Bethe Ansatz for the Propagator of the Multi-Species Totally Exclusion Process

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work in progress, joint with Luigi Cantini

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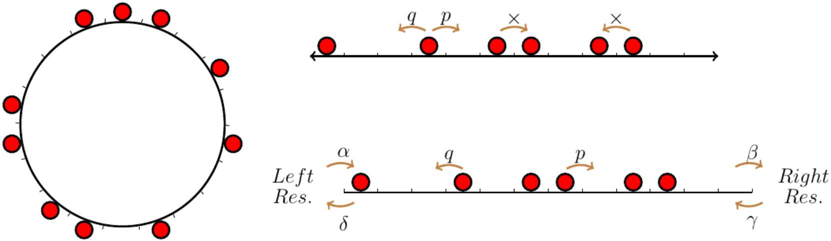
1. What is the exclusion process?
2. Conditional probability in the single species case
3. Conditional probability in the multi species case



- A gas of particles on the lattice, with the exclusion rule.
- Particular cases: Totally Asymmetric Simple Exclusion Process (TASEP)
- Paradigmatic transport model for out of equilibrium statistical physics.
- Pedagogical framework for quantum Integrability, which is **Markovian stochastic**

Mapping to XXZ Heisenberg chain

Time evolution for ASEP on the ring is governed by the master's equation:

$$\frac{dP(C)}{dt} = \sum_{C' \neq C} \underbrace{w_{C' \to C} P(C')}_{\text{gain}} - \sum_{C' \neq C} \underbrace{w_{C \to C'} P(C)}_{\text{loss}}$$

The Markov matrix can be written as a sum of local operator:

$$\frac{dP}{dt} = \left(\sum_{i=1}^{L} M_{i,i+1}\right) P$$

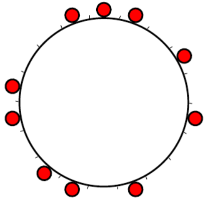
ASEP can be mapped to a non-Hermitian spin chain of the XXZ-type

$$M = \sum_{i=1}^{L} (p S_i^- S_{i+1}^+ + q S_i^+ S_{i+1}^- + \frac{1}{4} S_i^z S_{i+1}^z) - \frac{L}{4}$$

Bethe Ansatz can be used to diagonalise M [Gwa and Spohn, '92]. For a nice review [Golinelli,Mallick '06]

Impact on the boundary conditions

Imagine N particles with the two boundary conditions



N particles on the ring of L sites

- There is a steady state,(probability invariant measure)
- it's given by a uniform distribution

N particles on the line

- There is no steady state
- A reasonable question: starting at $t = 0$ at positions $x_1, x_2, ..., x_N$ what is the probability at time t to find the particles at positions $y_1, y_2, ..., y_N$

Conditional Probability for TASEP

How to compute the conditional probability
$P(\{y_1, ..., y_N\}; t|\{x_1, ..., x_N\})$?
[G. M. Schutz, 97]

$$\boxed{P(\{y_1, ..., y_N\}; t|\{x_1, ..., x_N\}) = \det((F_{i-j}(y_i - x_j; t))_{ij})}$$

$$F_p(n; t) := e^{-t} \oint e^{\frac{t}{z}} \frac{z^{n-1}}{(1-z)^p} dz$$

Conditional probability for TASEP ($q = 0, p = 1$)

How to compute the conditional probability $P(\{y_1, ..., y_N\}; t|\{x_1, ..., x_N\})$?

Example: $N = 2$

The master equation:

- For non neighbouring particles:

$$\frac{d}{dt}P(\{y_1, y_2\}; t) = P(\{y_1 - 1, y_2\}) + P(\{y_1, y_2 - 1\}) - 2P(\{y_1, y_2\})$$

- For neighbouring particles:

$$\frac{d}{dt}P(\{y_1, y_2\}; t) = P(\{y_1 - 1, y_2\}; t) - P(\{y_1, y_2\}; t)$$

It's possible to make the first equation account for the second if we define a non-physical boundary condition:

$$P(\{y_1, y_2 = y_1\}; t) = P(\{y_1, y_2 = y_1 + 1\}; t)$$

Conditional Probability for TASEP

The eigenvalue problem for the Markov matrix is:

$$\lambda P(\{y_1, y_2\}) = P(\{y_1 - 1, y_2\}) + P(\{y_1, y_2 - 1\}) - 2P(\{y_1, y_2\})$$

