# Quantum many-body spin ratchets

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# Background

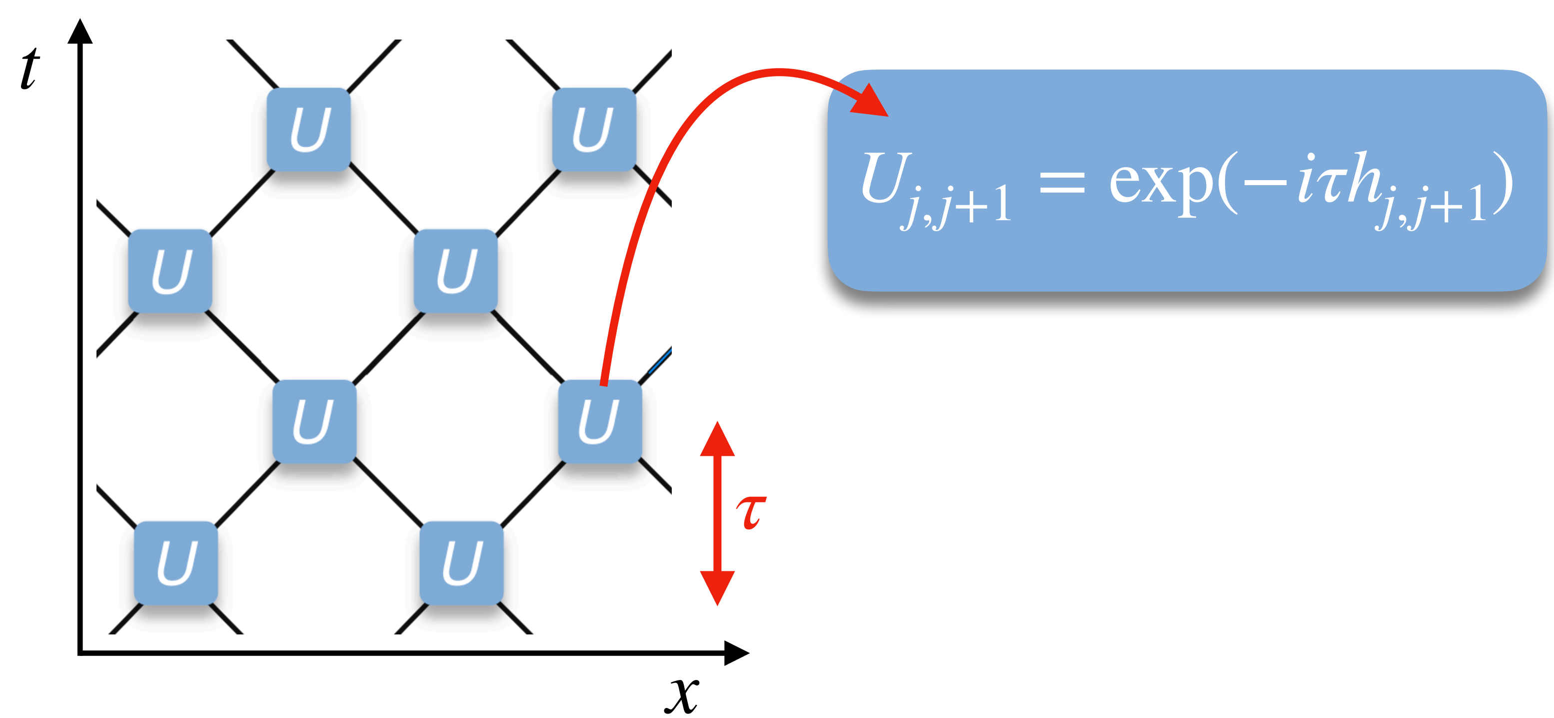
- Spin transport in Heisenberg magnet: requires understanding the large-time scaling of the dynamical susceptibility $\langle S^z(x,t)S^z(0,0)\rangle$ and spin current.

- Numerics: tensor-network techniques (TEBD), relying on the Trotter-Suzuki decomposition

$$H = \sum_{j=1}^{L} h_{j,j+1} = \sum_{\text{odd}\, j} h_{j,j+1} + \sum_{\text{even}\, j} h_{j,j+1} \equiv H_\text{o} + H_\text{e}$$

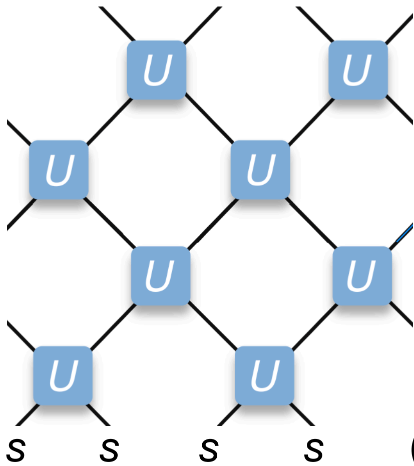
$$e^{-itH} = \lim_{\tau\to 0}\left(e^{-i\tau H_\text{e}}\, e^{-i\tau H_\text{o}}\right)^{t/\tau} =$$

*Lots of steps, while entanglement and complexity of the simulation grow!*



# Background

- Idea: study the circuit as an integrable cellular automaton (Floquet-driven model) with the same features as $\exp(-itH)$

