Exact overlaps for boundary states

Tamas Gombor

Based on: arXiv:2110.07960
arXiv:2311.04870
and recent unpublished work

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- Overlaps of two site states for $gl(2)$ spin chains
- Overlaps of two site states for $gl(N)$ spin chains
- Overlaps of matrix product states for $gl(N)$ spin chains

Motivation

Why boundary state overlaps?

boundary state

$$\langle \Psi | \bar{u} \rangle$$

Bethe state

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In statistical physics $\longrightarrow$ Time evolution from initial state $|\Psi\rangle$

The overlaps are input for Quench Action

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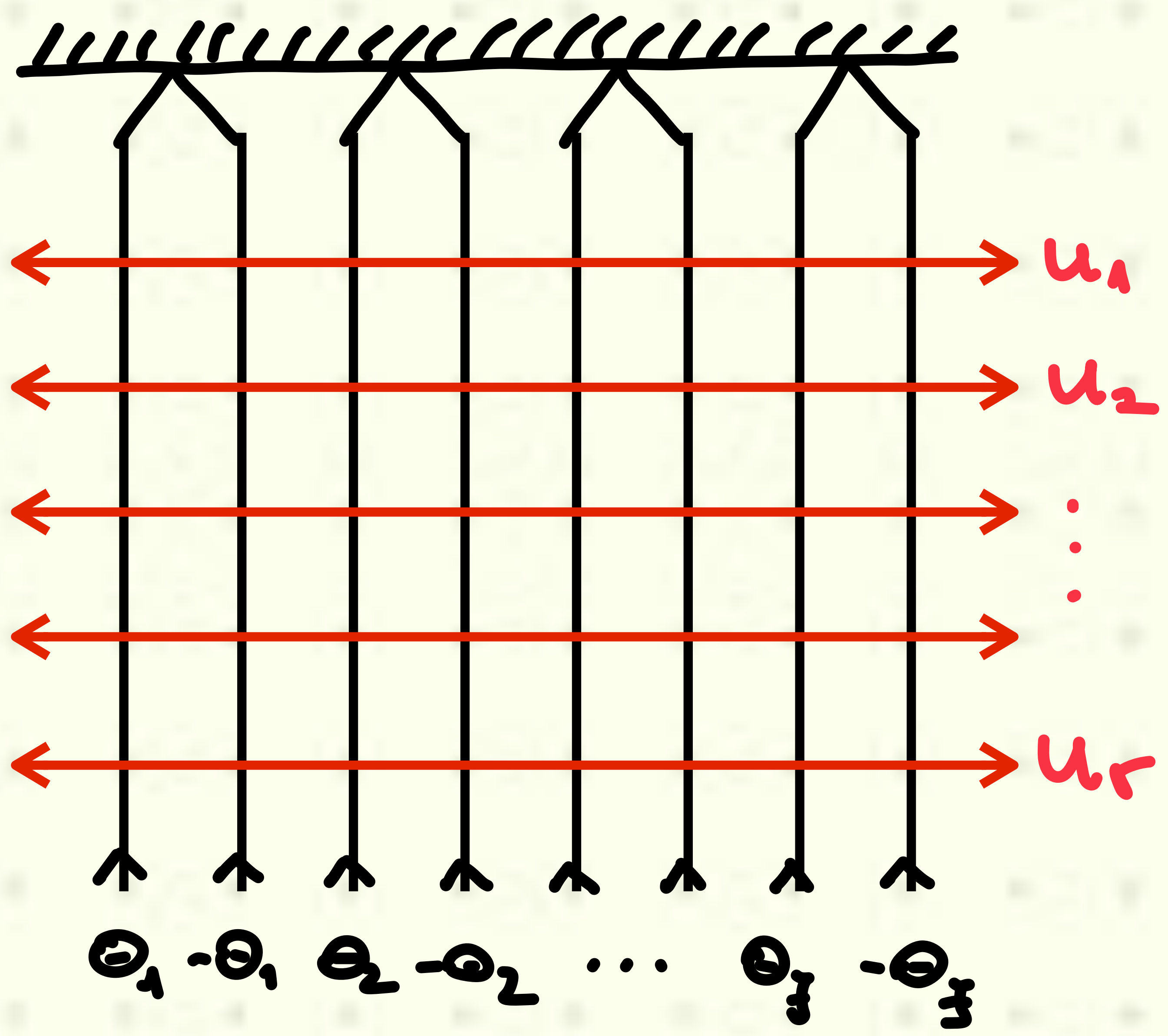
The overlaps are input for Quench Action

In AdS/CFT duality $\longrightarrow$ The overlaps correspond to 1-pt functions of defect theories two site states of gl(2) spin chains

Definitions

6-vertex model



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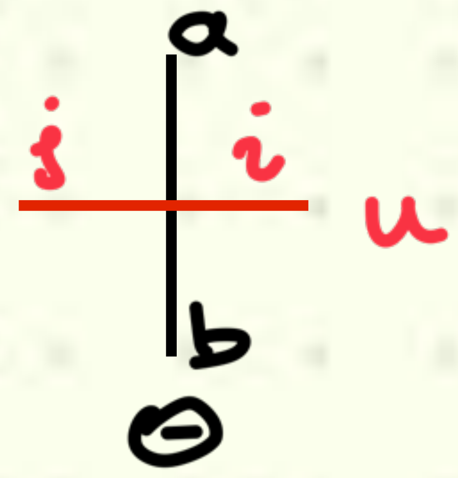
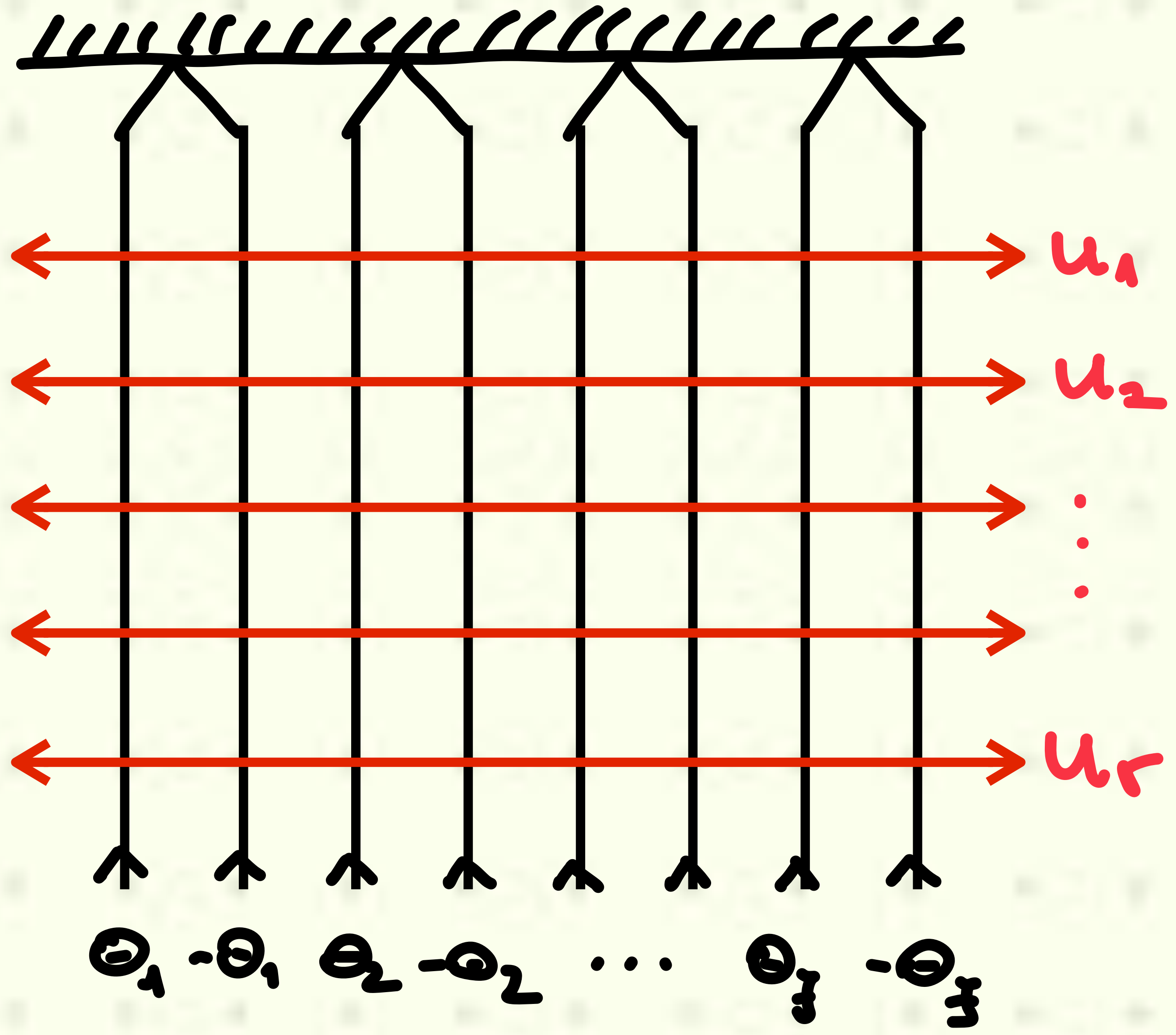
$$L^{j,b}_{i,a}(u-\theta) =$$





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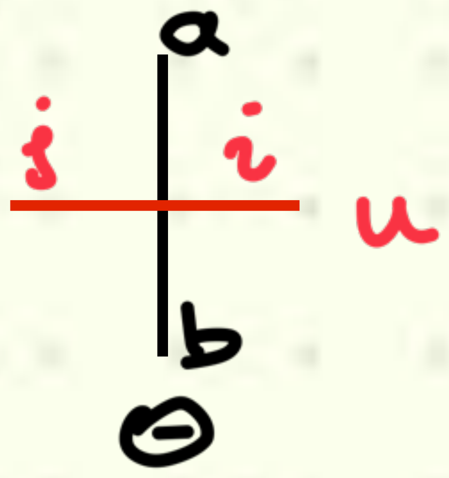
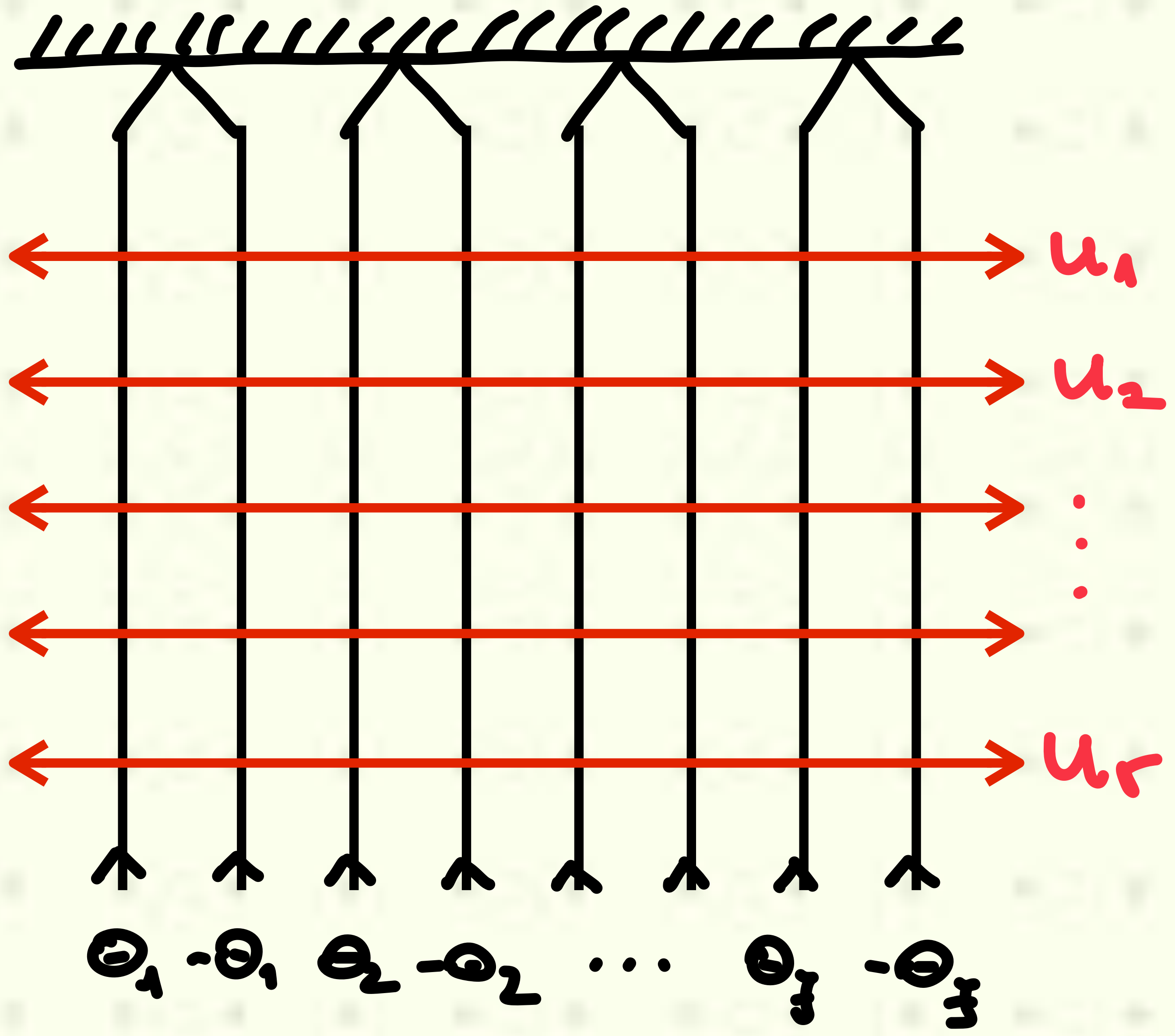
$L^{j,b}_{i,a}(u-\theta) =$ 



