Supersymmetric generalization of q-deformed long-range spin chains of Haldane-Shastry type

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Haldane-Shastry spin chain

Let $\mathcal{H} = (\mathbb{C}^n)^{\otimes L}$. Let P_{ij} be the permutation operator acting on the Hilbert space $\mathcal{H} = (\mathbb{C}^n)^{\otimes L}$ by permuting the i-th and the j-th tensor components.

The Hamiltonian of the Haldane-Shastry long-range spin chain:

$$H^{\mathrm{HS}} = \frac{\pi^2}{2} \sum_{i \neq j}^{L} \frac{1 - P_{ij}}{\sin^2(\pi(x_i - x_j))},$$

where $x_k = k/L$, $k = 1, \ldots, L$.

The Haldane-Shastry model is naturally extended to the supersymmetric case: the Hamiltonian keeps the same form but the permutation operator is defined on the graded space $\mathcal{H} = (\mathbb{C}^{N|M})^{\otimes L}$.

Haldane-Shastry spin chain and CMS model

Spin Calogero-Moser-Sutherland model

$$H^{\mathrm{spin\,CM}} = \frac{\mathrm{Id}}{2}\sum_{k=1}^{N}\eta^2\frac{\partial}{\partial^2_{z_k}} + \frac{1}{2}\sum_{i\neq j}^{N}(\hbar^2\mathrm{Id} - \eta\hbar P_{ij})\,U(z_i - z_j)$$

Polychronakos freezing trick: remove the terms with differential operators; fix the particles in the equilibrium positions $z_k \to x_k$ of the CMS spinless classical model

$$h^{\mathrm{CM}} = \frac{1}{2}\sum_{k=1}^{N}p_k^2 - \frac{\nu^2}{2}\sum_{i\neq j}^{N}U(z_i - z_j)\,,$$

Integrable long-range gl_M spin chains of HS type

$$H^{\mathrm{XXX}} = \frac{g}{2}\sum_{i\neq j}^{N}P_{ij}\,U(x_i - x_j) \in \mathrm{End}(\mathcal{H})$$

Spin chain and CMS model

Integrable long-range gl_M spin chains of HS type

$$H^{\mathrm{XXX}} = \frac{g}{2}\sum_{i\neq j}^{N} P_{ij}\, U(x_i - x_j) \in \mathrm{End}(\mathcal{H})\,,$$

Potential	Equilibrium positions	Spin chain
`elliptic` $U(z) = \wp(z)$	$x_k = \frac{k}{N}$	Inozemtzev
`trigonometric` $U(z) = \frac{1}{\sin^2(\pi z)}$	$x_k = \frac{k}{N}$	Haldane-Shastry
`rational` $U(z) = \frac{1}{z^2} + \frac{z^2}{2}$	zeros of Hermitiane polynomials	Polychronakos-Frahm

Integrability of the Haldane-Shastry spin chain

The integrability follows form the connection with many-body Calogero-Moser-Sutherland integrable systems.
[Bernard, Gaudin, Haldane, Pasquier 93], [Haldane Talstra 95]

- Dunkl-Cherednik operators
- Yangian symmetry
- R-matrix Lax pair approach [Sechin, Zotov 18]

To the q-deformed Haldane-Shastry spin chains

There is a relativistic generalization of the Calogero-Moser-Sutherland model, which is called the Ruijesenaars-Shneider model [RS 86].

Question: Are there any spin chains that can be extracted from the Ruijesenaars model?

Answer: spin Macdonald-Ruijesenaars model $\rightarrow$ q-deformed Haldane-Shastry chain

[Uglov 95], [Lamers 18], [Lamers, Pasquier, Serban 20]

Generalizations: elliptic [M.M., Zotov 22], [Klabbers, Lamers 23], supersymmetric [M.M., Zotov 23]

Ruijsenaars model

Let p_i be the shift operator defined by its action on a function $f(z_1, \dots, z_L)$:

$$(p_i f)(z_1, z_2, \dots z_L) = \exp\left(-\eta \frac{\partial}{\partial z_i}\right) f(z_1, \dots, z_L) = f(z_1, \dots, z_i - \eta, \dots, z_L)\,.$$

Denote by D_k the quantum Ruijsenaars operators

$$D_k = \sum_{|I|=k} \prod_{\substack{i \in I \\ j \notin I}} \phi(z_j - z_i) \prod_{i \in I} p_i, \qquad k = 1, \dots, L,$$

where the sum is taken over all subsets I of $\{1, \dots, L\}$ of size k, and

$$\phi(z) \equiv \phi(\hbar, z) = \frac{\vartheta'(0)\vartheta(z+u)}{\vartheta(z)\vartheta(u)} \stackrel{trig}{\rightarrow} \pi\cot(\pi z) + \pi\cot(\pi\hbar) \stackrel{rat}{\rightarrow} \frac{1}{z} + \frac{1}{\hbar}\,.$$