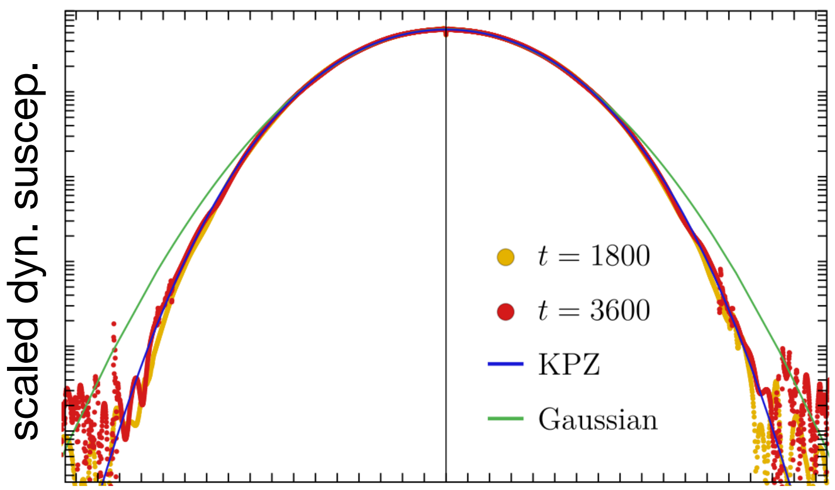



(s = 1/2)

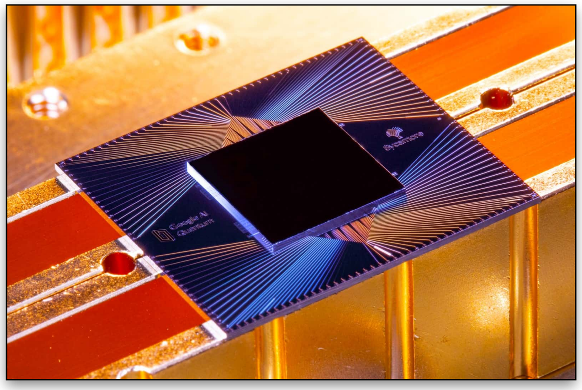
$$U_{j,j+1} = \exp(-i\tau h_{j,j+1})$$

*R matrix of the Heisenberg model*

[M. Vanicat, LZ, T. Prosen, PRL 121, 030606 (2018)]
[M. Ljubotina, LZ, T. Prosen, PRL 122, 150605 (2019)]




[M. Ljubotina, M. Žnidarič, T. Prosen, PRL 122, 210602 (2019)]



Google Quantum AI

[A. Morvan et al., Nature 612, 240 (2022)]

*The same structure as q-Hirota!*

[C. Destri, H. J. De Vega (1987)]
[L. D. Faddeev, A. Yu. Volkov (1994)]

[V. Gritsev, A. Polkovnikov, SciPost Phys. 2, 021 (2017)]
[A. J. Friedman et al., PRL 123, 210603 (2019)]
[P. W. Claeys et al., SciPost Phys. 12, 007 (2022)]
[E. Vernier et al., PRL 130, 260401 (2023)]
[Y. Miao, E. Vernier, Quantum 7, 1160 (2023)]

*… and many others …*

# Background

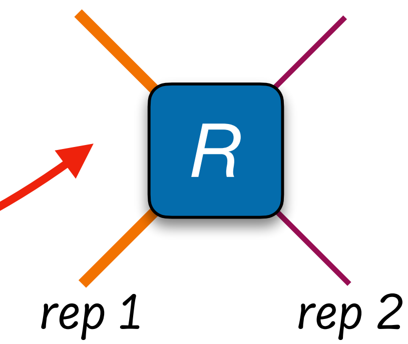
- Top-down: symmetries → models

[S. M. Khoroshkin, V. N. Tolstoy, CMP 141, 599 (1991)]

$R \in \mathfrak{A} \otimes \mathfrak{A}$, e.g. $\mathfrak{A}$ an extension of $sl_2$

↓ ↓

*representations (spins, q-oscillators, …)* →




- Examples

| *rep 1* | *rep 2* | *model/use* | |
|---|---|---|---|
| $s = 1/2$ | $s = 1/2$ | Trotterized Heisenberg magnet | $t$-discretized Heisenberg magnet |
| $q$-oscillator | $q$-oscillator | quantum Hirota equation | $t$-discretized quantum Volterra model<br>$x,t$-discretized sine-Gordon |
| $s = 1/2$ | $q$-oscillator | Q-operators for Heisenberg model<br>transfer matrices for $q$-Volterra model | |
| spin-$s$ | $s = 1/2$ | transfer matrices for Heisenberg model | |

*Our goal:*

*(1) circuit in which rep 1 and rep 2 are different spins*

*(2) what can we learn about physics in it?*

# Outline of the talk

1. Structure of a ratchet circuit
2. Physical features
3. Summary



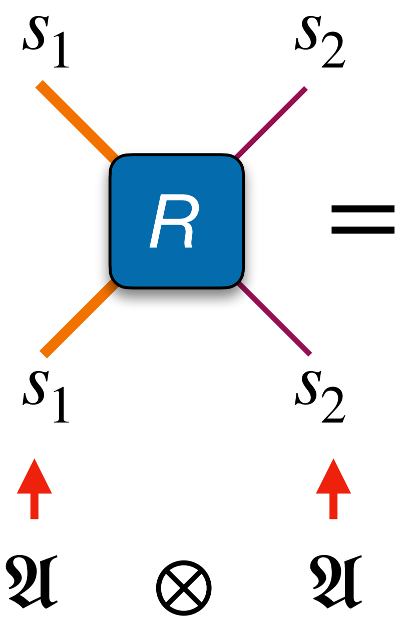

*Work done in collaboration with E. Ilievski, M. Ljubotina, Ž. Krajnik, T. Prosen.*