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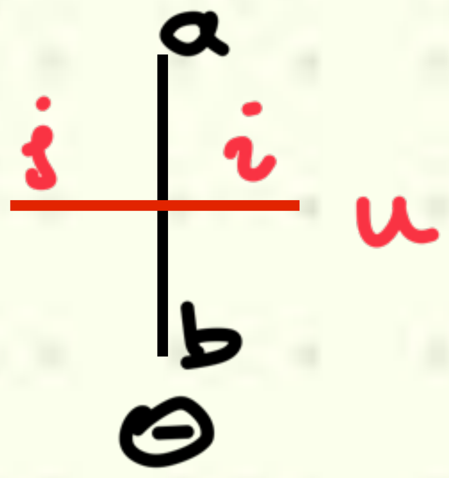

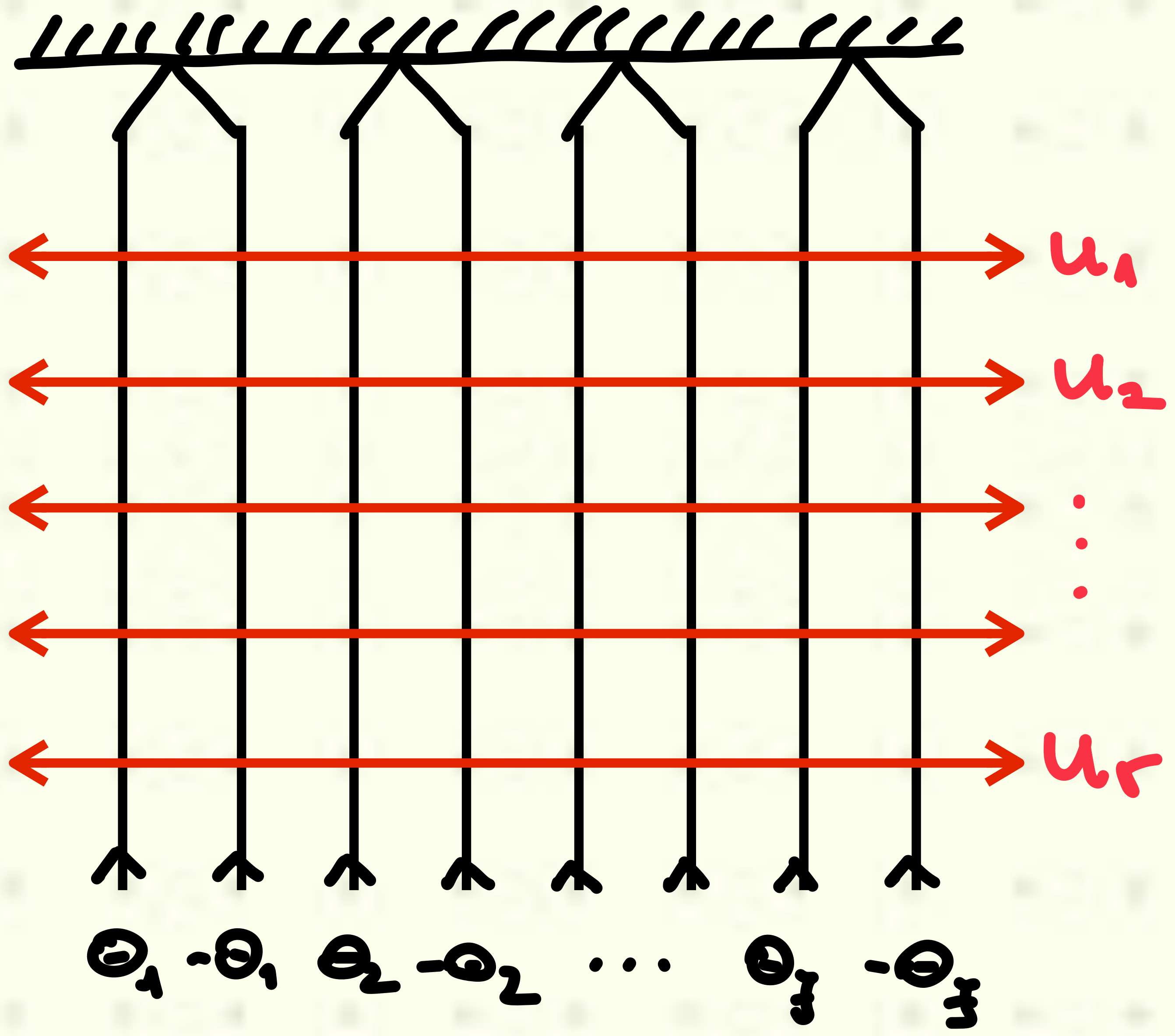

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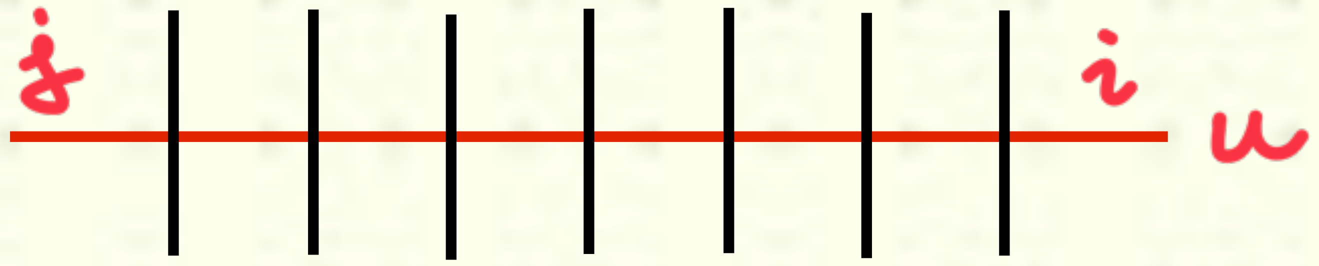

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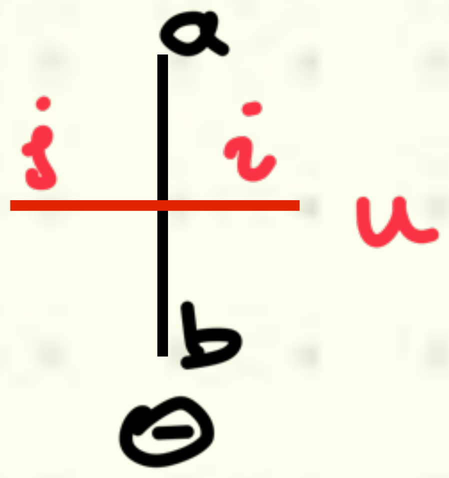
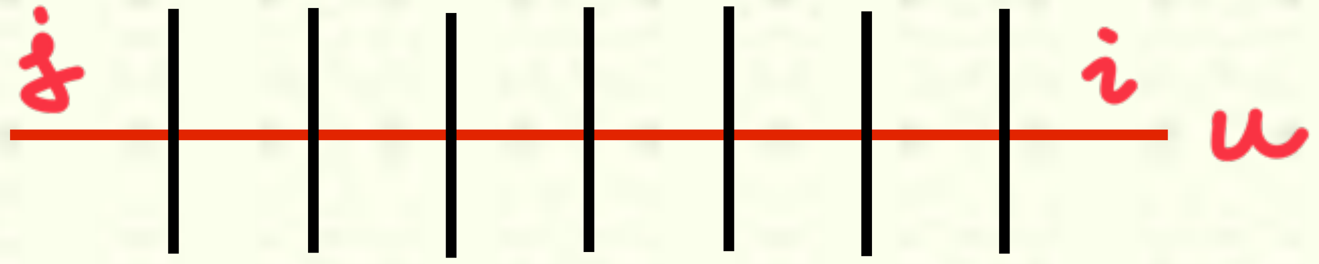
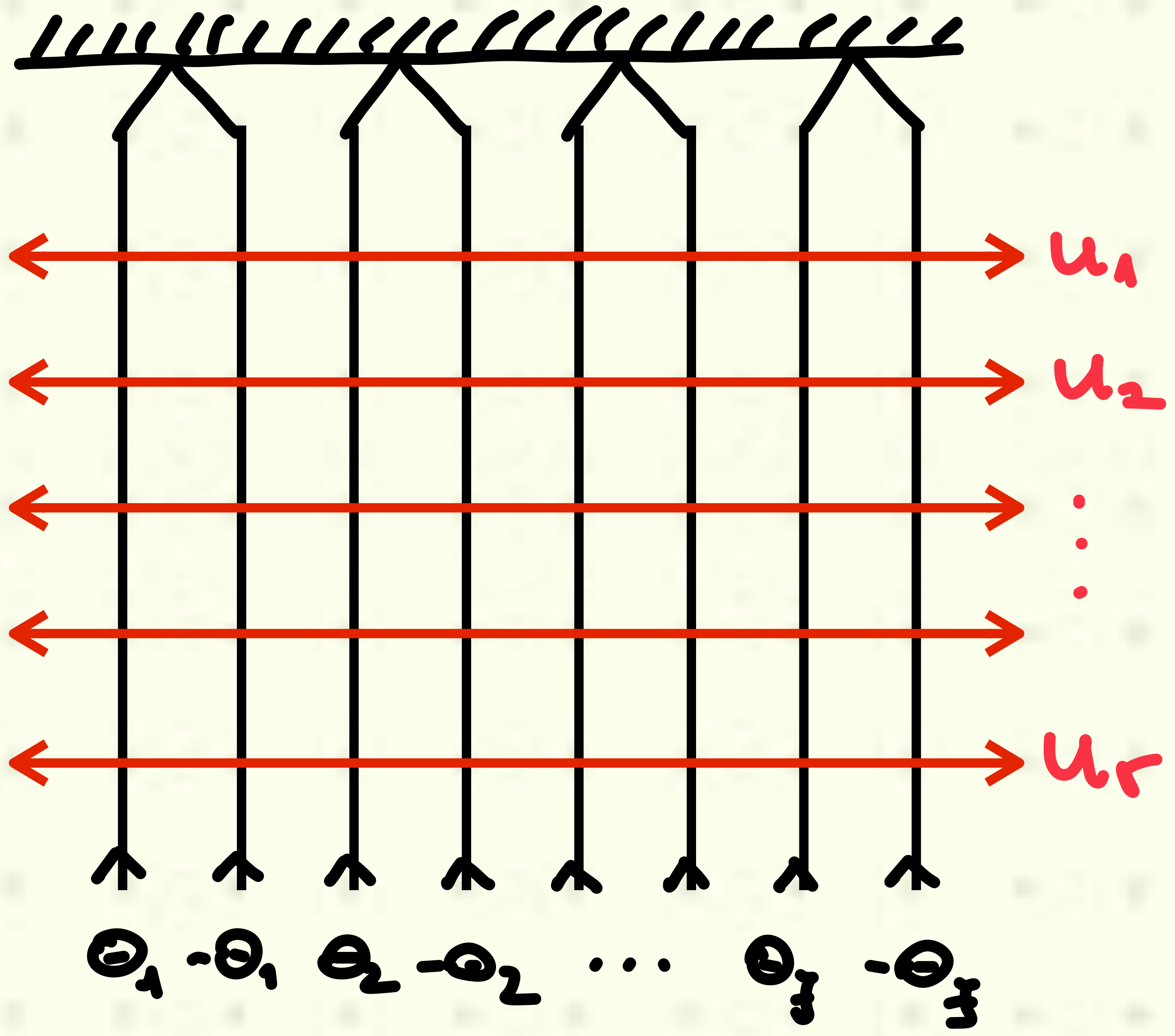
$L_{i,a}^{j,b}(u-\theta) =$ 