The commutativity of operators holds [Ruijsenaars 87]:

$$[D_k, D_l] = 0 \quad k, l = 1, ..., L\,.$$

Let $R^\hbar(u)$ be the function of a complex variable u valued in $\mathrm{End}(\mathcal{H})^{\otimes 2}$:

Quantum Yang-Baxter equation

$$R^\hbar_{12}(u)R^\hbar_{13}(u+v)R^\hbar_{23}(v) = R^\hbar_{23}(v)R^\hbar_{13}(u+v)R^\hbar_{12}(u)\,.$$

Unitarity property:

$$R^\hbar_{12}(z)R^\hbar_{21}(-z) = \phi(\hbar, z)\phi(\hbar, -z)\mathrm{Id}$$

Different normalization:

$$\bar{R}^\hbar_{12}(z) = \frac{1}{\phi(\hbar, z)}\,R^\hbar_{12}(z) = \frac{1}{\pi\cot(\pi\hbar) + \pi\cot(\pi z)}\,R^\hbar_{12}(z)\,.$$

Then unitarity property takes the form:

$$\bar{R}^\hbar_{12}(z)\bar{R}^\hbar_{21}(-z) = \mathrm{Id}\,.$$

R-matrix generalization of the Ruijesenaars model

$$\mathcal{D}_k = \sum_{1 \le i_1 < \ldots < i_k \le L} \left(\prod_{\substack{j=1 \\ j \neq i_1 \ldots i_{k-1}}}^{L} \phi(z_j - z_{i_1})\, \phi(z_j - z_{i_2}) \cdots \phi(z_j - z_{i_k}) \right) \times$$

$$\times \left(\overleftarrow{\prod_{j_1=1}^{i_1-1}} \bar{R}_{j_1 i_1} \overleftarrow{\prod_{\substack{j_2=1 \\ j_2 \neq i_1}}^{i_2-1}} \bar{R}_{j_2 i_2} \ldots \overleftarrow{\prod_{\substack{j_k=1 \\ j_k \neq i_1 \ldots i_{k-1}}}^{i_k-1}} \bar{R}_{j_k i_k} \right) \times$$

$$\times p_{i_1} \cdot p_{i_2} \cdots p_{i_k} \times \left(\overrightarrow{\prod_{\substack{j_k=1 \\ j_k \neq i_1 \ldots i_{k-1}}}^{i_k-1}} \bar{R}_{i_k j_k} \overrightarrow{\prod_{\substack{j_{k-1}=1 \\ j_{k-1} \neq i_1 \ldots i_{k-2}}}^{i_{k-1}-1}} \bar{R}_{i_{k-1} j_{k-1}} \ldots \overrightarrow{\prod_{j_1=1}^{i_1-1}} \bar{R}_{i_1 j_1} \right)$$

Commutativity and R-matrix Identities

Theorem [M.M., Zotov 22]

The operators $\mathcal{D}_k$ commute with each other

$$[\mathcal{D}_m, \mathcal{D}_l] = 0 \quad m, l = 1, ..., L \tag{1}$$

if and only if the set of R-matrix identities holds true.

What R-matrices hold these identities?

- elliptic: Baxter-Belavin R-matrix
- trigonometric: $\hat{\mathcal{U}}_q(\mathrm{gl}(N))$ R-matrix (6-vertex R-matrix), trigonometric limits form Baxter-Belavin R-matrix, graded version: $\hat{\mathcal{U}}_q(\mathrm{gl}(N|M))$ R-matrix, graded extension of $\mathbb{Z}_n$-invariant R-matrix
- rational Yang R-matrix, its supersymmetric version

Felder dynamical R-matrix [Klabbers, Lamers 23]

q-deformed Haldane-Shastry spin chain

Using the Polychronakos freezing trick we can obtain spin chains form the R-matrix generalization of the Ruijsenaars model.

The first Hamiltonian has the form

$$H = \sum_{k<i}^{L} \bar{R}_{i-1,i} \ldots \bar{R}_{k+1,i} \bar{R}_{k,i} \bar{F}_{i,k} \bar{R}_{i,k+1} \ldots \bar{R}_{i,i-1} \,,$$

where we use the short notations $\bar{F}^{\hbar}_{ij}(z) = \partial_z \bar{R}^{\hbar}_{ij}(z)$, $\bar{F}_{ij} = \bar{F}^{\hbar}_{ij}(x_i - x_j)$ and $\bar{R}_{ij} = \bar{R}^{\hbar}_{ij}(x_i - x_j)$.

$\mathbb{Z}_2$-graded algebras. Notations.