*arXiv: 2406.01571, to appear in PRX Quantum*

# 1. Structure of a ratchet circuit

- A chain of alternating spins $s_1$, $s_2$

[L. D. Faddeev, arXiv:hep-th/9605187 (1996)]
[A. G. Bytsko, A. Doikou, J. Phys. A 37, 4465 (2004)]




$$J(J+\mathbb{1}) = (\boldsymbol{S}_1+\boldsymbol{S}_2)^2$$

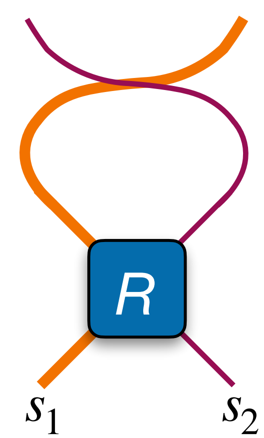
$$R^{s_1,s_2}(\lambda)=(-1)^{J+j}\frac{\Gamma(j+1+i\lambda)\Gamma(J+[1-i\lambda]\mathbb{1})}{\Gamma(j+1-i\lambda)\Gamma(J+[1+i\lambda]\mathbb{1})}$$

$$R^{s_1,s_2}(\tau)R^{s_1,s_2}(-\tau) = \mathbb{1}, \quad \lim_{\tau\to\infty} R^{s_1,s_2}(\tau) = \mathbb{1}$$

$$[R^{s_1,s_2}(\lambda)]^T = R^{s_1,s_2}(\lambda)$$

- Quantum unitary gate

$$U = P^{s_1,s_2}R^{s_1,s_2}(\tau) =$$

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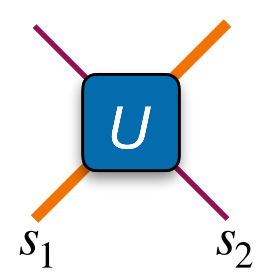


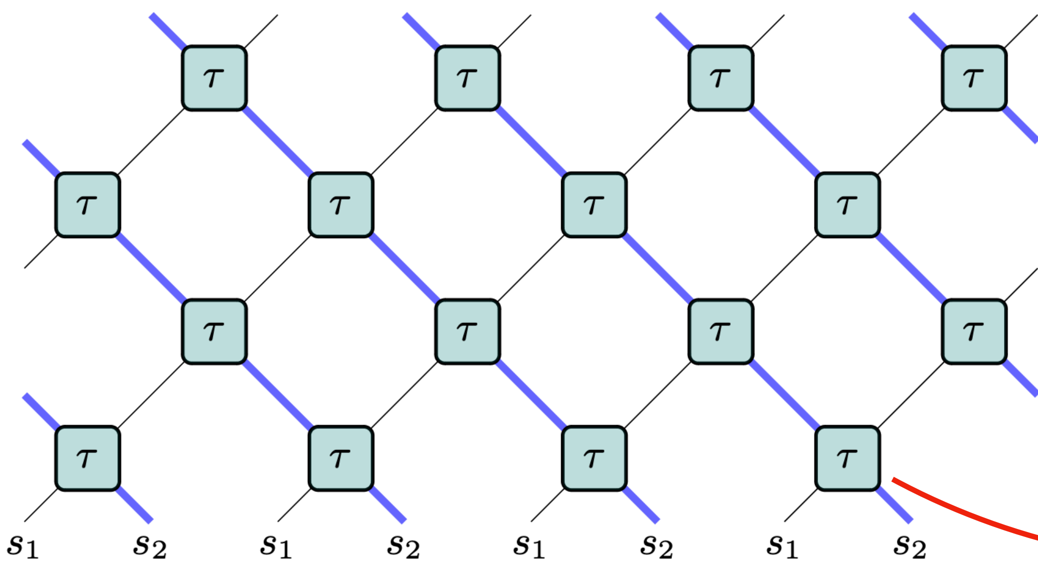

*Experiment: CA -> unitaries*

$$\left(V_{\frac{1}{2}} \otimes V_{\frac{1}{2}}\right) \otimes V_{\frac{1}{2}} = \left(V_1 \otimes V_{\frac{1}{2}}\right) \oplus \left(V_0 \otimes V_{\frac{1}{2}}\right)$$

M. Ringbauer's group, trapped ion qudits
[arXiv:2310.12110]

# 1. Structure of a ratchet circuit

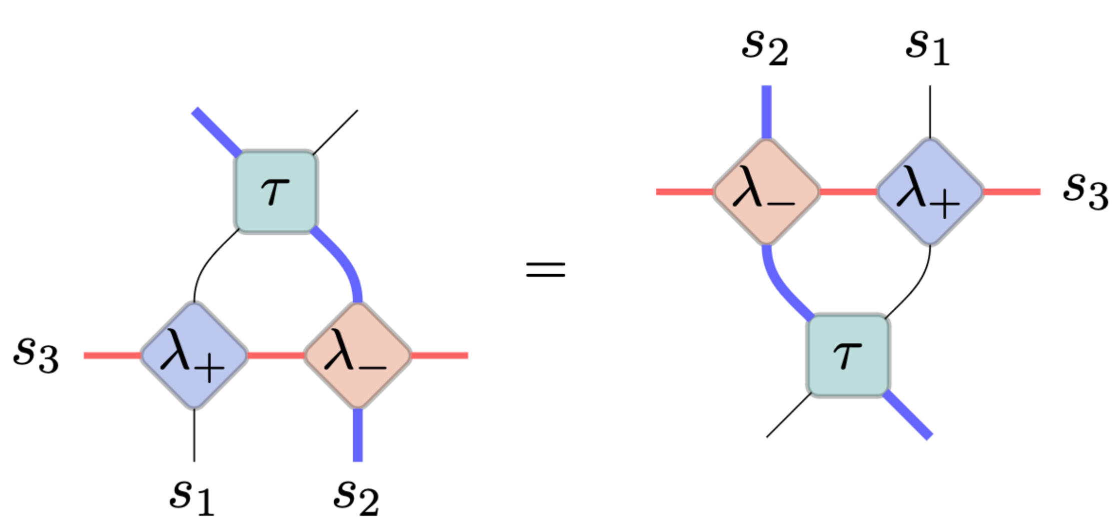
- Brickwork architecture:




$$\mathbb{U} = \mathbb{U}_{\mathrm{o}}\,\mathbb{U}_{\mathrm{e}},$$

$$\mathbb{U}_{\mathrm{o}} = \prod_{j=1}^{L/2} U_{2j,2j+1}, \qquad \mathbb{U}_{\mathrm{e}} = \prod_{j=1}^{L/2} U_{2j-1,2j}$$