$T_{ij}(u) =$ 



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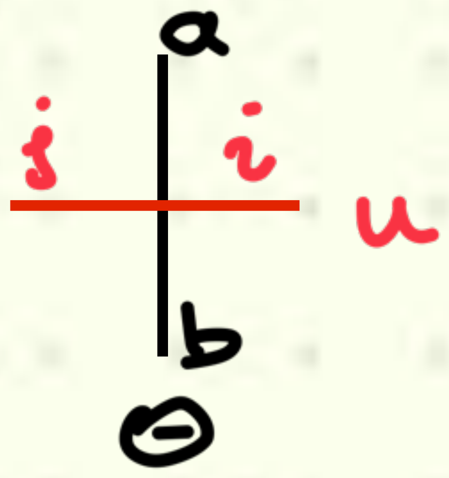
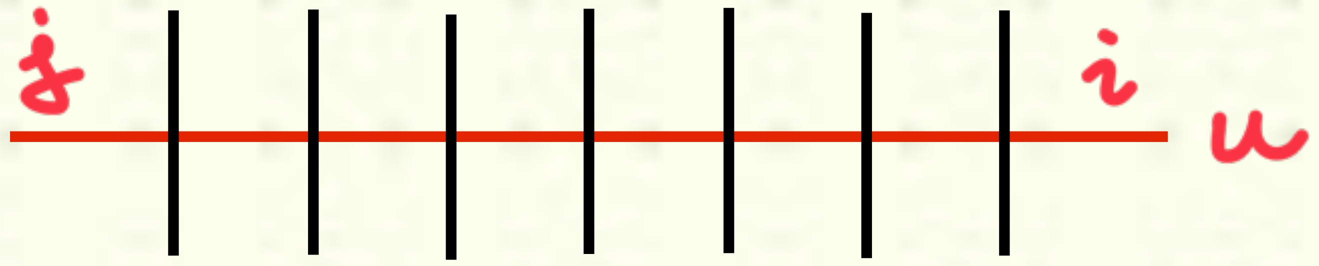
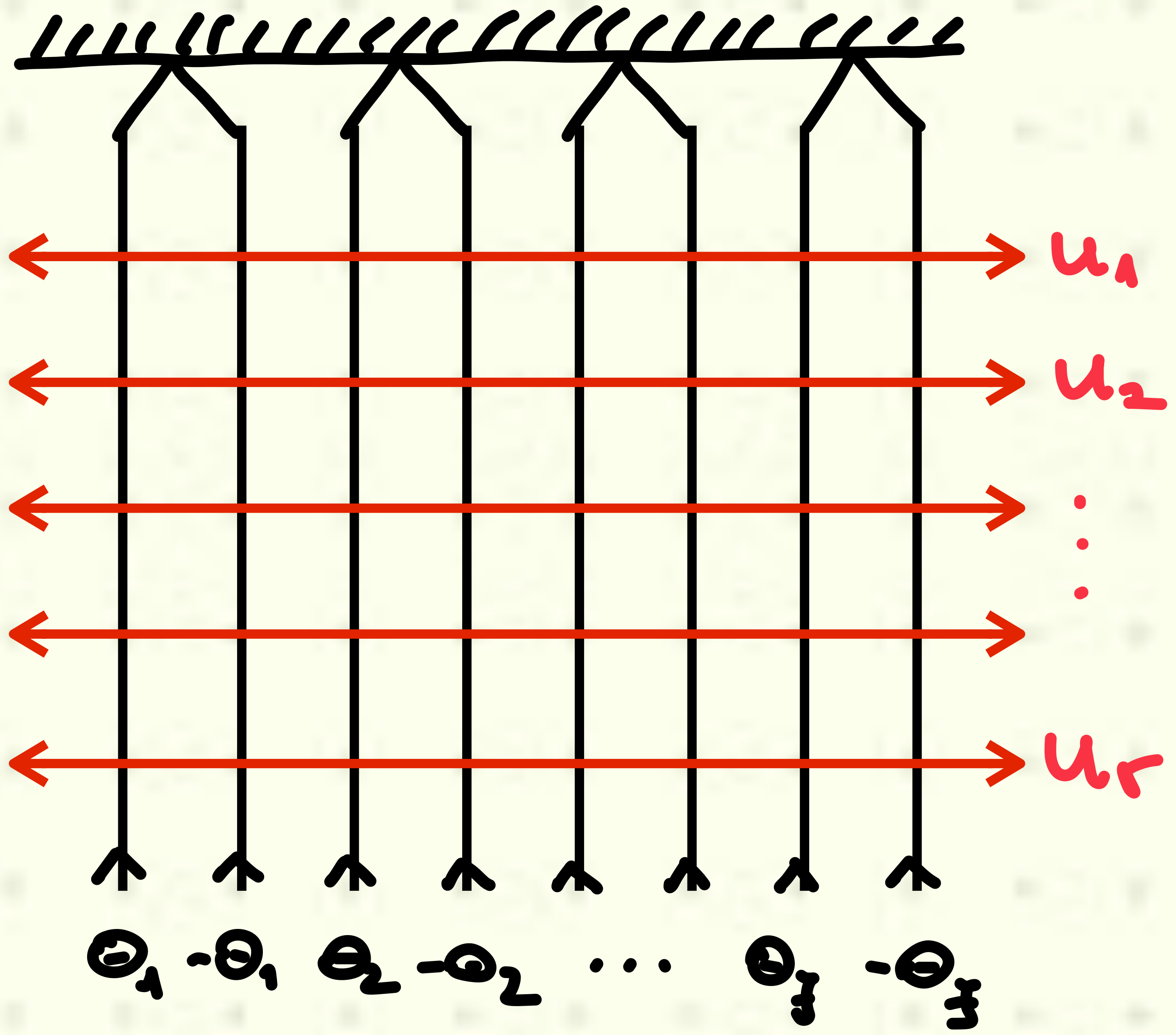
$L^{j,b}_{i,a}(u-\theta) =$ 

$T_{i,j}(u) =$ 



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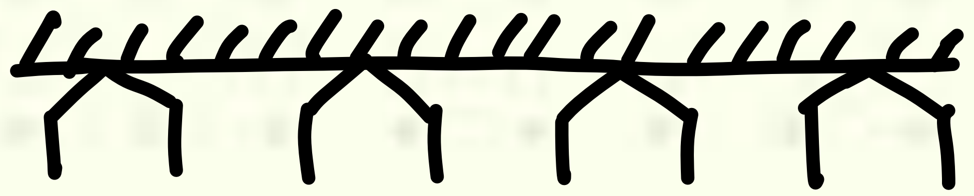
$\langle\Psi| =$ 

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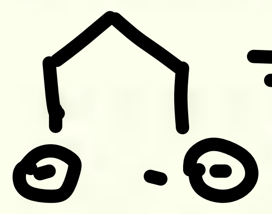
$\langle\varphi(\theta)| =$  $= \sum_{a,b} \varphi_{a,b}(\theta)\langle a| \otimes \langle b|$

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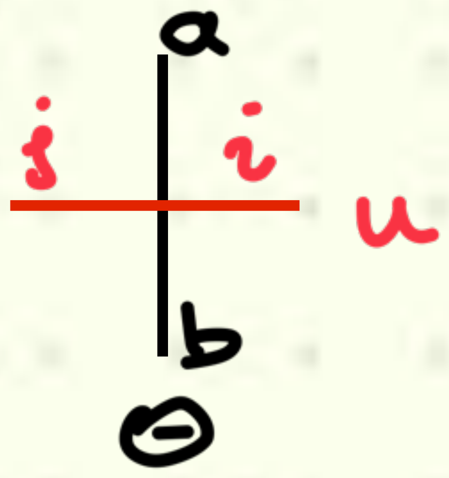
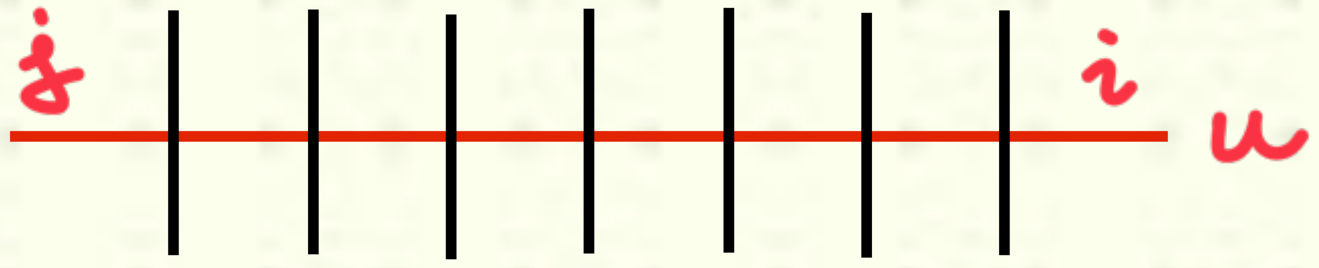
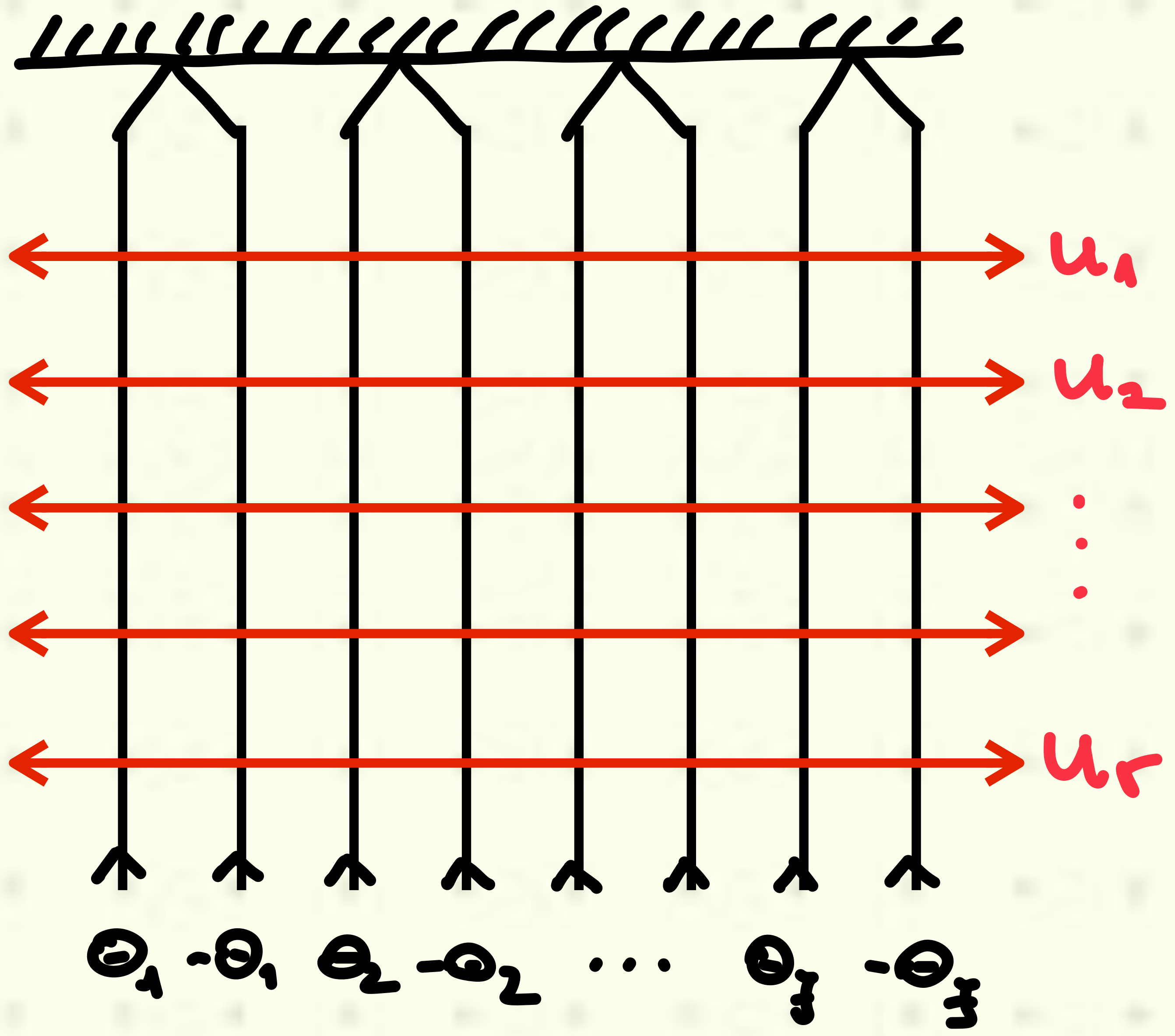
$L_{i,a}^{j,b}(u-\theta) =$ 

$T_{ij}(u) =$ 



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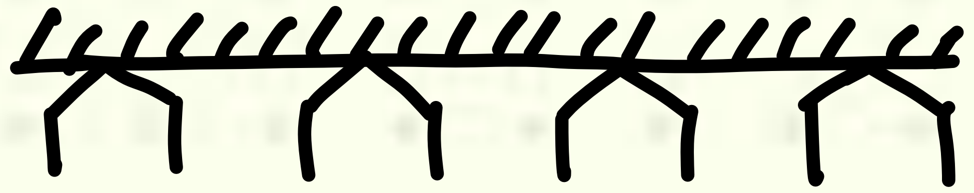
$\langle \Psi | =$ 

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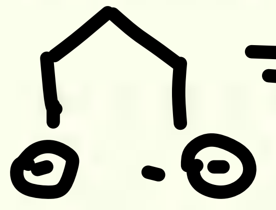
$\langle \varphi(\theta)| =$ 

