ* have the same dimension. Coupled equations for the mom polynomials, we can recast Eq. (1) into the following (in-

$$
\frac{\partial \mathcal{L}_n}{\partial \tau} = -\frac{1}{\tau} \left[a_n \mathcal{L}_n + b_n \mathcal{L}_{n-1} + c_n \mathcal{L}_{n+1} \right] - \frac{\mathcal{L}_n}{\tau_R} \quad (n \ge 1)
$$
\n(Free streaming)

\n(collisions)

\n
$$
\frac{\partial \mathcal{L}_0}{\partial \tau} = -\frac{1}{\tau} \left[a_0 \mathcal{L}_0 + c_0 \mathcal{L}_1 \right]
$$

 $\frac{1}{2}$

0

 \mathcal{L}

 \mathcal{L}_{1}^{\prime}

 \int

 $\boldsymbol{\mathcal{L}}$ $\mathbf \circ$

ر
-

- *a*₀ = 4/3_{*, c*₀ = 2/3 **)**} where the coecients *an, bn, cⁿ* are pure numbers • The coefficients a_n, b_n, c_n are pure numbers ($a_0 = 4/3$ $c_0 = 2/3$)
- α resti α .
2017 - 14¹ coup (4*ⁿ* 1)(4*ⁿ* + 3) *, bⁿ* ⁼ \cdot equations, with ne • Interesting system of coupled linear equations, with nearest neighbour
couplings *aⁿ* = **couplings** (4*ⁿ* 1)(4*ⁿ* + 3) *, bⁿ* ⁼ (4*ⁿ* 1)(4*ⁿ* + 1) *,*
- \overline{c} *r* sol ution provides exact value exact solution proviaes exact values for the energy aensity and
pressures, but does not allow the complete reconstruction of the entirely determined by the free streaming part of the ki-**distribution function** FIG. 1. Comparison of the *L*-moment equations obtained re and the re • Exact solution provides exact values for the energy density and **component of the energy of the set of the set** FIG. 1. Comparison of the *L*-moment equations obtained from various truncation of \mathbb{R}
- ne competition between expansion and collisions is made o and the enconce of collisional demning for the energy density with the distance of connection damping for the exergy dent entirely determined by the free streaming part of the ki-• The competition between expansion and collisions is made obvious. Note the absence of collisional damping for the energy density.

 Γ ^{*n*}_{Γ} Γ ^{*n*} Γ with *n* 1. In fact, if one ignores the expansion, i.e., set *Effective theory obtained by 'eliminating' moments* $\mathcal{L}_{n>1}$

Two-moment truncation (effective theory)

$$
\frac{\partial}{\partial \tau} \begin{pmatrix} \mathcal{L}_0 \\ \mathcal{L}_1 \end{pmatrix} = -\frac{1}{\tau} \begin{pmatrix} a_0 & c_0 \\ b_1 & a_1 + \frac{\tau}{\tau_R} \end{pmatrix} \begin{pmatrix} \mathcal{L}_0 \\ \mathcal{L}_1 \end{pmatrix}
$$

- **Contains second order viscous hydrodynamics à la Israel-Stewart**
- **Views hydrodynamics as a coupled mode problem**
- **Amenable to analytic solution, bringing insight into the notions of attractors, general features of the gradient expansion, and its resummations in terms of trans-series, etc. [not discussed here] [for analytic solution see JPB and L. Yan, PLB 820:136478 (2021)]**
- **Captures most important features of more sophisticated approaches, and can be made quantitatively accurate with a simple renormalization of a second order transport coefficient (a1) [see later].**

Free streaming fixed points

One can transform the coupled linear equations into a single non linear differential equation for the quantity $g_0(\tau) =$ τ $\left| \frac{\partial \mathcal{L}_0}{\partial \tau} \right| \quad \left| \frac{P_L-P_T}{\varepsilon} = -\frac{1}{c_0} \right|$ $(a_0 + g_0)$

$$
\tau \frac{dg_0}{d\tau} + g_0^2 + (a_0 + a_1) g_0 + a_1 a_0 - c_0 b_1 - \left[c_0 c_1 \frac{\mathcal{L}_2}{\mathcal{L}_0} \right] = 0
$$

 \mathcal{L}_0

!

Write this as

FIG. 6. The function (*g*0) in Eq. (3.12). The full line corresponds to Eq. (3.12), and the dots • This fixed point structure is only moderately affected by higher moments The two dashed lines correspond to the shifts proportional to *L*² in Eq. (3.14), which induce minor • This structure is approximately captured by Israel-Stewart hydrodynamics

Including collisions

$$
w \frac{d g_0}{d w} = \beta(g_0, w)
$$

$$
\begin{aligned} w &\equiv \tau/\tau_R \\ \beta(g_0) & = -g_0^2 - (a_0 + a_1 + w) \, g_0 - a_1 a_0 + c_0 b_1 - a_0 w + \left[c_0 c_1 \frac{\mathcal{L}_2}{\mathcal{L}_0} \right] \end{aligned}
$$

This non linear equation is formally identical to that resulting from Israel-Stewart formulation of second order viscous hydrodynamics [Heller, Spalinski , 2015]

 $w \gg 1 \; (\tau \gg \tau_R)$ $g_0 + a_0 = 0$, $g_0 = -4/3$ hydrodynamic fixed point **one recovers the two free streaming fixed points** $w \ll 1$ ($\tau \ll \tau_R$)

The attractor solution is the particular solution that starts from the stable collisionless fixed point at small time and evolves "slowly" to the hydrodynamic fixed point at late time.