In the supersymmetric case consider the $\mathbb{Z}_2$-graded vector space $\mathbb{C}^{N|M} = \mathbb{C}^N \oplus \mathbb{C}^M$ with grading

$$p_i = \begin{cases} 0 & \text{for } 1 \le i \le N, \\ 1 & \text{for } N < i \le N+M. \end{cases}$$

Denote by e_{ij} the generators of the Lie superalgebra $\mathrm{gl}(N|M)$, $\deg e_{ij} = p_i + p_j$. We regard e_{ij} as the elementary matrix units acting on the space $\mathbb{C}^{N|M}$. Multiplication in the tensor product $\mathbb{C}^{N|M} \otimes \mathbb{C}^{N|M}$ is defined by

$$(\mathbb{I} \otimes e_{ij})(e_{kl} \otimes \mathbb{I}) = (-1)^{(p_i+p_j)(p_k+p_l)} e_{kl} \otimes e_{ij}$$

and the graded permutation operator acting on $\mathbb{C}^{N|M} \otimes \mathbb{C}^{N|M}$ is given by

$$P_{12} = \sum_{i,j=1}^{N+M} (-1)^{p_j} e_{ij} \otimes e_{ji} \, .$$

Supersymmetric R-matrix $\hat{\mathcal{U}}_q(\mathrm{gl}(N|M))$

$$\mathcal{R}^{\hbar}_{12}(z) = \pi \sum_{a=1}^{N+M} \Big((-1)^{p_a} \cot(\pi z) + \coth(\pi\hbar) \Big) e_{aa} \otimes e_{aa} + \frac{\pi}{\sin(\pi\hbar)} \sum_{a\neq b}^{N+M} e_{aa} \otimes e_{bb}$$

$$+ \frac{\pi}{\sin(\pi z)} \sum_{a<b}^{N+M} \Big((-1)^{p_b} e_{ab} \otimes e_{ba}\, e^{\pi \imath z} + (-1)^{p_a} e_{ba} \otimes e_{ab}\, e^{-\pi \imath z} \Big) .$$

Supersymmetric R-matrix $\hat{\mathcal{U}}_q(\mathrm{gl}(N|M))$

Example $\hat{\mathcal{U}}_q(\mathrm{gl}(2))$:

$$\mathcal{R}^{\hbar}_{12}(z) = \pi\Big(\cot(\pi z) + \coth(\pi\hbar)\Big)\Big(e_{11}\otimes e_{11} + e_{22}\otimes e_{22}\Big) +$$

$$+\frac{\pi}{\sin(\pi\hbar)}\Big(e_{11}\otimes e_{22} + e_{22}\otimes e_{11}\Big) + \frac{\pi}{\sin(\pi z)}\Big(e_{12}\otimes e_{21}e^{\pi\imath z} + e_{21}\otimes e_{12}e^{-\pi\imath z}\Big)$$

Example $\hat{\mathcal{U}}_q(\mathrm{gl}(1|1))$:

$$\mathcal{R}^{\hbar}_{12}(z) = \pi\Big(\cot(\pi z) + \coth(\pi\hbar)\Big)e_{11}\otimes e_{11} + \pi\Big(-\cot(\pi z) + \coth(\pi\hbar)\Big)e_{22}\otimes e_{22}$$

$$+\frac{\pi}{\sin(\pi\hbar)}\Big(e_{11}\otimes e_{22} + e_{22}\otimes e_{11}\Big) + \frac{\pi}{\sin(\pi z)}\Big(-e_{12}\otimes e_{21}e^{\pi\imath z} + e_{21}\otimes e_{12}e^{-\pi\imath z}\Big)$$

Supersymmetric q-deformed Haldane-Shastry Hamiltonian

In case of the supersymmetric R-matrix the Hamiltonian takes the form:

$$H = -\pi \sum_{k<i}^{L} \frac{\sin(\pi\hbar)}{\sin\pi(\hbar + x_i - x_k)\sin\pi(\hbar - x_i + x_k)} \times$$

$$\times \bar{\mathcal{R}}_{i-1,i} \dots \bar{\mathcal{R}}_{k+1,i} C^{\mathrm{susy}}_{k,i} \bar{\mathcal{R}}_{i,k+1} \dots \bar{\mathcal{R}}_{i,i-1}.$$

where

$$C^{\mathrm{susy}}_{12} = \big(e^{-\imath\pi\hbar} e_{11} \otimes e_{22} + e_{12} \otimes e_{21} -$$

$$e_{21} \otimes e_{12} + e^{\imath\pi\hbar} e_{22} \otimes e_{11} + 2\cos(\pi\hbar) e_{22} \otimes e_{22}\big)\,.$$

In the limit $\hbar \to 0$ we have $\bar{\mathcal{R}}^{\hbar}_{ij} \to \mathrm{Id}$ and $C^{\mathrm{susy}}_{ij} \to (1 - P_{ij})$, thus

$$\lim_{\hbar\to 0} \frac{1}{\hbar} H = \pi^2 \sum_{k<i}^{L} \frac{1}{\sin^2(\pi(x_i - x_k))} (1 - P_{ik}) = H^{\mathrm{HS}}.$$