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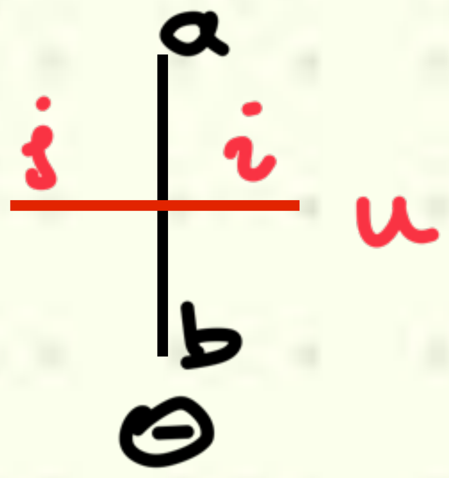
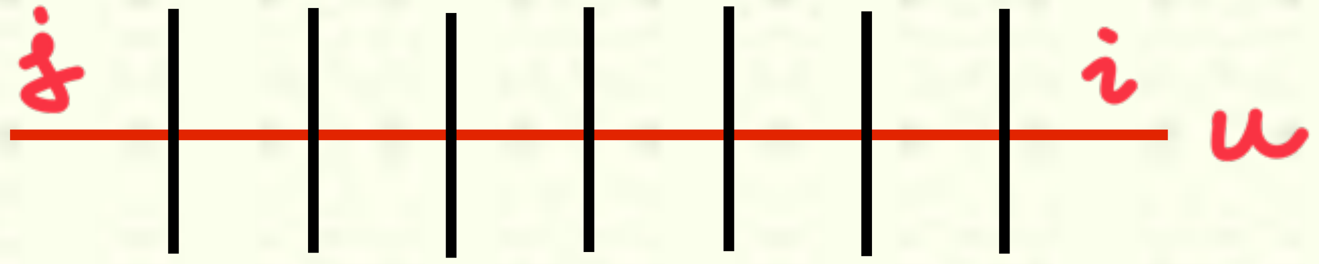
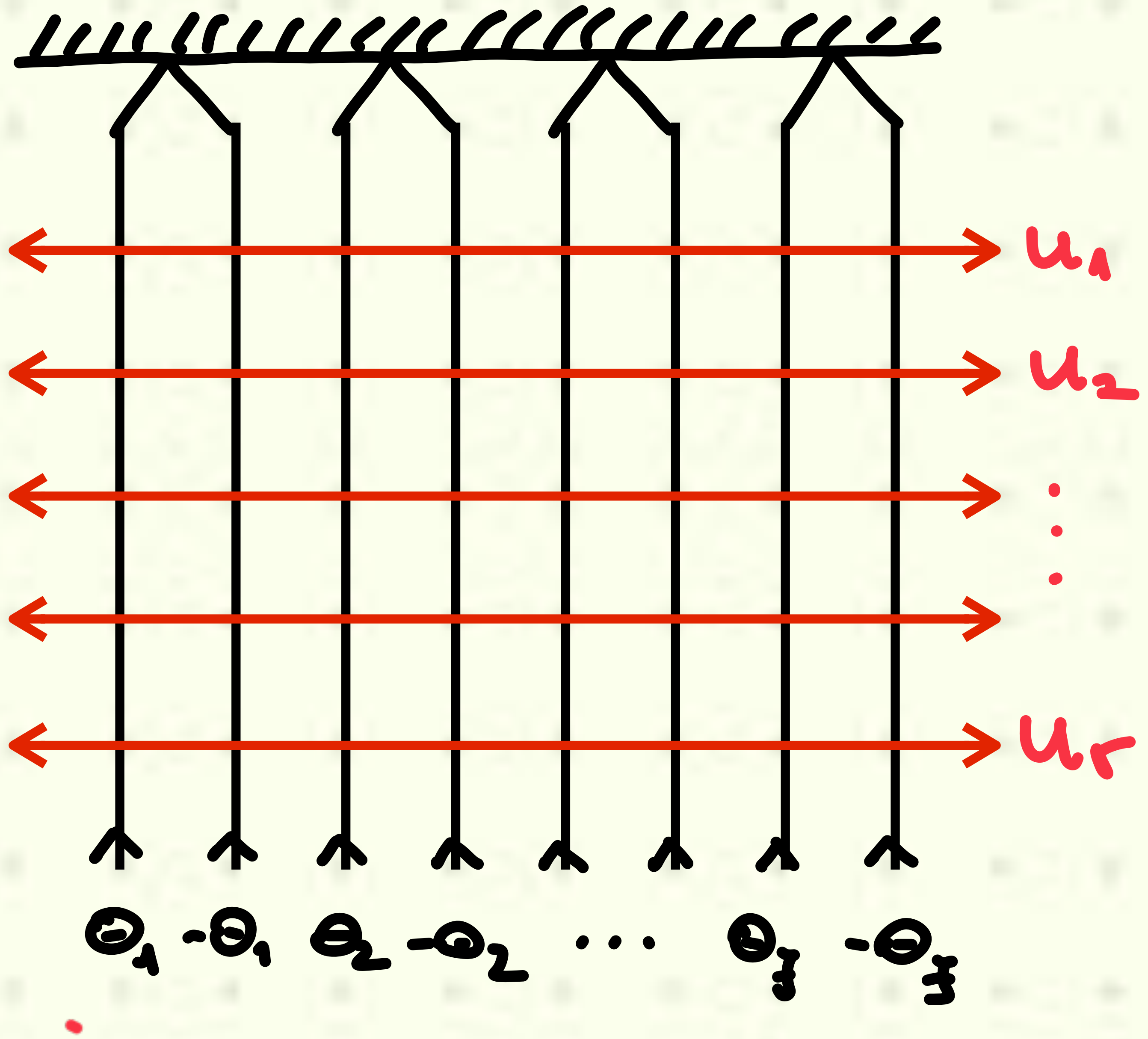
$L_{i,a}^{j,b}(u-\theta) =$ 

$T_{ij}(u) =$ 



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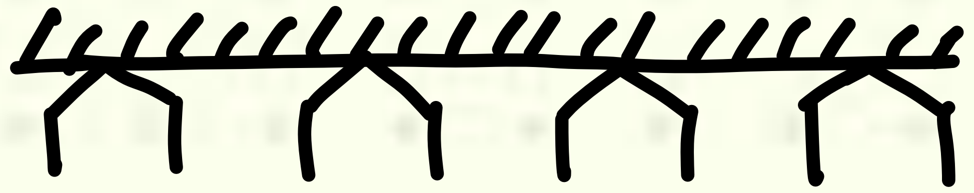
$\langle \Psi | =$ 

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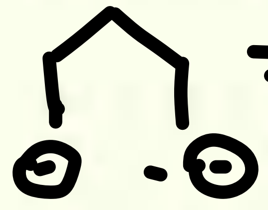
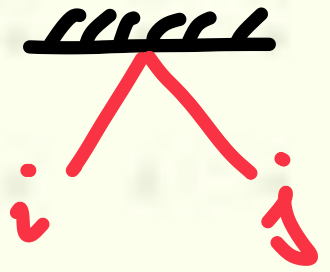
$\langle \varphi(\theta)| =$  $= \sum_{a,b} \varphi_{a,b}(\theta) \langle a| \otimes \langle b|$

$K_{i,j}(u) =$ 

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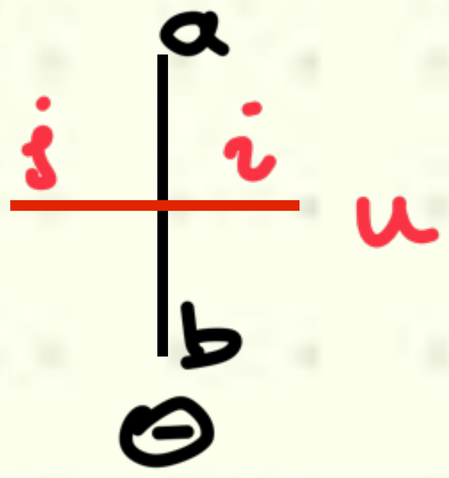
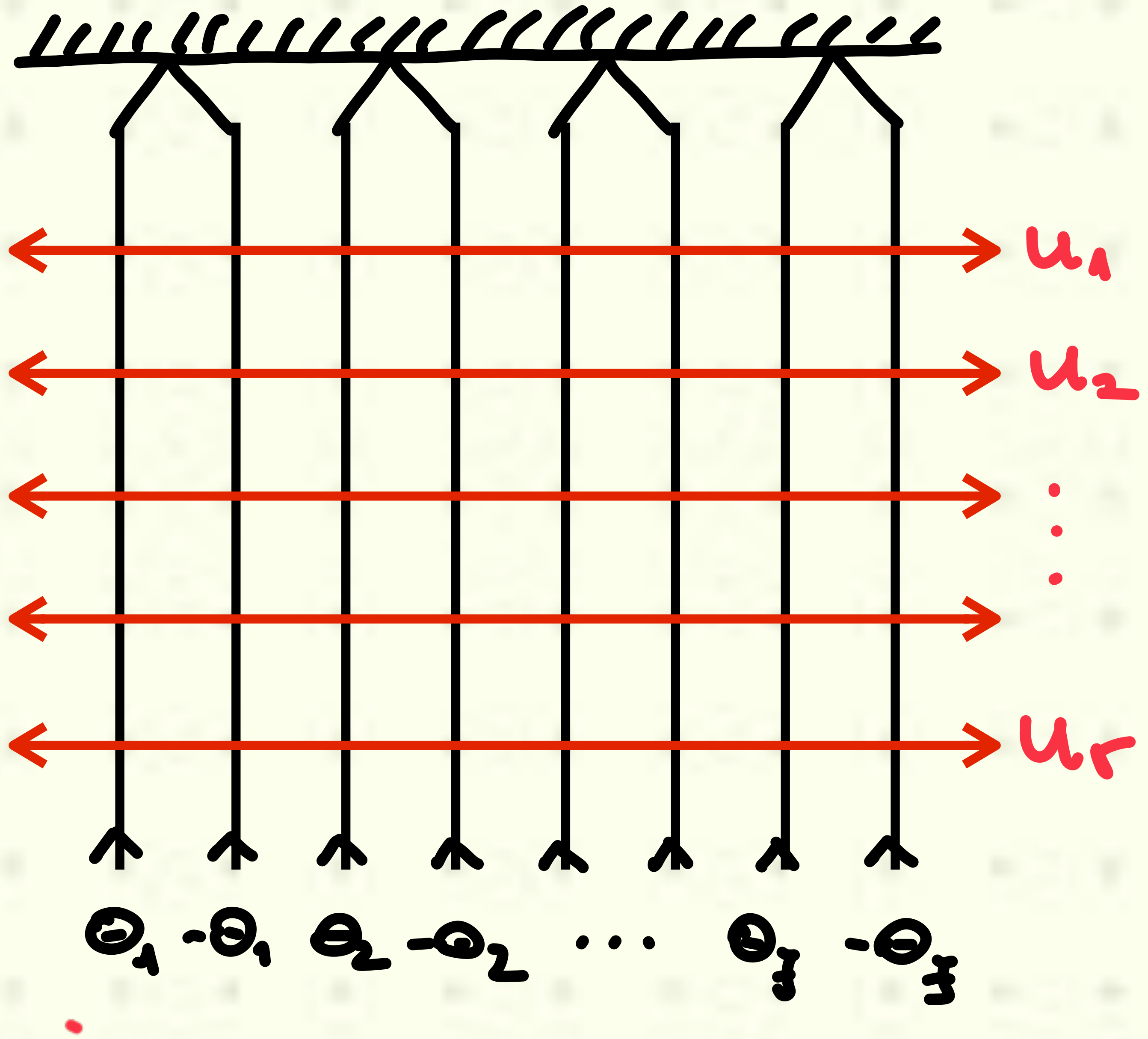
$L^{j,b}_{i,a}(u-\theta) =$ 



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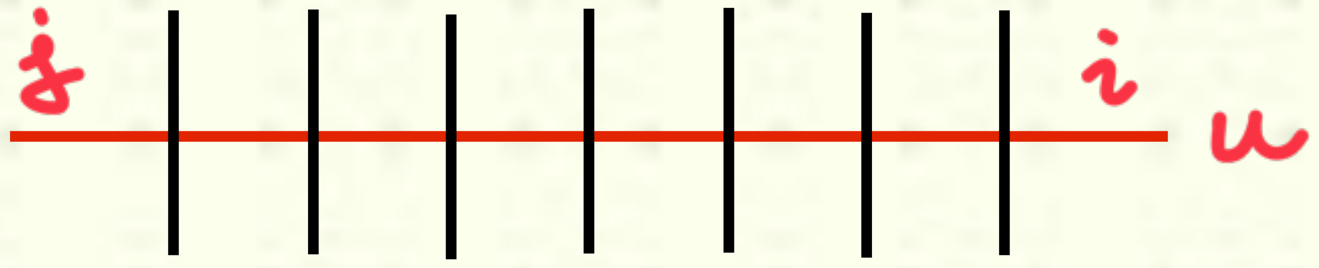
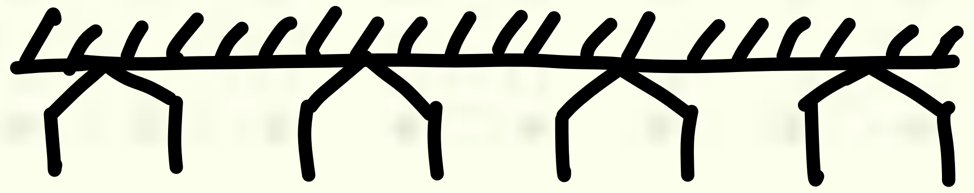
$T_{i,j}(u) =$ 

$\langle \Psi | =$ 

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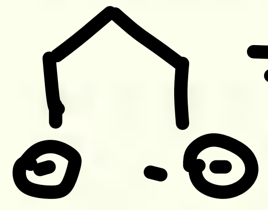
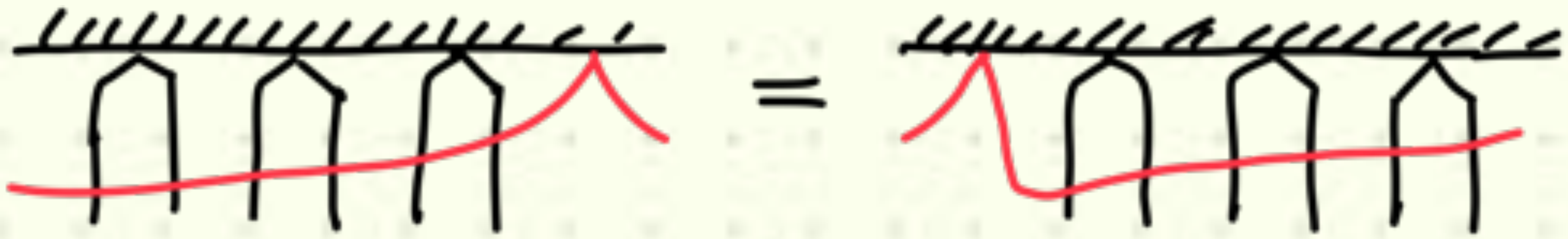
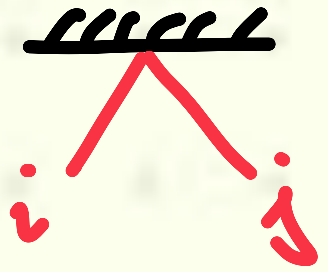
$\langle \psi(\theta)| =$  $= \sum_{a,b} \psi_{a,b}(\theta) \langle a | \otimes \langle b |$