- Yang-Baxter integrability



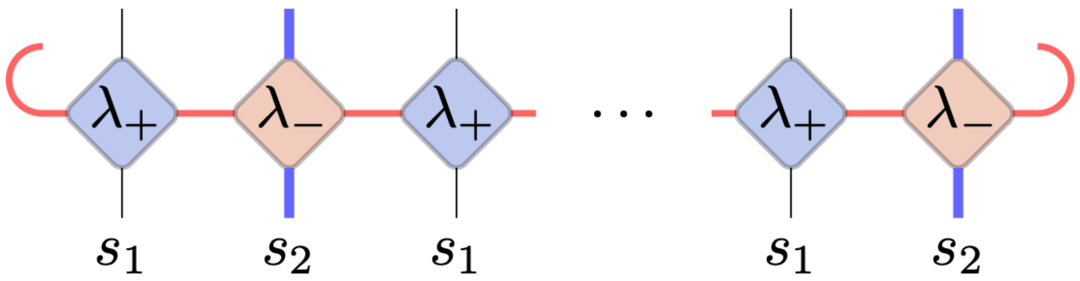

$$U_{1,2} R_{1,3}^{s_1,s_3}(\lambda_+) R_{2,3}^{s_2,s_3}(\lambda_-) =$$

$$= R_{1,3}^{s_2,s_3}(\lambda_-) R_{2,3}^{s_1,s_3}(\lambda_+) U_{1,2}$$

$$(\lambda_\pm = \lambda \pm \tau/2)$$

# 1. Structure of a ratchet circuit

- Transfer matrices:




$$T_{s_3}(\lambda) = \mathrm{Tr}_a\left(\prod_{1\le j\le L/2}^{\rightarrow} R^{s_1,s_3}_{2j-1,a}(\lambda_+)R^{s_2,s_3}_{2j,a}(\lambda_-)\right)$$

- Propagator and lattice shift are reproduced from such transfer matrices!

$$\mathbb{U} = T_{s_2}(\tfrac{\tau}{2})\left[T_{s_1}(-\tfrac{\tau}{2})\right]^{-1},$$
$$\mathbb{T} = T_{s_2}(\tfrac{\tau}{2})T_{s_1}(-\tfrac{\tau}{2}).$$

*Diagonalization via Bethe ansatz: quasi-momenta, quasi-energies, and quasi-particle content can be characterized.*

$$\mathbb{T}\,|\{\lambda_j\}\rangle = e^{-2i\sum_{j=1}^{N}p(\lambda_j)}\,|\{\lambda_j\}\rangle \qquad \mathbb{U}\,|\{\lambda_j\}\rangle = e^{i2\sum_{j=1}^{N}\varepsilon(\lambda_j)}\,|\{\lambda_j\}\rangle$$

$$p(\lambda) = \frac{1}{2}\left[p^{(2s_1)}(\lambda_+) + p^{(2s_2)}(\lambda_-)\right] \qquad \varepsilon(\lambda) = \frac{1}{2}\left[p^{(2s_1)}(\lambda_+) - p^{(2s_2)}(\lambda_-)\right]$$

$$p^{(2s)}(\lambda) \equiv i\log\left(\frac{\lambda+is}{\lambda-is}\right)$$

→ *Quasi-momentum in a homogeneous spin-s Heisenberg model.*

# 1. Structure of a ratchet circuit

- Limiting cases:

| | |
|---|---|
| $\tau \to \infty$ | *brickwork circuit of permutations* |
| $s_1 = s_2$ | *integrable Trotterization* |
| $s_1 \neq s_2$ | *alternating spins Hamiltonian?* |

$\mathbb{U}(0)^{-1}\partial_\tau \mathbb{U}(\tau = 0)$

[P. Richelli, K. Schoutens, A. Zorzato, arXiv:2402.18440]

[A. G. Bytsko, A. Doikou, J. Phys. A 37, 4465 (2004)]

- *T*-reversal symmetry vs. *PT* symmetry: $\mathcal{I}[\mathbb{U}_\mathrm{o}\mathbb{U}_\mathrm{e}] = \mathbb{U}_\mathrm{e}^{-1}\mathbb{U}_\mathrm{o}^{-1}, \qquad \mathcal{I} =?$

$$U = P^{s_1,s_2} R^{s_1,s_2}(\tau)$$

$$P^{s_1,s_2} = \sum_{\eta=-s_1}^{s_1} \sum_{m=-s_2}^{s_2} |m\eta\rangle\langle\eta m|$$

$$\mathcal{P}(U) = U^T$$

1. a spatial reflection $\mathcal{P}$, which transposes each unitary gate;
2. an anti-unitary conjugation which, together with $\mathcal{P}$, inverts each unitary gate;
3. a one-site lattice shift which exchanges the order of half-steps as follows: $\mathbb{U}_\mathrm{o}^{-1}\mathbb{U}_e^{-1} \mapsto \mathbb{U}_\mathrm{e}^{-1}\mathbb{U}_0^{-1}$.

$\mathcal{T}$

*The model is not time-reversal symmetric, but has a PT symmetry!*

# 2. Physical features

How is dynamics affected by the broken *P* and *T* symmetries?