All solutions converge, soon or later depending on the initial conditions, towards the attractor (hence to hydrodynamics) at late time.

Attractor

Under the effect of collisions, the stable collisionless fixed point evolves "slowly" into the hydrodynamic fixed point 36

The "attractor" is the solution $\,\mathcal{B}^{0(\tau)}$ that joins the (stable) collisionless point corresponds to the hydrodynamic fixed point fixed point at early time to the hydrodynamic fixed point at late time.

The transition from free streaming to hydrodynamics (Attractor solution)

Early and late times are controlled by the free streaming and the hydrodynamic fixed points, respectively

The transition region occurs when the collision rate is comparable to the expansion rate $(\tau \sim \tau_R)$ for various initial conditions set up at the initial value *w*⁰ = \sim 11 answers region occurs when the comsion rate is **Time dependent relaxation time**

 $\tau_R \sim \tau^{1-\Delta}$

controls the "speed" of the transition

17

Müller-Israel-Stewart hydrodynamics in the context of Bjorken flow

$$
T^{\mu\nu} = (\varepsilon + P)u^{\mu}u^{\nu} - Pg^{\mu\nu} + \pi^{\mu\nu}
$$

(viscous pressure)

Gradient expansion

$$
\pi = \frac{4\eta}{3} \frac{1}{\tau} \longrightarrow \qquad \text{NS eqn.} \quad \partial_{\tau} \varepsilon = -\frac{a_0}{\tau} \left(\varepsilon - \frac{\eta}{\tau} \right)
$$

MIS hydro

$$
\partial_{\tau}\pi + \frac{a_1^{IS}}{\tau}\pi = -\frac{1}{\tau_{\pi}}\left(\pi - \frac{4\eta}{3}\frac{1}{\tau}\right)
$$

τ ime dependent relaxation time $(\tau_R \sim \tau)$ significant, i.e., in the transition region that connects the free streaming fixed point and the e dependent relaxa

changing al (a second order 'transport coefficient') does not "improve" hydrodynamics, but rather improves the location of the collisionless fixed point a possible negative negative internative longitudinal pressure accurate accurate *L*⁰ her improves the location of the collisionless fixed point we can absorb the term ϵ

Renormalization of al

where Eq. (4.3) has been used, and to observe that we can absorb the term *c*¹

$$
-c_0c_1\frac{\mathcal{L}_2}{\mathcal{L}_0} = -c_1c_0\frac{A_2}{A_1}\frac{\mathcal{L}_1}{\mathcal{L}_0} = c_1\frac{A_2}{A_1}(g_0 + a_0) \qquad a_1 \mapsto a_1' = a_1 + c_1\frac{A_2}{A_1} = \frac{31}{15}
$$

In the massless case, this second strategy is the preferred one as it does not at α

Renormalizing a1 cures unphysical features of two-moment truncation (and other Israel-Stewart calculations)

$$
\tau \frac{dg_0}{d\tau} + g_0^2 + \left(a_0 + a_1 + \frac{\tau}{\tau_R}\right)g_0 + a_1a_0 - c_0b_1 + \frac{a_0\tau}{\tau_R} = 0
$$

w

Conclusions

The solution of a simple kinetic equation for Bjorken flow was analysed in terms of special moments of the distribution function.

The simplest two moment-truncation yields an 'effective' theory that captures the main qualitative features of the dynamics, in particular the transition from the collisionless regime to hydrodynamics. It encompasses all versions of second order (Israel-Stewart) hydrodynamics

The collisionless regime is characterized by two fixed points, one stable, the other unstable. The effect of the collisions is to move "slowly" the stable free streaming fixed point into the (universal) hydrodynamic fixed point.

Conclusions

The "attractor" emerges as the solution that joins the collisionless fixed point at t=0 to the hydrodynamic fixed point at large time. The vicinities of the two fixed points are easy to control (known ratios of moments in free streaming, Navier-Stokes in hydrodynamics). Large deviations from the hydrodynamic fixed point involves information about the collisionless fixed point.

Terminologies "hydrodynamic attractor", "early time attractor" are somewhat misleading. Vicinity of hydro fixed point is genuine hydro. Early time fixed point exists in collisionless regime of kinetic theory, not in holographic descriptions.

Hydrodynamic behavior emerges when it is supposed to do so, i.e. within kinetic theory when the collision rate is comparable to the expansion rate.

By 'improving' the transition region between the fixed points (i.e., adjusting the collisionless fixed point), one does not 'improve hydrodynamics'!

The present analysis extends with 'minor' modifications to the non-conformal case (2208.02750)

Holographic description of a boost invariant plasma with the expectation of the species community for

(Heller, Janik, Witaszczyk, [1103.3452])

Viscous hydro can cope with "partial thermalization", and "large" differences between longitudinal and transverse pressures.
International constants constants and momentum distribution is was helped out. While the momentum distribution

The hydrodynamic attractor. When we combine the two e α ects, expansion and collisions, one may expect support