Integrable boundary states

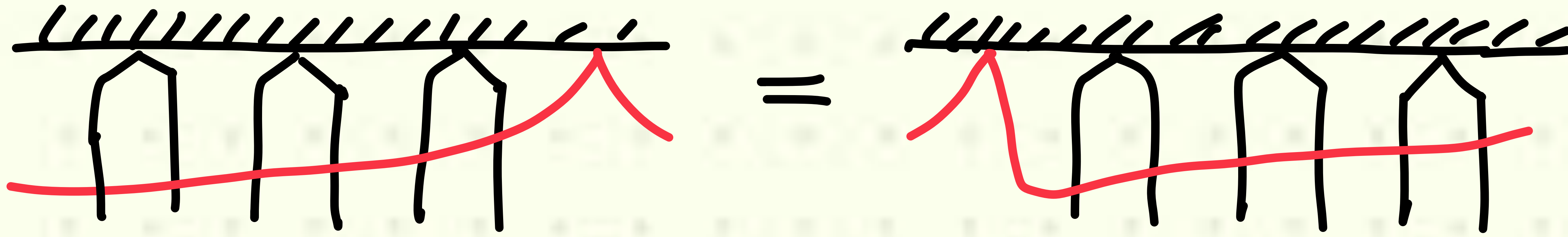


$$\boxed{\sum_{k} K_{i,k}(u)\,\langle \Psi | T_{k,j}(u) = \sum_{k} \langle \Psi | T_{i,k}(-u)\,K_{k,j}(u)}$$

$K_{i,j}(u) =$ 

Properties of the KT-relation

$$K_0(z)\langle\Psi|T_0(z)=\langle\Psi|T_0(-z)K_0(z)$$



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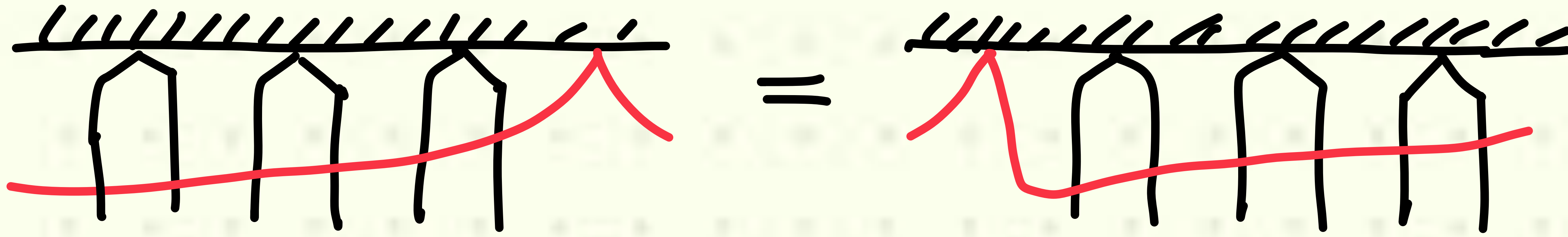

Compatibility with the RTT-relation $\quad R_{12}(u-v)\,T_1(u)\,T_2(v) = T_2(v)\,T_1(u)\,R_{12}(u-v)$

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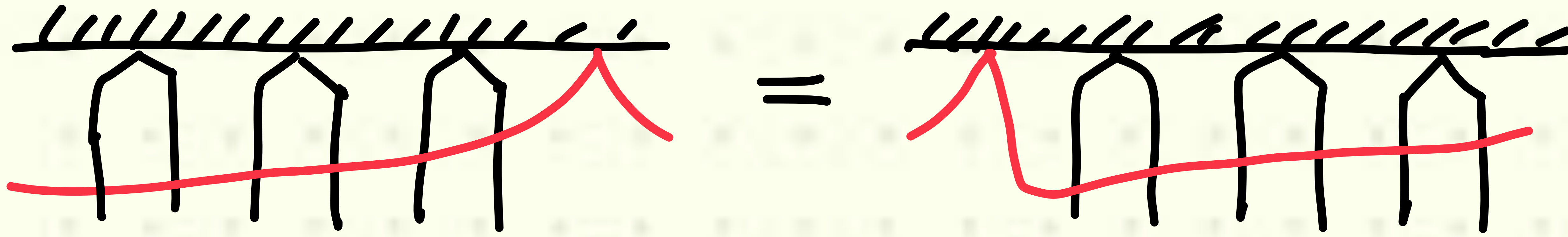

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$$\langle\Psi|\left[T_1(z_1)T_2(z_2)\right] = R^{-1}\langle\Psi|T_2(z_2)T_1(z_1)R = \ldots = (\ldots)\langle\Psi|T_1(-z_1)T_2(-z_2)(\ldots)$$

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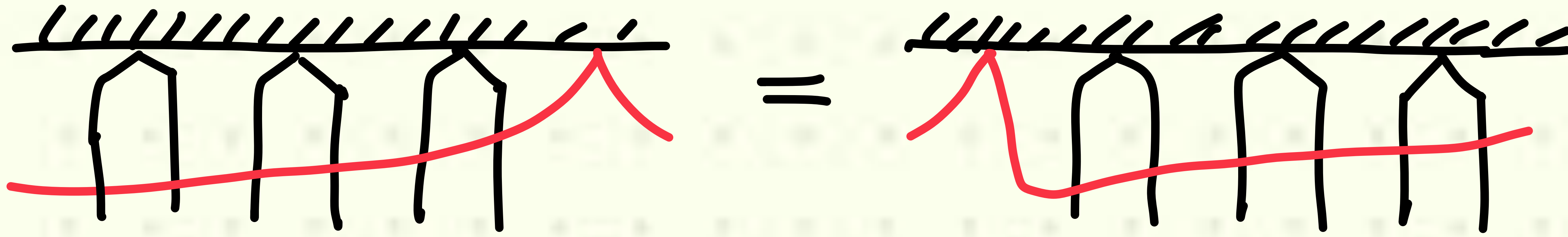

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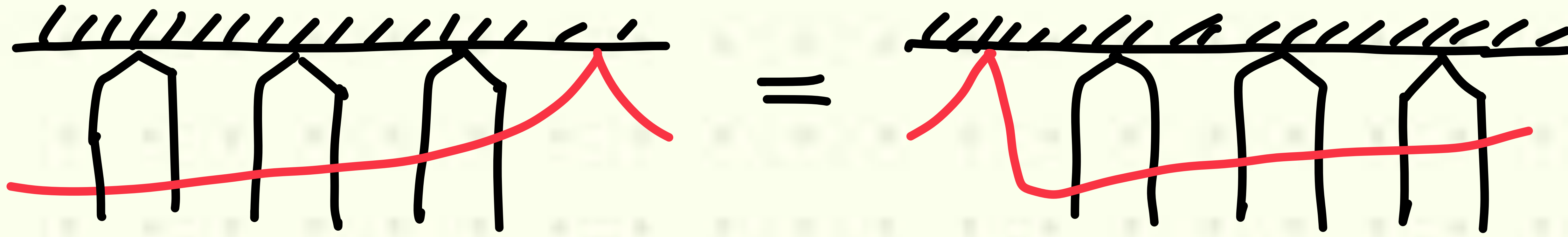

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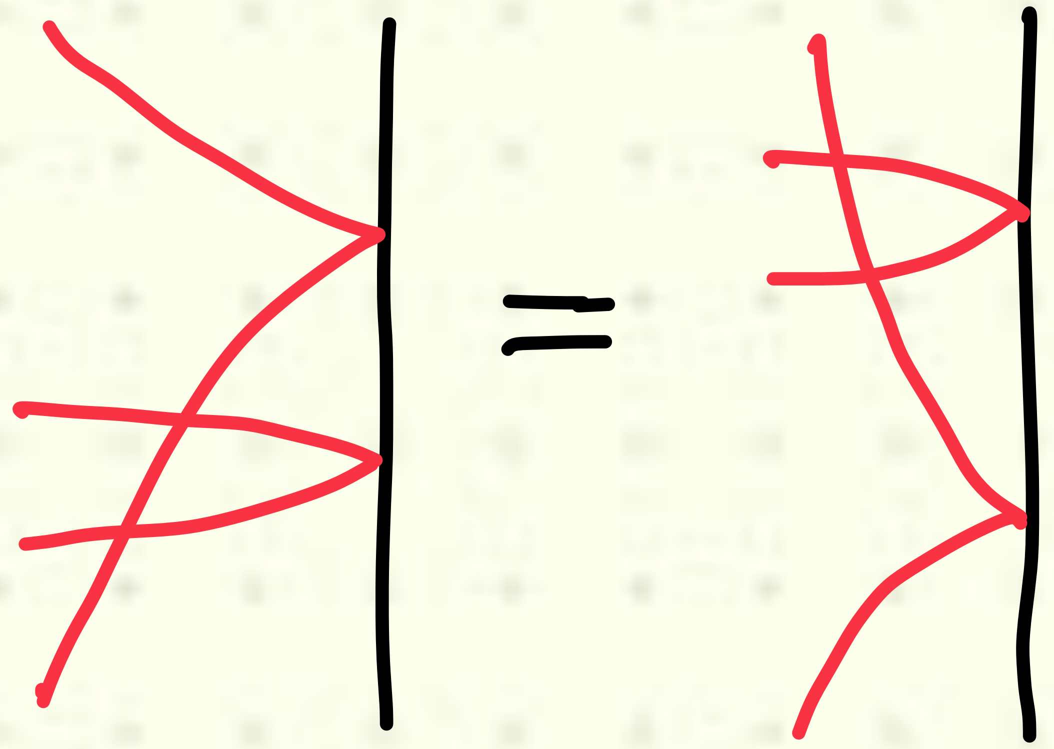
$\Longrightarrow$ reflection equation $R_{12}(u-v)K_1(-u)R_{12}(u+v)K_2(-v) = K_2(-v)R_{12}(u+v)K_1(-u)R_{12}(u-v)$



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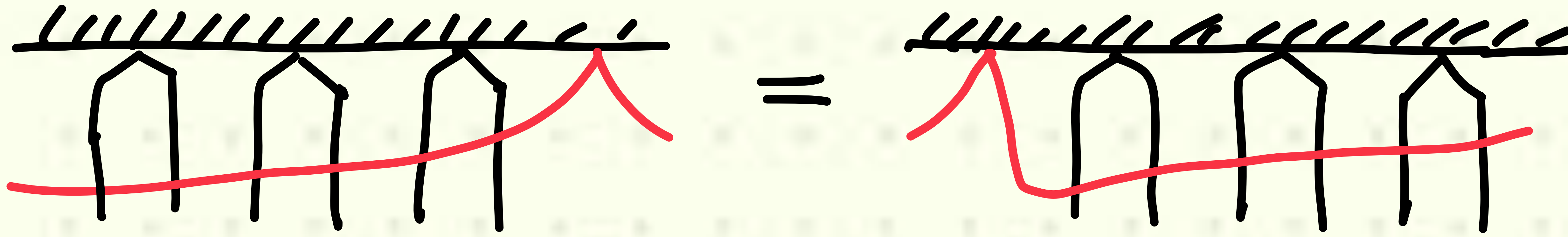

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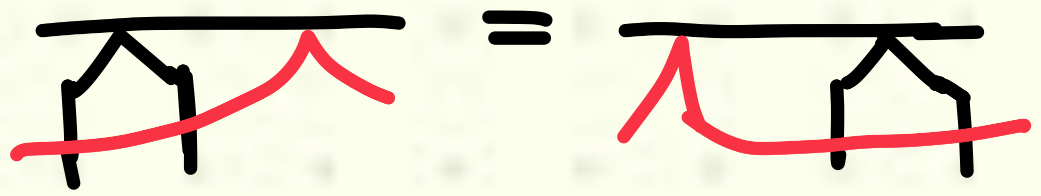
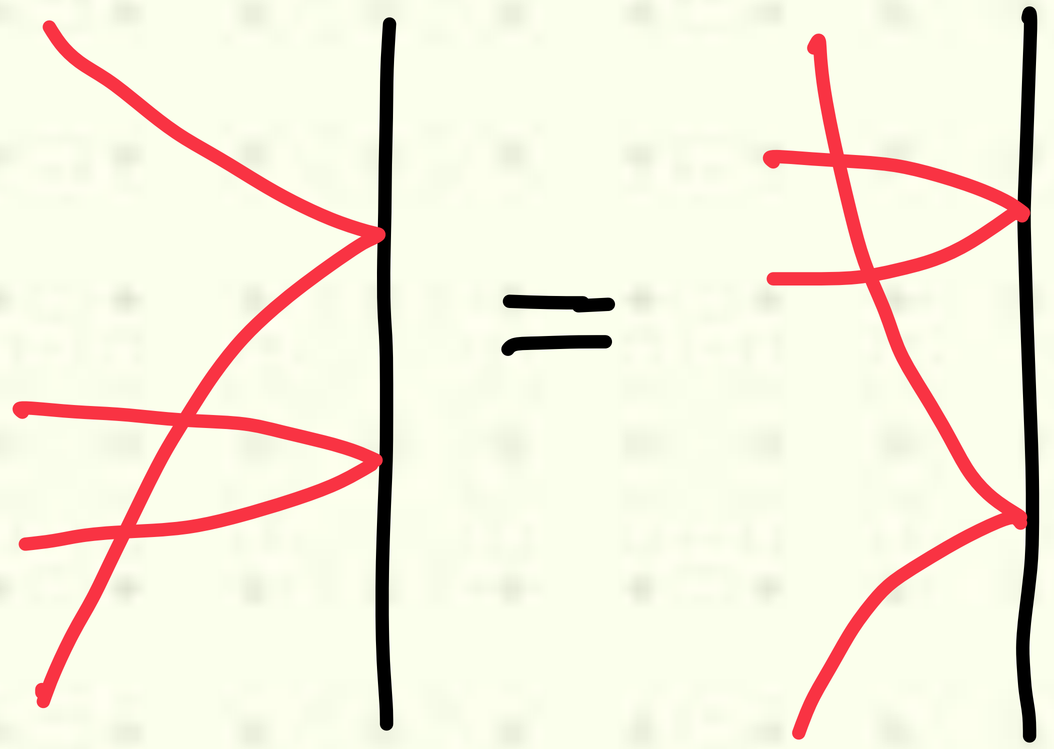
 $= \langle\psi| \longrightarrow$ higher rep K-matrix