1. Structure factor → its scaling tells about charge transport properties!

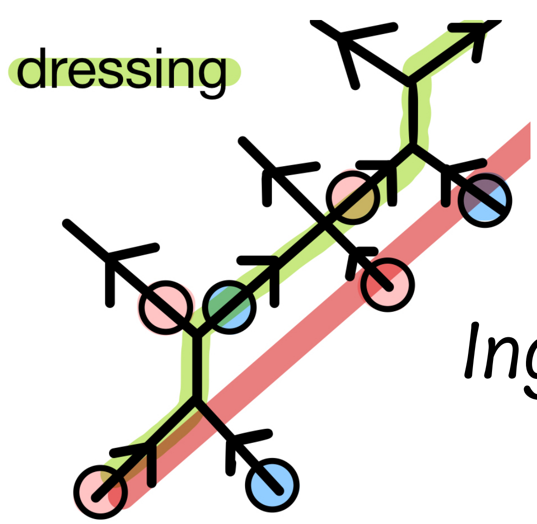
$$S(x,t) = \langle q(x,t) q(0,0) \rangle^c, \quad q = S^z$$

*U(1) charge = magnetization*

2. Distribution of the spin current values (from scaled cumulants)

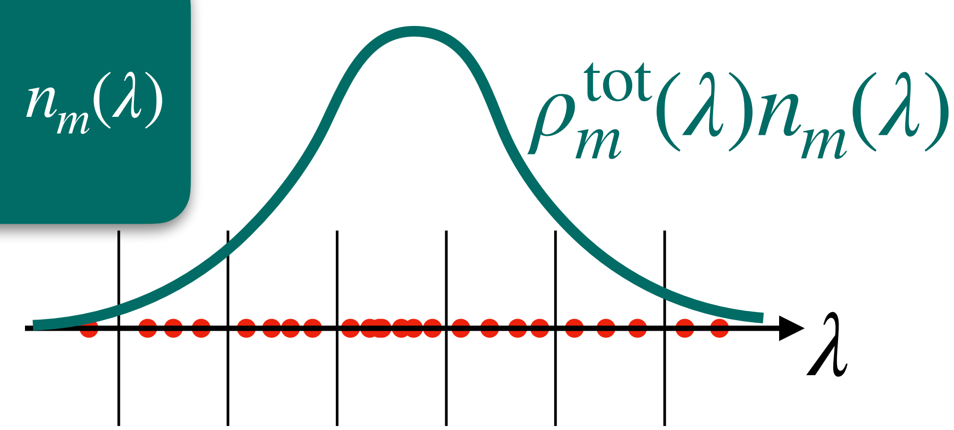
$$\mathcal{J}_t = \int_0^t \mathrm{d}\tau \, [j(x=0,\tau) - \langle j \rangle] \qquad c_k^{(\mathrm{sc})} \equiv \lim_{t\to\infty} \frac{\langle \mathcal{J}_t^k \rangle^c}{t}$$

Both can be investigated in the hydrodynamic picture and its generalizations!




$$v_m^{\mathrm{eff}}(\lambda) = \frac{[\varepsilon_m'(\lambda)]^{\mathrm{dr}}}{[p_m'(\lambda)]^{\mathrm{dr}}}, \; \rho_m^{\mathrm{tot}}(\lambda), \; n_m(\lambda)$$

*Ingredients from TBA and continuity equations.*




[O. Castro-Alvaredo, B. Doyon, T. Yoshimura, PRX 6, 041065 (2016)]
[B. Bertini, M. Collura, J. De Nardis, M. Fagotti, PRL 117, 207201(2016)]
*and others …*

# 2. Physical features

**Ingredients for hydro:** combining the infinite-$T$ TBA solutions of Heisenberg spin-$s$ model (see appendix in arXiv:2406.01571):

$b = 2s$

$$(\mathbf{1} + \mathbf{Kn})\boldsymbol{\rho}^{\mathrm{tot}(b)} = \boldsymbol{K}^{(b)} \qquad K_m^{(b)}(\lambda) \equiv \frac{1}{2\pi}\partial_\lambda p_m^{(b)}(\lambda)$$

*Floquet system ⇒ infinite-T! The occupancy function is independent of the spin and of the rapidity ⇒ TBA equations become linear.*

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{p}_+^{(b_1)} - \boldsymbol{p}_-^{(b_2)}\right)$$

$$\boldsymbol{p} = \frac{1}{2}\left(\boldsymbol{p}_+^{(b_1)} + \boldsymbol{p}_-^{(b_2)}\right)$$

$$(\boldsymbol{\varepsilon}')^{\mathrm{dr}} = \pi\left[\boldsymbol{\rho}_+^{\mathrm{tot}(b_1)} - \boldsymbol{\rho}_-^{\mathrm{tot}(b_2)}\right]$$

$$(\boldsymbol{p}')^{\mathrm{dr}} \equiv 2\pi\boldsymbol{\rho}^{\mathrm{tot}} = \pi\left[\boldsymbol{\rho}_+^{\mathrm{tot}(b_1)} + \boldsymbol{\rho}_-^{\mathrm{tot}(b_2)}\right]$$