A solution:

$$P(\{y_1, y_2\}) = e^{i(p_1 y_1 + p_2 y_2)}$$

but is not compatible with the boundary condition.
Bethe Ansatz: Try a superposition of the possible permutations of the momentum

$$P_{\{p_1,p_2\}}(\{y_1, y_2\}) = A_{1,2} e^{i(p_1 y_1 + p_2 y_2)} + A_{2,1} e^{i(p_2 y_1 + p_1 y_2)}$$

This indeed verifies the boundary condition is the coefficients verify:

$$\frac{A_{1,2}}{A_{2,1}} = -\frac{1 - e^{ip_1}}{1 - e^{ip_2}}$$

And the corresponding eigenvalue would be: $\lambda_{\{p_1,p_2\}} = e^{-ip_1} + e^{-ip_2} - 2 = \lambda_{p_1} + \lambda_{p_2}$

For N particles: The Bethe wave function would be composed of $N!$ term:

$$P_{\mathbf{p}}(\mathbf{y}) = \sum_{\sigma \in \mathcal{S}_N} A_\sigma \prod_{i=1}^{N} e^{ip_{\sigma(i)} y_i}$$

Where $\mathbf{p} = \{p_1, ..., p_N\}$, $\mathbf{y} = \{y_1, ..., y_N\}$ and S_N is the symmetric group. The restriction on the coefficients generalizes to:

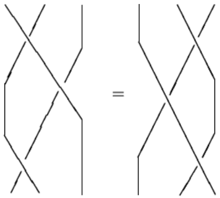
$$\frac{A_\sigma}{A_{\sigma\tau_{i,i+1}}} = -\frac{1 - e^{ip_{\sigma(i)}}}{1 - e^{ip_{\sigma(i+1)}}}$$

Where $\tau_{i,i+1}$ is the transposition applied on positions i and $i+1$.
Fix $A_{Id} = 1$, that defines the value of any A_σ for any σ
Compatibility test the value of A_σ should be the same regardless of how do we decompose σ into transpositions.

Yang Baxter equation

It happens that we don't need to check for all the possible paths and it's enough to verify that the path is taken by next neighbor transpositions:

$$\tau_{1,3} = \tau_{1,2}\tau_{2,3}\tau_{1,2} = \tau_{2,3}\tau_{1,2}\tau_{2,3}$$



This is the Yang-Baxter equation. We can check that it is verified:

$$A_{\tau_{1,3}} = A_{\tau_{1,2}\tau_{2,3}\tau_{1,2}} = A_{\tau_{2,3}\tau_{1,2}\tau_{2,3}}$$

1. What is the exclusion process?
2. Conditional probability in the single species case
3. **Conditional probability in the multi species case**
   - Integrability and restrictions on the rates

Multi-Species Exclusion Process

- Each species has a jumping rate: $(\bullet\circ \to \circ\bullet)$ with rate $r_{\bullet}$
- two species exchange at a rate: $(\bullet{\color{red}\bullet} \to {\color{red}\bullet}\bullet)$ with rate $r_{\bullet{\color{red}\bullet}}$

- Let $I = \{i_1 < \ldots < i_N\}$ be the set of positions corresponding to the species $\alpha = \{\alpha_1 < \ldots < \alpha_N\}$.
- Let $P(I, \alpha | J, \beta; t)$ be the probability of having the state (I, α) at time t starting from the initial state (J, β).

The Bethe wave vector:

$$\psi^I_\alpha = r^I_\alpha \sum_{\sigma \in \mathcal{S}_N} F^\sigma_\alpha \sigma[\mathbf{z}^I] \qquad \sigma[\mathbf{z}^I] = \prod_k z^{i_k}_{\sigma(k)} \quad z_k = e^{ip_k}$$

Multi species

For two particles, define: $F^\sigma := (F^\sigma_{\alpha,\alpha}, F^\sigma_{\alpha,\beta}, F^\sigma_{\beta,\alpha}, F^\sigma_{\beta,\beta}) \in \mathbb{C}^2 \otimes \mathbb{C}^2$.
Applying the boundary condition:

$$F^{(12)} M(z_2) + F^{Id} M(z_1) = 0 \tag{1}$$

$$M(z) = \begin{pmatrix} 1 - r_\alpha z & 0 & 0 & 0 \\ 0 & 1 - (r_\alpha - r_{\alpha,\beta})z & -\frac{r_\beta}{r_\alpha} r_{\alpha,\beta} z & 0 \\ 0 & -\frac{r_\alpha}{r_\beta} r_{\beta,\alpha} z & 1 - (r_\beta - r_{\beta,\alpha})z & 0 \\ 0 & 0 & 0 & 1 - r_\beta z \end{pmatrix}$$

$$F^{(12)} = F^{Id} \check{R}(z_1, z_2)$$

$$\check{R}(z_1, z_2) = -M(z_1) M(z_2)^{-1}$$

$$F^{\sigma \tau_{i,i+1}} = F^\sigma \check{R}_{i,i+1}(z_{\sigma(i)}, z_{\sigma(i+1)})$$