KT for $z=1$





Calculation of the Off-shell overlap

(2,2) component of the KT-relation

$$K_{2,1}(z)\langle\Psi|T_{1,2}(z)+K_{2,2}(z)\langle\Psi|T_{2,2}(z)=\langle\Psi|T_{2,1}(-z)K_{1,2}(z)+\langle\Psi|T_{2,2}(-z)K_{2,2}(z)$$

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Assuming $K_{2,1}\neq 0$ we can express $\langle\Psi|T_{1,2}$ with $\langle\Psi|T_{2,2}$ or $\langle\Psi|T_{2,1}$

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Creation diagonal annihilation

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off-shell overlap $S_2(\bar{u})=\langle\psi|\bar{u}\rangle$

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off-shell overlap $S_n(\bar{u}) = \langle\psi|\bar{u}\rangle$ $|\bar{u}\rangle = \prod_{j=1}^{n} T_{1,2}(u_j)|0\rangle$

Calculation of the Off-shell overlap

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off-shell overlap $S_n(\bar{u}) = \langle\Psi|\bar{u}\rangle$ $|\bar{u}\rangle = \prod_{j=1}^{n} T_{1,2}(u_j)|0\rangle$ $T_{i,i}(u)|0\rangle = \lambda_i(u)|0\rangle$

Calculation of the Off-shell overlap

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$$S_\lambda(\{z,\bar{u}\})=\sum(\dots)S_\lambda(\bar{w})$$

$\#\bar{u}\geq\#\bar{w}$ $\qquad \bar{w}\subset\{z,-z\}\cup\bar{u}$

Properties of the off-shell overlaps

1) sum formula $S_{\bar\lambda}(\bar u) = \sum_{\bar u = \bar u_{\mathrm{I}} \cup \bar u_{\mathrm{II}}} W(\bar u_{\mathrm{I}} | \bar u_{\mathrm{II}}) \lambda_1(\bar u_{\mathrm{I}}) \lambda_2(\bar u_{\mathrm{II}})$

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1) KT-relation $\langle \Psi | T_{1,2} \to \langle \Psi | T_{2,2}$ & $\langle \Psi | T_{2,1}$

2) recurrence equation $|\{z, \bar u\}\rangle = T_{1,2}(z) | \bar u \rangle$

3) action formula $T_{i,j}(z) | \bar u \rangle = \sum (\dots) | \bar w \rangle$

4) co-product formula $| \bar u \rangle = \sum (\dots) | \bar u_{\mathrm{I}} \rangle^{(1)} \otimes | \bar u_{\mathrm{II}} \rangle^{(2)}$

On-shell limit

transfer matrix

$$\mathcal{T}(u) = \sum_{i=1}^{2} T_{i,i}$$

On-shell limit

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$$\Upsilon(u)=\sum_{i=1}^{2} T_{i,i} \longrightarrow \langle\Psi|\Upsilon(u)=\langle\Psi|\Upsilon(-u)$$

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$$\langle\Psi|\bar{u}\rangle \neq 0 \longrightarrow \tau(z|\bar{u}) = \tau(-z|\bar{u})$$

pair structure

$$\longrightarrow \{u_j\}_{j=1}^{r} = \{-u_j\}_{j=1}^{r}$$

$$Q(z) = (-1)^r Q(-z)$$

on-shell limit

Bethe ansatz equations

$$e^{\Phi_j} := \frac{\lambda_1(u_j)}{\lambda_2(u_j)} \prod_{k\neq j} \frac{u_j - u_k - i}{u_j - u_k + i} = 1$$

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Korepin's criteria $\longrightarrow \quad \langle \psi | \bar{u} \rangle \sim \det G^+$

Normalized on-shell overlaps

$$\frac{\langle \psi | \bar{u} \rangle}{\sqrt{\langle \bar{u} | \bar{u} \rangle}} = \mathcal{F}(\bar{u}^+) \sqrt{\frac{\det G^+}{\det G^-}}$$

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$$\frac{\langle \Psi | \bar{u} \rangle}{\sqrt{\langle \bar{u} | \bar{u} \rangle}} = \mathcal{F}(\bar{u}^+) \sqrt{\frac{\det G^+}{\det G^-}}$$

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general solution of the reflection equation

$$K(u) = \frac{\alpha}{u}\mathbb{1} + A = \begin{pmatrix} \frac{\alpha}{u} + b_{11} & b_{12} \\ b_{21} & \frac{\alpha}{u} - b_{11} \end{pmatrix}$$

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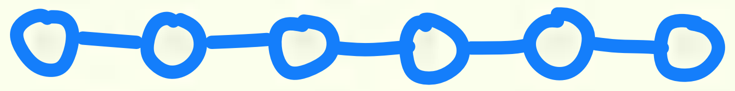
$\mathfrak{gl}(2) \longrightarrow \mathfrak{gl}(N)$

$\bar{u}$



$\bar{u}^1 \quad \bar{u}^2 \quad \bar{u}^3 \quad \dots \quad \bar{u}^{N-1}$

$|\bar{u}\rangle$

$|\bar{u}\rangle \equiv |\bar{u}^1, \bar{u}^2, \dots, \bar{u}^{N-1}\rangle$

Generalisation to gl(N) spin chains

two types of KT-relations

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compatibility conditions

$$R_{12}(u-v)K_1(-u)R_{12}(u+v)K_2(v) = K_2(-v)R_{12}(u+v)K_1(-u)R_{12}(u-v)$$

$$R_{12}(u-v)K_1(-u)\bar{R}_{12}(u+v)K_2(v) = K_2(-v)\bar{R}_{12}(u+v)K_1(-u)R_{12}(u-v)$$

Symmetries and pair structures

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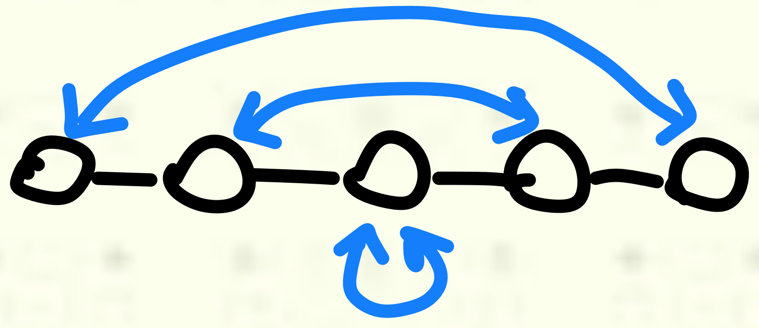

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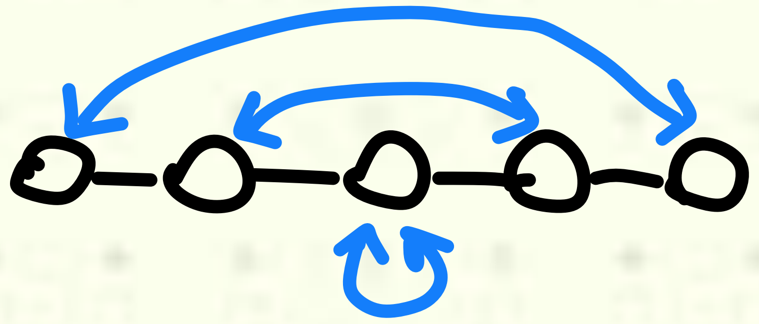

Crossed KT $\quad \langle\psi|\bar{u}\rangle \neq 0 \iff \tau(z|\bar{u}) = \hat{\tau}(-z|\bar{u})$

Symmetries and pair structures

Non-crossed K-matrices $\quad K(u)=\frac{a}{u}\mathbb{1}+A \qquad A^2=\mathbb{1}$ $\qquad$ residual symmetry

$$A\sim \mathrm{diag}(\underbrace{+,+,\dots,+}_{N-M},\underbrace{-,-,\dots,-}_{M}) \qquad \mathfrak{gl}(M)\oplus\mathfrak{gl}(N-M)$$

Crossed K-matrices $\quad K(u)=V \qquad V^t=\pm V$ $\qquad$ $O(N)$, $Sp(N)$

Non-crossed KT $\quad \langle\Psi|\bar{u}\rangle\neq 0 \iff \tau(z|\bar{u})=\tau(-z|\bar{u})$

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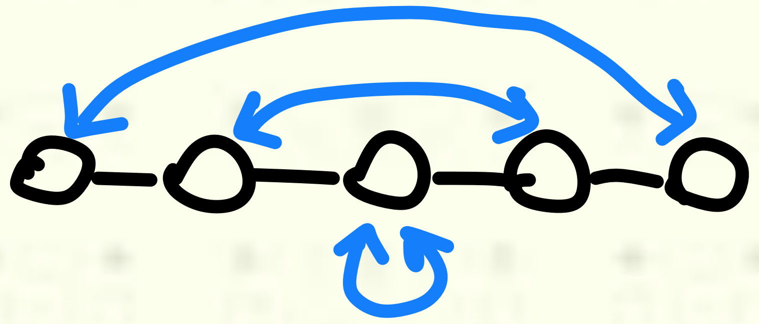

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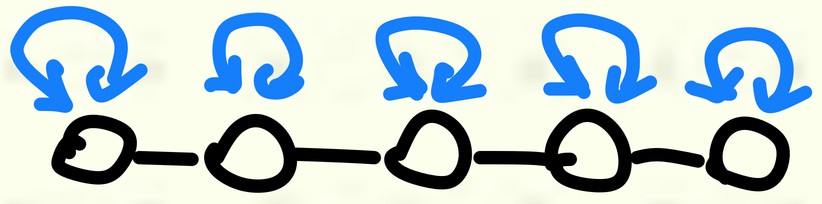

Off-shell overlaps

List of criteria for off-shell overlaps

$$S_{\bar\lambda}(\bar u) = \sum W(\bar u_{\mathrm{I}} | \bar u_{\mathrm{II}}) \prod_{\nu=1}^{N-1} \lambda_\nu(\bar u^\nu_{\mathrm{I}}) \lambda_{\nu+1}(\bar u^\nu_{\mathrm{II}})$$

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$\Longrightarrow$ **Hutsalyuk, Liashyk, Pakuliak, Ragoucy, Slavnov '16, '17, '20**

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Recursion: $\langle\psi|\{z,\bar{u}^1\},\bar{u}^2,\dots\rangle = \sum_j(\dots)\langle\psi|T_{1,j}|\bar{u}^1,\bar{v}^2,\dots\rangle = \sum_j(\dots)\langle\psi|\bar{w}^1,\bar{w}^2,\dots\rangle$

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On-shell overlaps without twists

Korepin's criteria $\longrightarrow$

$$\frac{\langle\psi|\bar{u}\rangle}{\sqrt{\langle\bar{u}|\bar{u}\rangle}} = \prod_{\nu} F_{\nu}(\bar{u}^{+\nu})\sqrt{\frac{\det G^{+}}{\det G^{-}}}$$

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- $gl(M) \oplus gl(N-M)$ $$\frac{Q_M(a)}{\sqrt{Q_n(0) Q_n(\frac{i}{2})}} \sqrt{\frac{\det G^{+}}{\det G^{-}}} \qquad n = \frac{N}{2}$$