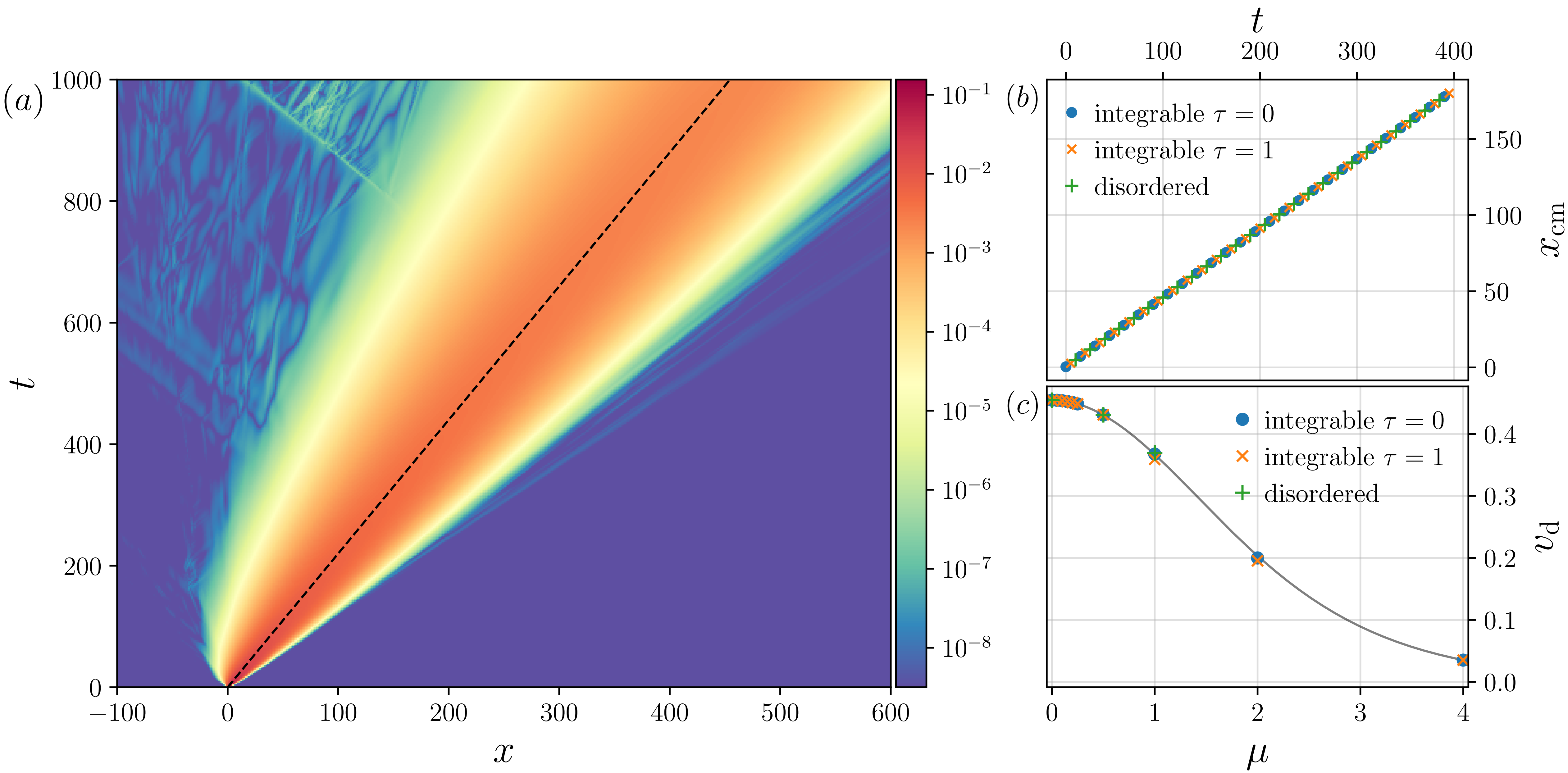
*The ratio is the quasi particle velocity!*

$$\hat{\rho}_m^{\mathrm{tot}(b)}(k;z) = \frac{\mathcal{X}_m}{\mathcal{X}_b\mathcal{X}_{m-1}\mathcal{X}_{m+1}}\hat{\Xi}_{\min(m,b)}^{(\max(m,b))}(k;z)$$

$$\hat{\Xi}_m^{(b)}(k;z) = e^{-(m+b+1)\frac{|k|}{2}}\sum_{j=0}^{m}\Big[\mathcal{X}_{b+1}(z)\mathcal{X}_j(z)\mathcal{X}_{m-j-1}(z) - \mathcal{X}_{b-1}(z)\mathcal{X}_{j-1}(z)\mathcal{X}_{m-j}(z)\Big]e^{(m-j)|k|}$$

$$\mathcal{X}_m(\mu) = \frac{\sinh\left([m+1]\frac{\mu}{2}\right)}{\sinh\left(\frac{\mu}{2}\right)} \qquad z \equiv e^{\mu/2}$$

# 2. Physical features: drift velocity




$$S(x,t) \simeq \sum_m \int \mathrm{d}\lambda\, \delta(x - v_m^{\mathrm{eff}}(\lambda)t)\Big(\rho_m^{\mathrm{tot}}(\lambda) n_m(\lambda)[1 - n_m(\lambda)](q_m^{\mathrm{dr}})^2\Big)$$

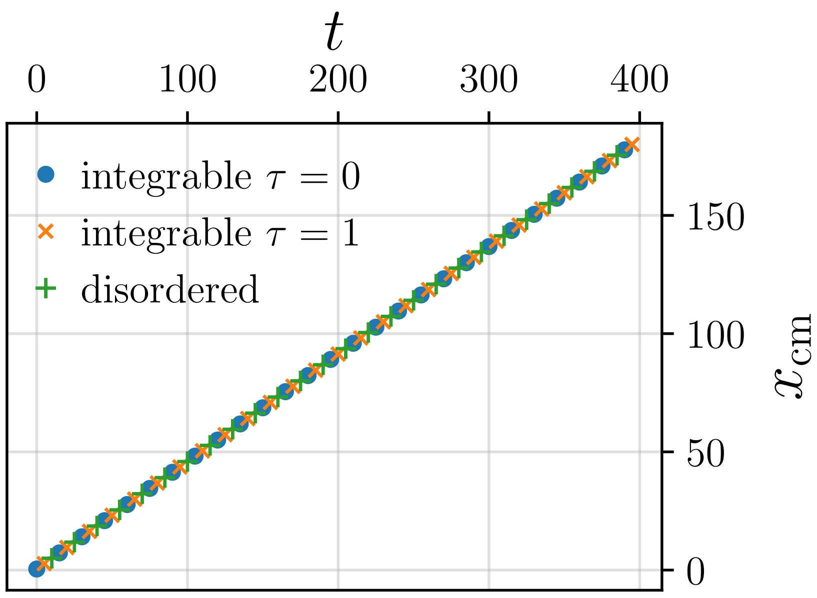
*Euler-scale dynamical structure factor of a U(1) charge.*