Graded extension of $\mathbb{Z}_n$-invariant R-matrix

$$R^{\hbar}_{12}(z) = \pi \sum_{a=1}^{N+M} \Big((-1)^{p_a} \cot(\pi z) + \cot(\pi \hbar) \Big) e_{aa} \otimes e_{aa} +$$

$$+\pi \sum_{a \neq c}^{N+M} e_{aa} \otimes e_{cc} \frac{\exp\left(\frac{\pi \imath \hbar}{N+M} \Big(2(a-c) - (N+M)\mathrm{sign}(a-c) \Big) \right)}{\sin(\pi \hbar)} +$$

$$+\pi \sum_{a \neq c}^{N+M} (-1)^{p_c} e_{ac} \otimes e_{ca} \frac{\exp\left(\frac{\pi \imath z}{N+M} \Big(2(a-c) - (N+M)\mathrm{sign}(a-c) \Big) \right)}{\sin(\pi z)} .$$

Drinfeld twist

The relation between the R-matrix of $\hat{\mathcal{U}}_q(\mathrm{gl}(N|M))$ and the graded extension of $\mathbb{Z}_n$-invariant R-matrix is the following:

$$R^{\hbar}_{12}(u-v) = G_1(u)G_2(v)F_{12}(\hbar)\mathcal{R}^{\hbar}_{12}(u-v)F^{-1}_{21}(\hbar)G^{-1}_1(u)G^{-1}_2(v),$$

Here $G_1(u) = G(u) \otimes 1_{N+M}$, $G_2(v) = 1_{N+M} \otimes G(v)$,

$$G(u) = \sum_{j=1}^{N+M} \exp\left(\frac{2\pi\imath(j-1)u}{N+M}\right) e_{jj}\,.$$

$$F_{12}(\hbar) = \sum_{i,j=1}^{N+M} \exp\left(\frac{\pi\imath\hbar\left(2(i-j)-(N+M)\mathrm{sign}(i-j)\right)}{2(N+M)}\right) e_{ii} \otimes e_{jj}\,,$$

$$\mathrm{sign}(i-j) = \begin{cases} 1, & i > j\,, \\ 0, & i = j\,, \\ -1, & i < j \end{cases}$$

Supersymmetric R-matrices

Example $\mathrm{GL}(2)$:

$$R^{\hbar}_{12}(z) = \pi\Big(\cot(\pi z) + \coth(\pi\hbar)\Big)\Big(e_{11}\otimes e_{11} + e_{22}\otimes e_{22}\Big) +$$

$$+\frac{\pi}{\sin(\pi\hbar)}\Big(e_{11}\otimes e_{22} + e_{22}\otimes e_{11}\Big) + \frac{\pi}{\sin(\pi z)}\Big(e_{12}\otimes e_{21} + e_{21}\otimes e_{12}\Big)\,.$$

Example $\mathrm{GL}(1|1)$:

$$R^{\hbar}_{12}(z) = \pi\Big(\cot(\pi z) + \coth(\pi\hbar)\Big)e_{11}\otimes e_{11} + \pi\Big(-\cot(\pi z) + \coth(\pi\hbar)\Big)e_{22}\otimes e_{22}$$

$$+\frac{\pi}{\sin(\pi\hbar)}\Big(e_{11}\otimes e_{22} + e_{22}\otimes e_{11}\Big) + \frac{\pi}{\sin(\pi z)}\Big(-e_{12}\otimes e_{21} + e_{21}\otimes e_{12}\Big)\,.$$

Non-relativistic limit in $GL(1|1)$ case

By taking the non-relativistic limit in the Hamiltonian

$$H = \sum_{k<i}^{L} \bar{R}_{i-1,i} \dots \bar{R}_{k+1,i} \bar{R}_{k,i} \bar{F}_{i,k} \bar{R}_{i,k+1} \dots \bar{R}_{i,i-1} \,,$$

we obtain the anisotropic spin chain:

$$\lim_{\hbar \to 0} \frac{1}{\hbar} H = \pi^2 \sum_{k<i}^{L} \frac{1}{\sin^2(\pi(x_i - x_k))} \Big(e_{11} \otimes e_{22} + e_{22} \otimes e_{11} + 2 e_{22} \otimes e_{22} \Big) +$$

$$+ \pi^2 \sum_{k<i}^{L} \frac{\cos(\pi(x_i - x_k))}{\sin^2(\pi(x_i - x_k))} \Big(e_{12} \otimes e_{21} - e_{21} \otimes e_{12} \Big) \,,$$

which is graded version of [Hikami,Wadati 93], [Sechin, Zotov 18] spin chain.

Thank you for your attention!