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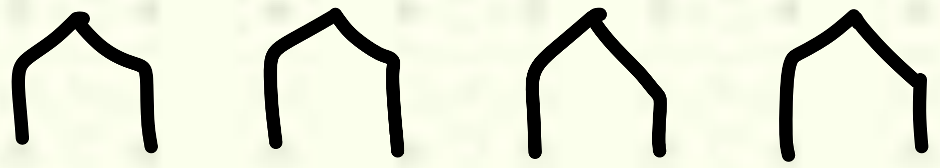
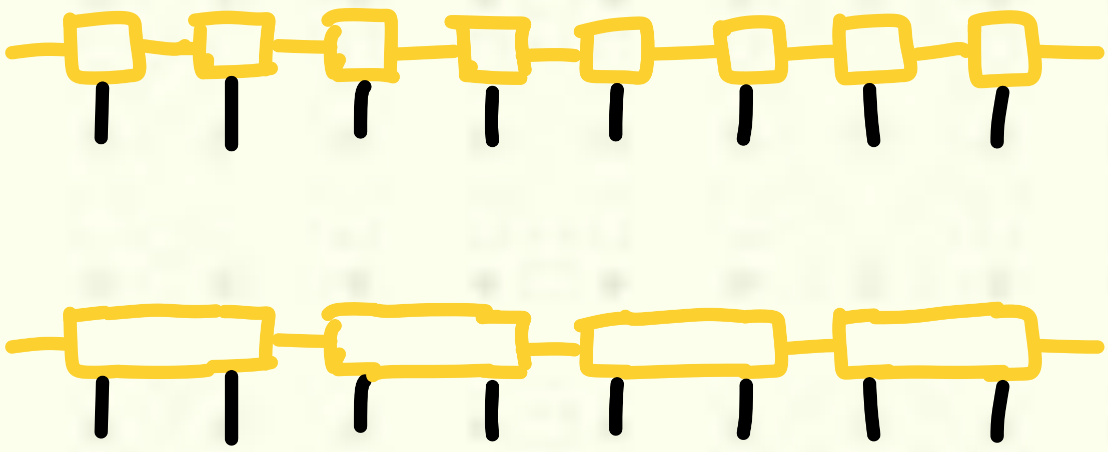
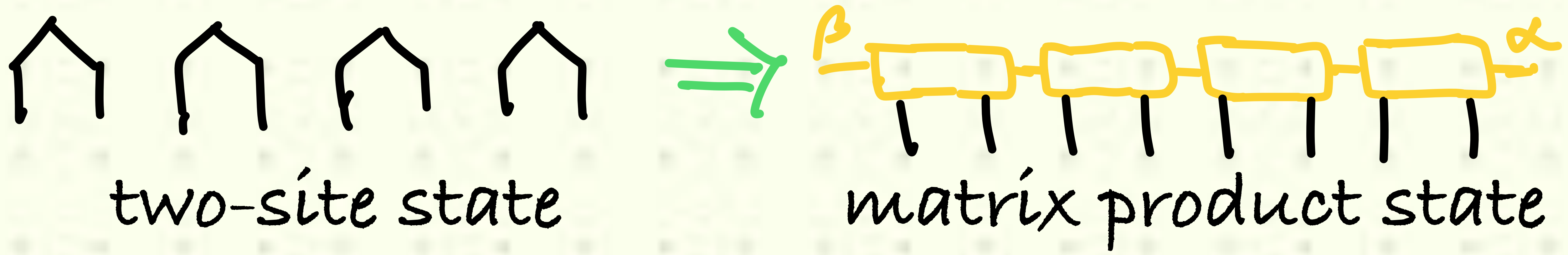
two-site states → matrix product states





Generalisation to MPs



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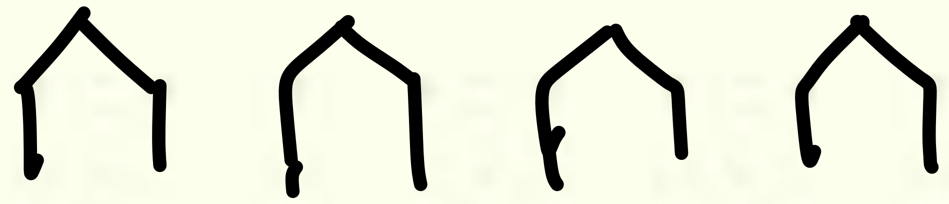
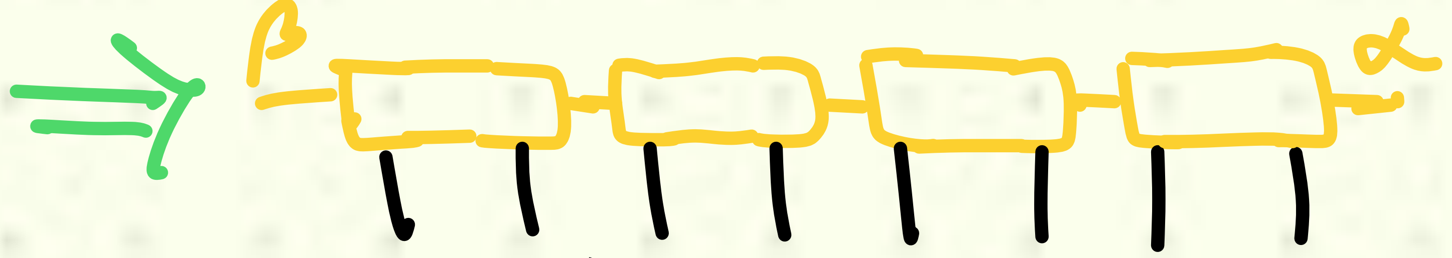


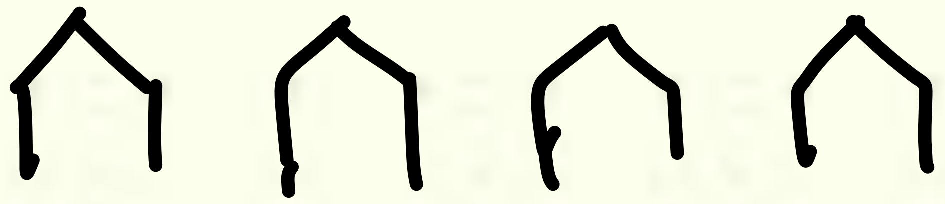
two-site state

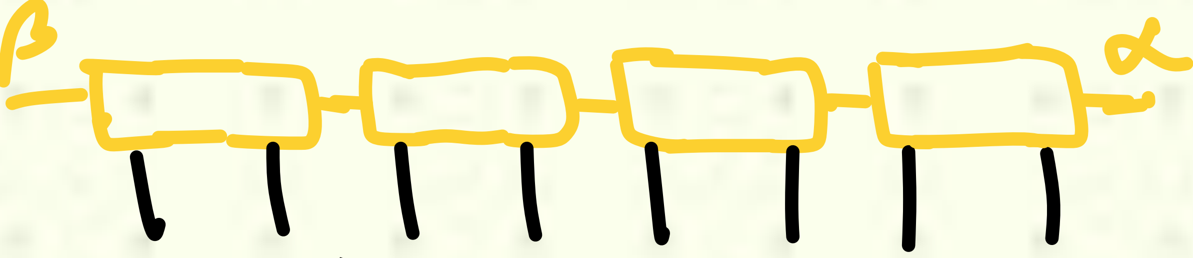


matrix product state

$$\langle \Psi_{\alpha\beta}| = \sum \Big(M^{\cdot}_{i_{2g-1},i_{2g}}(\theta_g) \ldots M^{\cdot}_{i_1,i_2}(\theta_1) \Big)_{\alpha\beta} \langle i_1, i_2, \ldots, i_{2g-1}, i_g|$$

Generalisation to MPS





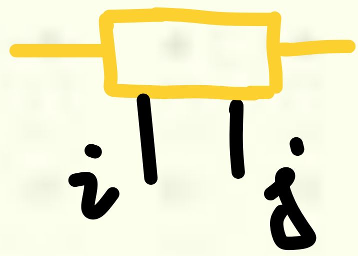
two-site state $\Rightarrow$ matrix product state

$$\langle \Psi_{\alpha\beta}| = \sum \left(M_{i_{2J-1}, i_{2J}}(\theta_J) \ldots M_{i_1, i_2}(\theta_1) \right)_{\alpha\beta} \langle i_1, i_2, \ldots, i_{2J-1}, i_{2J}|$$

$M_{ij}(\theta) \in \mathrm{End}(\mathcal{H}_B)$

$i = 1, \ldots, d$

$\mathcal{H} = [\mathbb{C}^d]^{\otimes 2J}$



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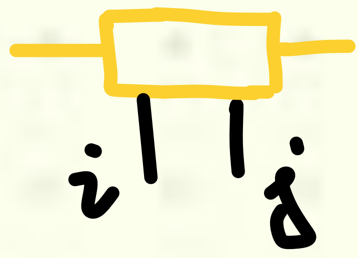

$$\langle \Psi | = \sum_{\alpha, \beta} \langle \psi_{\alpha,\beta} | \otimes e_{\alpha,\beta} \in \mathcal{H}^* \otimes \mathrm{End}(\mathcal{H}_B)$$

Generalisation to MPS

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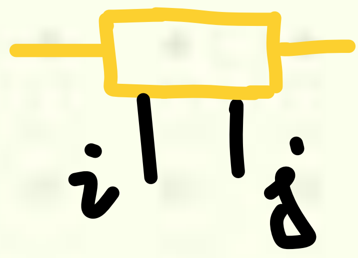

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Generalisation to MPS

two-site state ⟹ matrix product state

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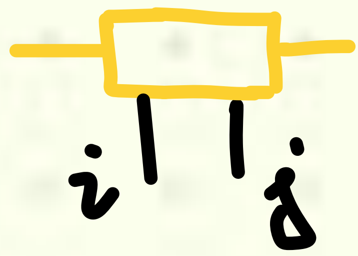

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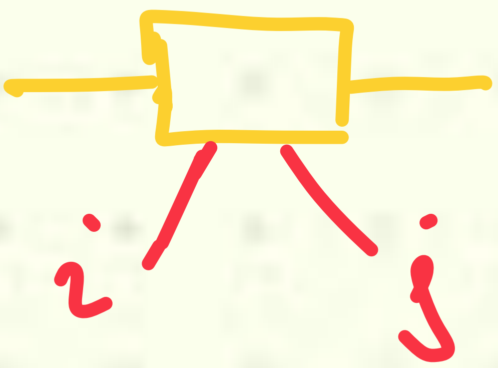

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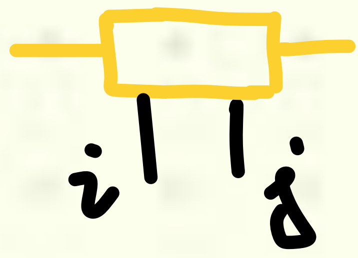

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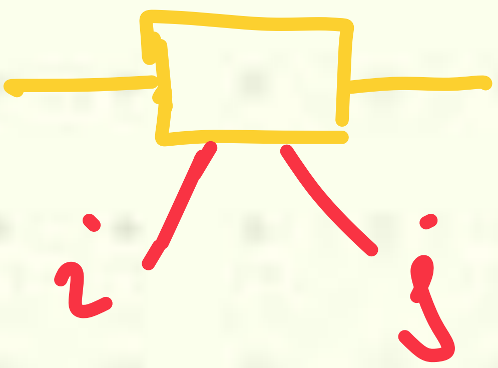
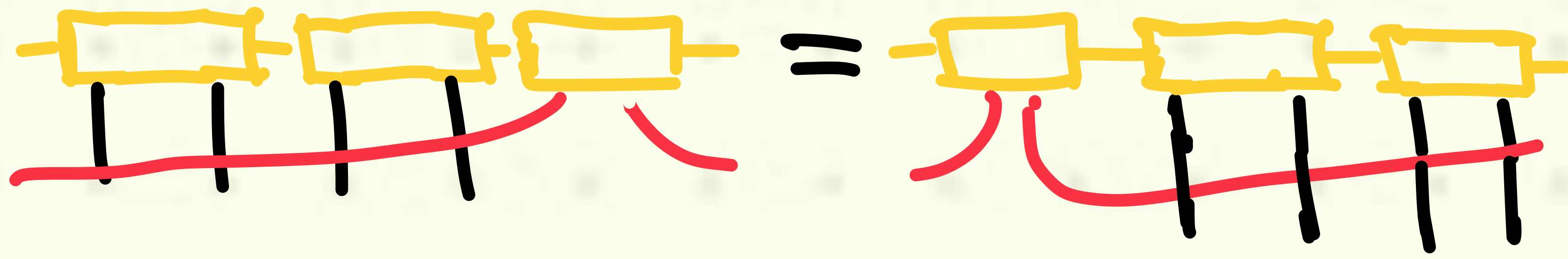


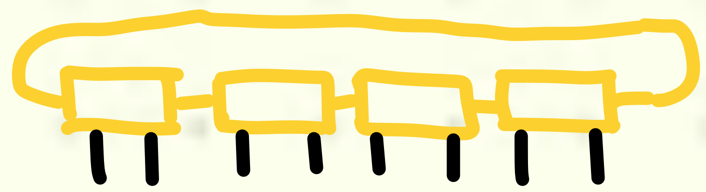
KT- relation

$$\sum_{j=1}^{N} K_{ij}(z)\langle \Psi| T_{j,k}(z) = \sum_{j=1}^{N} \langle \Psi| T_{i,j}(-z) K_{j,k}(z)$$



Integrability condition

Integrability condition

$\langle MPS| =$  $= \sum_{\alpha} \langle \psi_{\alpha\alpha}|$

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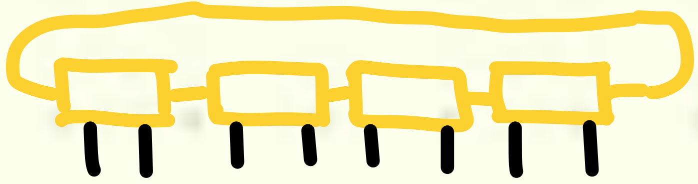
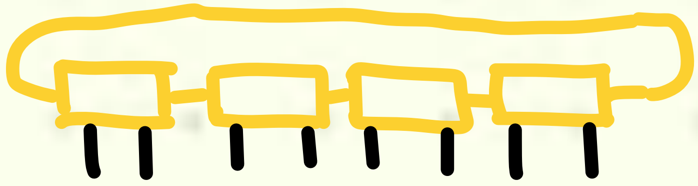
$\langle \mathrm{MPS}| =$  $= \sum_{\alpha} \langle \Psi_{\alpha\alpha}|$

$$\Rightarrow \quad \langle \mathrm{MPS}|T(u) = \langle \mathrm{MPS}|T(-u) \quad \text{or} \quad \langle \mathrm{MPS}|T(u) = \langle \mathrm{MPS}|\widehat{T}(-u)$$