Multi species

Fixing F^{Id}, we can write F^{σ}: $F^{\sigma} = F^{Id}\check{R}_{\sigma}(\boldsymbol{z})$

The matrix $\check{R}$ needs to obey the Braided Yang-Baxter equation, namely:

$$\check{R}_{12}(z_1,z_2)\check{R}_{23}(z_1,z_3)\check{R}_{12}(z_2,z_3) = \check{R}_{23}(z_2,z_3)\check{R}_{12}(z_1,z_3)\check{R}_{23}(z_1,z_2)$$

Explicit expansion of this equation shows that we need to impose hierarchy over the species. A given species can hope only over lower ones in the hierarchy:

$$r_{\alpha,\beta} = 0 \quad \text{if} \quad \alpha > \beta$$

In addition to this restriction, an additional one is needed, for $\alpha > \beta > \gamma$

$$r_{\gamma,\alpha} - r_{\beta,\alpha} = r_{\gamma} - r_{\beta}$$

Which means:

$$r_{\alpha,\beta} = (r_{\alpha} + \nu_{\beta})\mathbb{1}_{\beta>\alpha}$$

The matrix M verifies Braid-like algebra related to the one introduced in [Crampe,Frappat,Ragoucy,Vanicat '17]

The propagator

Theorem: [Cantini, Zahra, to come] The propagator of the process is given by:

$$P_t(I,\alpha|J,\beta) = \oint_0 \cdots \oint_0 \prod_{k=1}^{n} \frac{dx_k}{2\pi i z_k} e^{\lambda_\beta(x)t} (r_\beta z)^{-J} r_\alpha^I \prod_{k=1}^{n} (z_k)^{i_{\sigma_k}} \sum_{\sigma\in S_n} \left(\check{R}_\sigma(z)\right)_\alpha^\beta$$

In order for this expression to be the propagator, it need to fulfil:

- The evolution equation.
- The initial condition:

$$r_I^\alpha r_J^{-\beta} \sum_{\sigma\in S_n} \oint_0 \cdots \oint_0 \prod_{k=1}^{n} \frac{dz_k}{2\pi i x_k} z^{-J} \left(\check{R}_\sigma(z)\right)_\alpha^\beta z_\sigma^I = \delta_{\alpha,\beta}\delta_{I,J}.$$

That we can prove to be hold.

A case where computations are explicit

One case where we can bring computations till the end for arbitrary number of particles. ${\color{green}\bullet} > {\color{red}\bullet}$

$$G(I, {\color{green}\bullet}{\color{red}\bullet}\ldots{\color{red}\bullet} | J, {\color{red}\bullet}\ldots{\color{red}\bullet}{\color{green}\bullet}) = \left(\frac{r_{\color{green}\bullet}(r_{\color{red}\bullet} + \nu_{\color{green}\bullet})}{r_{\color{red}\bullet}}\right) \oint e^{\frac{t}{z} - r_{\color{green}\bullet} t} (r_{\color{green}\bullet} z)^{i-j} det[m_{k,h}(z,t)]_{hk} \frac{dz}{2\pi i z}$$

Where:

$$m_{k,h}(z,t) := e^{-r_{\color{red}\bullet} t} \oint \frac{e^{\frac{t}{x}} (1 - x r_{\color{red}\bullet})^{h-k} (r_{\color{red}\bullet} x)^{i_k - j_h} (x - z)}{(-1 + r_{\color{green}\bullet} x)(1 + \nu_{\color{green}\bullet} x)} \frac{dx}{2\pi i x}$$

if $r_{\color{green}\bullet} > r_{\color{red}\bullet}$, then we can compute the (absorbing) probability

$$G({\color{green}\bullet}{\color{red}\bullet} | J, {\color{red}\bullet}{\color{green}\bullet}; t = \infty) = \frac{r_{\color{red}\bullet} + \nu_{\color{green}\bullet}}{r_{\color{green}\bullet} + \nu_{\color{green}\bullet}} \left(\frac{r_{\color{red}\bullet}}{r_{\color{green}\bullet}}\right)^{j - j_1 + 1}$$

The formula matches the one obtained using purely probabilistic method [Cantini, Zahra, to come]

Thanks!