Integrability condition

$\langle MPS| =$  $= \sum_\alpha \langle \Psi_{\alpha\alpha}|$

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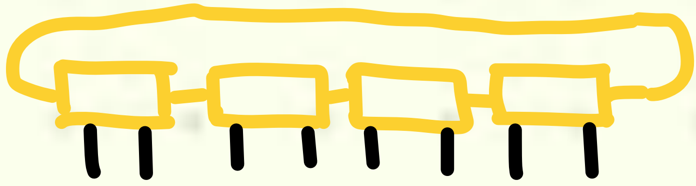
homogeneous limit

Integrability condition

$\langle MPS| =$  $= \sum_{\alpha} \langle \Psi_{\alpha\alpha}|$

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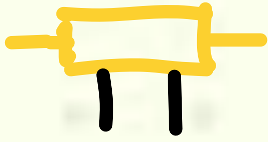
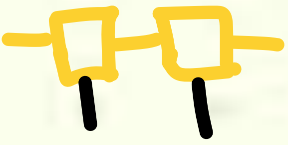
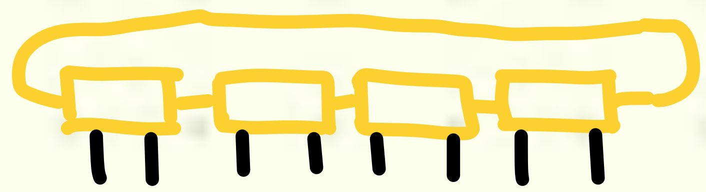
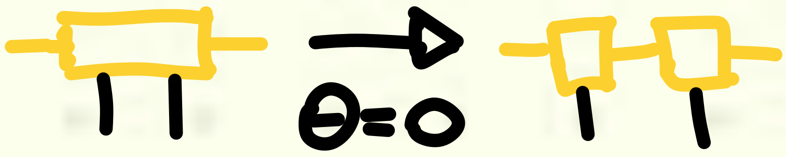
If  $\xrightarrow[\theta=0]{}$ 

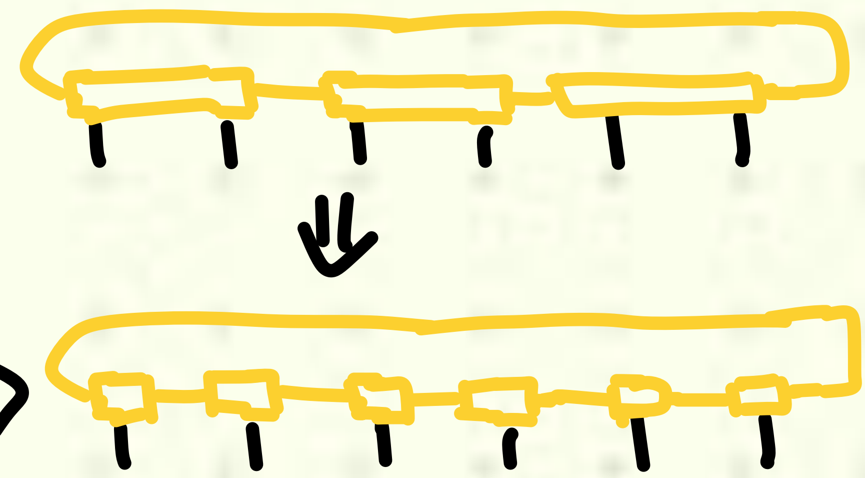
Integrability condition

$\langle MPS| =$  $= \sum_\alpha \langle \Psi_{\alpha\alpha}|$

$\Rightarrow \quad \langle MPS|T(u) = \langle MPS|T(-u) \quad$ or $\quad \langle MPS|T(u) = \langle MPS|\hat{T}(-u)$

homogeneous limit

If 

$\Rightarrow$ 

Classification of the K-matrices

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(bYBe)

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compatibility of the KT with RTT → Reflection equation (bYBe)

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compatibility of the KT with RTT $\longrightarrow$ Reflection equation $\quad RKRK = KRKR$

(bYBe) $\qquad \bar{R} \qquad \bar{R}$

$\downarrow$

K-matrices are representations of reflection algebras

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Recursion for off-shell overlaps

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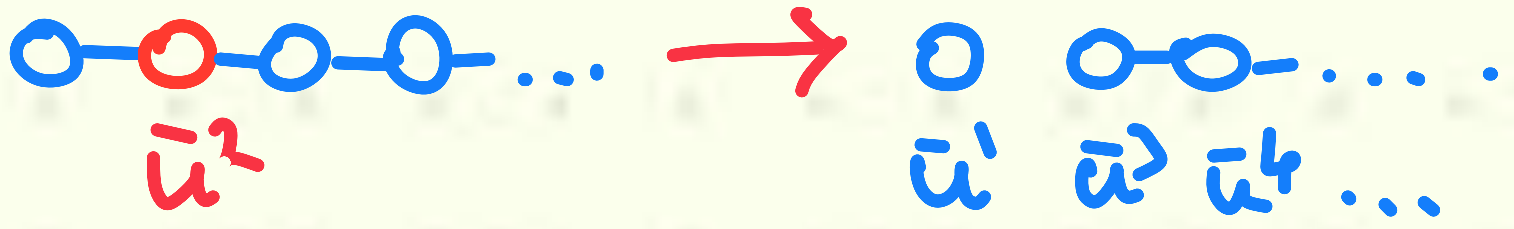

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$\bar{u}^2 \qquad \bar{u}^2 \qquad \bar{u}^1 \; \bar{u}^3 \; \bar{u}^4 \ldots$

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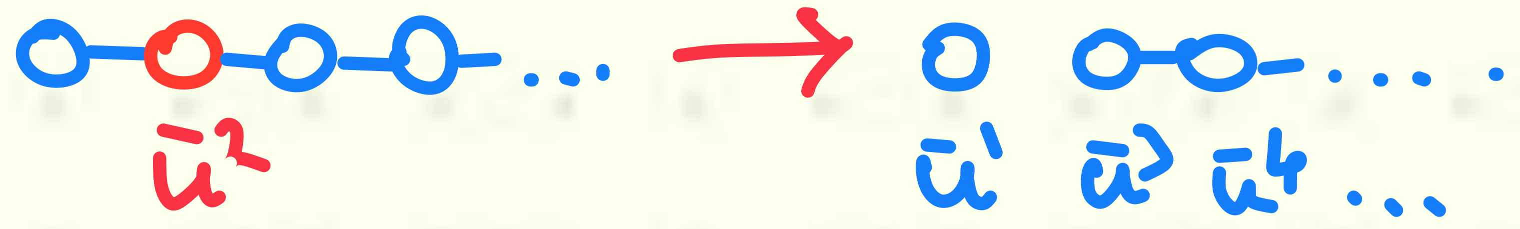

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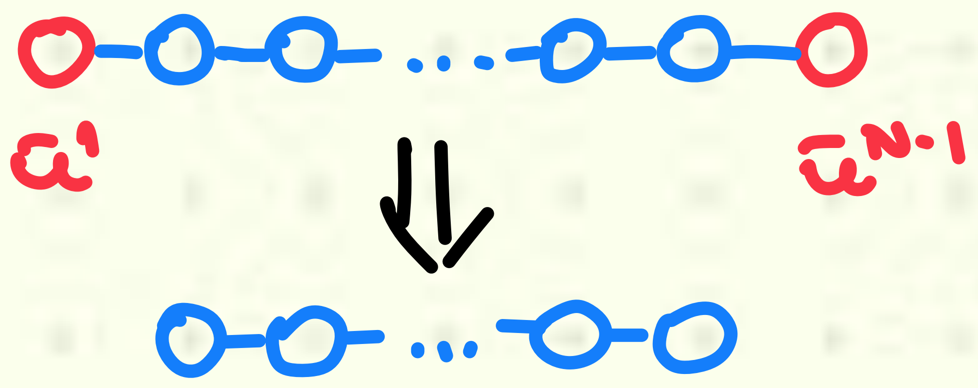

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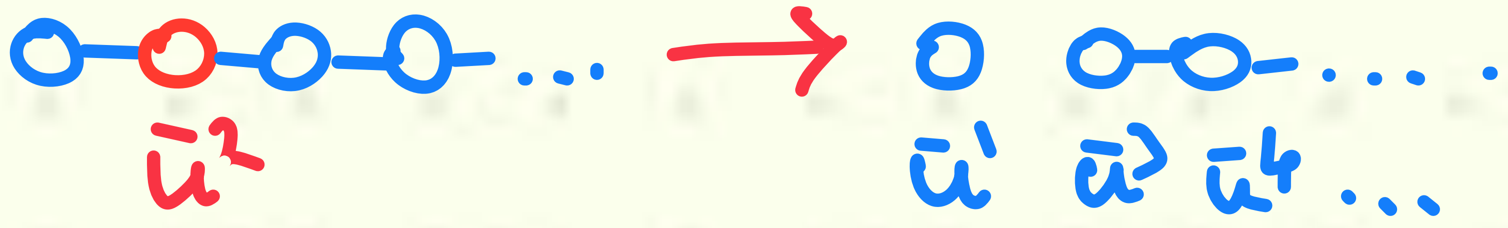

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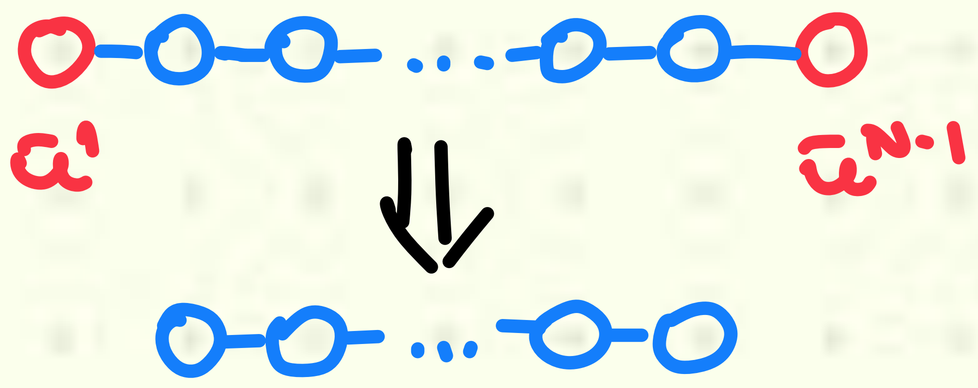

$$\mathfrak{gl}(N) \to \mathfrak{gl}(N-2) \to \dots \to \begin{matrix}\mathfrak{gl}(2)\\ \mathfrak{gl}(3)\end{matrix} \to \mathfrak{gl}(1)$$

$K^{(1)} \equiv K$	$K^{(2)}$	$K^{(n)}$	$K^{(n+1)}$
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$n = \lfloor N/2 \rfloor$

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Crossed

$$\tilde{\mathcal{F}}_\alpha^{(s)}(u) = \mathcal{F}_\alpha^{(s)}\left(u - i\frac{s}{2}\right)\sqrt{\frac{u^2}{u^2+1/4}}$$

Non-crossed

$$\tilde{\mathcal{F}}_\alpha^{(s)}(u) = \begin{cases} \mathcal{F}_\alpha^{(s)}(u - c_s) \\ \mathcal{F}_\alpha^{(\frac{N-1}{2})}\left(u - c_{\frac{N-1}{2}}\right)\sqrt{\frac{u^2}{u^2+1/4}} \end{cases}$$

Other spin chains

The on-shell overlaps are extended to other rational spin chains without proofs

Symmetry of the spin chain	Type of refl.	Residual symmetry	Pair structure
$gl(N)$	AI	$o(N)$	Chiral
	AII	$sp(N)$	Chiral
	AIII	$gl(M)+gl(N-M)$	Achiral
$o(2n+1)$	BI	$o(M)+o(N-M)$	Chiral
$sp(2n)$	CI	$sp(2m)+sp(2n-2m)$	Chiral
	CII	$gl(n)$	Chiral
$o(2n)$	DI	$o(M)+o(2n-M)$	$n-M=0$ (mod 2) chiral
			$n-M=1$ (mod 2) achrial
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o(2n)	DI	o(M)+O(2n-M)	n-M=0 (mod 2) chiral
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The overlaps are also conjectured for graded spin chains, including gl(m|n) and osp(m|2n)

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where $\tilde{\mathcal{F}}^{(s)}_{\alpha}(u)$ are the eigenvalues of the operators $\mathcal{F}^{(s)}(u)$

Proof is possible if 1) KT- relation: creation to annihilation

2) recurrence formula $|\{z,\bar{u}^1\},\bar{u}^2,\dots\rangle = \sum(\dots)T_{1,j}(z)|\bar{u}^1,\bar{u}^2,\dots\rangle$

3) action formula $T_{i,j}(z)|\bar{u}\rangle = \sum(\dots)|\bar{w}\rangle$

4) co-product formula $|\bar{u}\rangle = \sum(\dots)|\bar{u}_{I}\rangle^{(1)}\otimes|\bar{u}_{II}\rangle^{(2)}$