# 2. Physical features: drift velocity

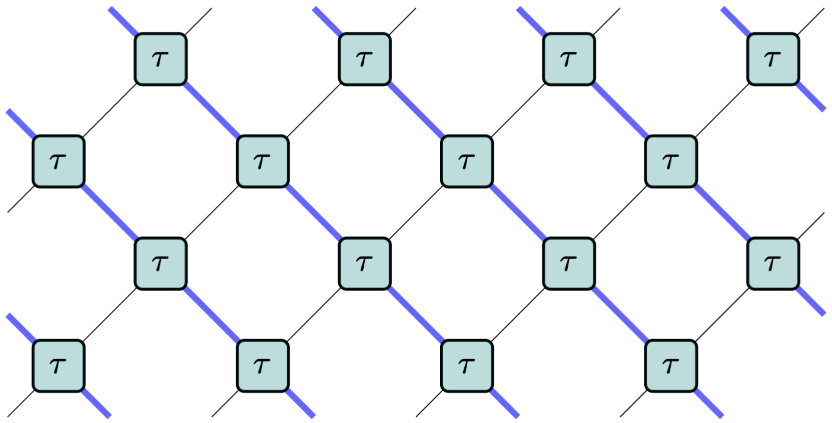
- The first moment of the structure factor — the "drift velocity"

*Exact calculation at infinite-T and half-filling.*

$$v_{\mathrm{d}} = \frac{\int \mathrm{d}x \left(\frac{x}{t}\right) S(x,t)}{\int \mathrm{d}x S(x,t)} = \frac{s_1(s_1+1) - s_2(s_2+1)}{s_1(s_1+1) + s_2(s_2+1)}$$




For $\tau \to \infty$ we get permutations: linear response from a slightly biased initial state reproduces the formula!




$$e^{(\delta\mu)S^z} \otimes e^{-(\delta\mu)S^z}$$

*Breaking integrability but retaining the symmetry and permutations does not affect the drift velocity!*

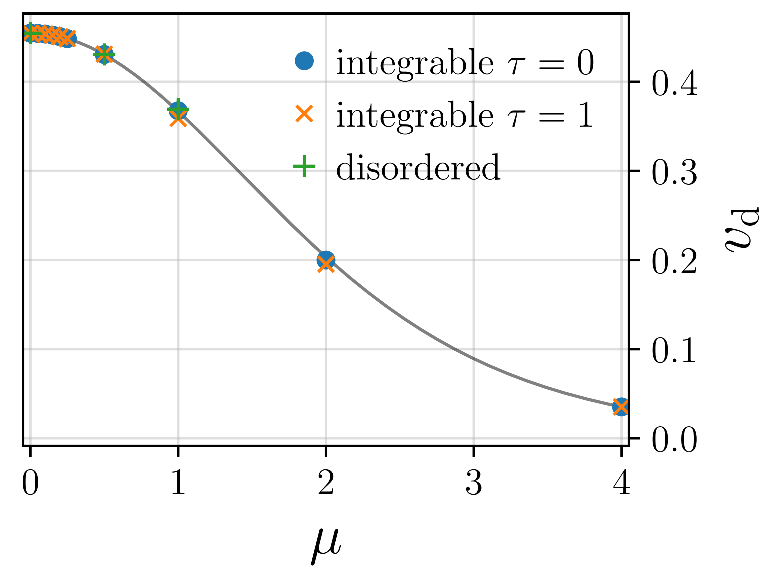
- *Case 1:* $\mathbb{U} \to \mathbb{U}_{\mathrm{o}}(\tau_1)\mathbb{U}_{\mathrm{e}}(\tau_2)$
- *Case 2: all $\tau$ random i.i.d.*

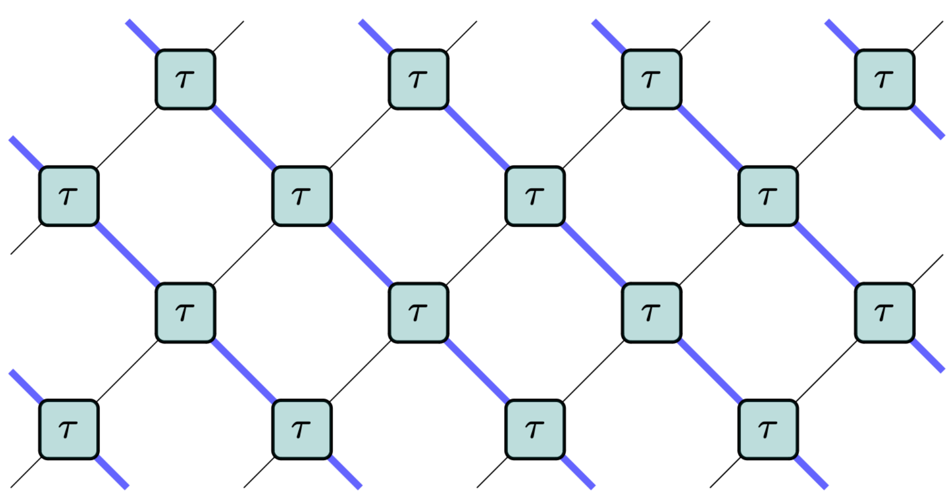
# 2. Physical features: drift velocity

- Linear response also yields the formula away from half-filling:

$$v_{\mathrm{d}}(\mu) = \frac{\partial_\mu \langle S_1^z \rangle - \partial_\mu \langle S_2^z \rangle}{\partial_\mu \langle S_1^z \rangle + \partial_\mu \langle S_2^z \rangle}$$

$$\partial_\mu \langle S^z \rangle = \{(s+\tfrac{1}{2})\operatorname{csch}(\mu[s+\tfrac{1}{2}])\}^2 - [\tfrac{1}{2}\operatorname{csch}(\tfrac{\mu}{2})]^2$$




$$e^{(\mu+\delta\mu)S^z} \otimes e^{(\mu-\delta\mu)S^z}$$

*Again confirmed by hydrodynamics (this time numerically)!*

$$v_{\mathrm{d}} = \frac{1}{\chi} \sum_{m=1}^{\infty} \int \mathrm{d}\lambda \chi_m(\lambda) v_m^{\mathrm{eff}}(\lambda) (q_m^{\mathrm{dr}})^2$$

$$\chi = \sum_{m=1}^{\infty} \int \mathrm{d}\lambda \, \chi_m(\lambda) (q_m^{\mathrm{dr}})^2$$

# 2. Some physical features of the model

- We are interested in large fluctuations of the accumulated current in an

$$\mathcal{J}_t = \int_0^t \mathrm{d}\tau \, [j(x\!=\!0,\tau) - \langle j \rangle]$$

infinite-$T$ state: $\mathbb{P}(\mathcal{J} = \mathrm{j}t|t) \asymp e^{-tI(\mathrm{j})}$ ⟷ $c_k^{(\mathrm{sc})} \equiv \lim_{t\to\infty} \frac{\langle \mathcal{J}_t^k \rangle^c}{t}$

Example:

$$c_3^{(\mathrm{sc})} = c_{3;1}^{(\mathrm{sc})} + c_{3;2}^{(\mathrm{sc})}$$

$$c_{3;1}^{(\mathrm{sc})} = \sum_{m=0}^{\infty} \int \frac{\mathrm{d}\lambda}{2\pi} w_m^{(3)} (\varepsilon_m')^{\mathrm{dr}}(\lambda)(q_m^{\mathrm{dr}})^3,$$

$$c_{3;2}^{(\mathrm{sc})} = 3 \sum_{m=0}^{\infty} \int \frac{\mathrm{d}\lambda}{2\pi} w_m^{(2)} \sigma_m(\lambda)\gamma_m(\lambda)(\varepsilon_m')^{\mathrm{dr}}(\lambda) q_m^{\mathrm{dr}}.$$

$$w_m^{(k)} = \sum_{r=1}^{\infty} (-1)^{r-1} r^{k-1} \left( \frac{n_m}{1-n_m} \right)^r$$

$$\gamma_m(\lambda) \equiv - \left[ (1-n_m)\sigma_m (q_m^{\mathrm{dr}})^2 \right]^{\mathrm{scr}} (\lambda)$$

[D.-L. Vu, arXiv:2008.06901]
[J. Myers et al., SciPost Phys. 8, 007 (2020)]
[B. Doyon et al., SciPost Phys. 15, 136 (2023)]
[B. Bertini et al., PRL 131, 140401 (2024)]

*Nonzero 3rd scaled cumulant!*

# 2. Some physical features of the model

- The 3rd scaled cumulant is nonzero (broken Gallavotti-Cohen symmetry):

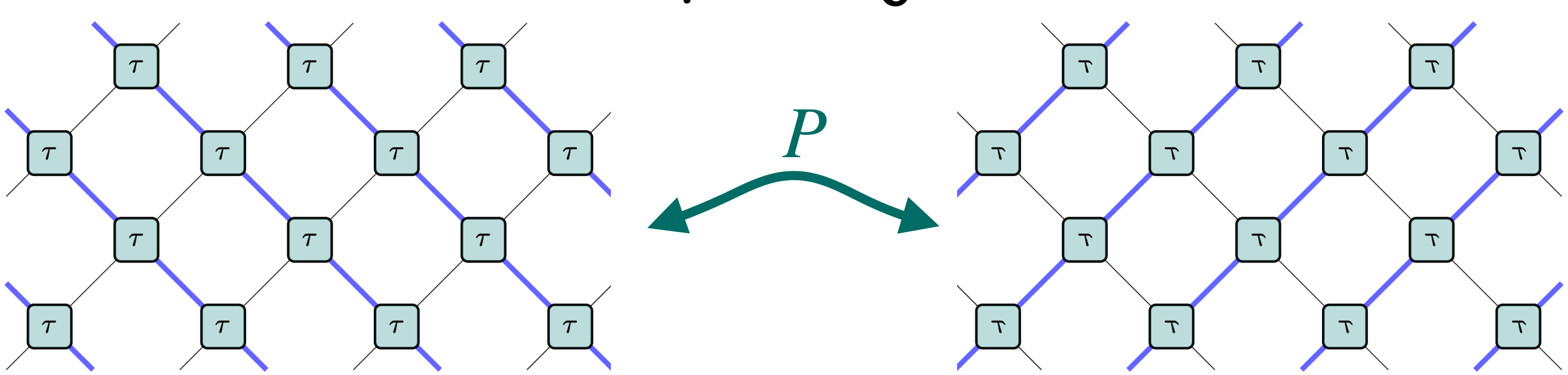
$$\frac{\mathbb{P}(J = jt\,|\,t)}{\mathbb{P}(J = -jt\,|\,t)} \asymp e^{-t[I(j)-I(-j)]} \neq 1$$

*No detailed balance in a grand-canonical ensemble of a ratchet!*

- A generalized version of the fluctuation symmetry instead holds!

$$\frac{\mathbb{P}_{s_1,s_2}(J = jt\,|\,t)}{\mathbb{P}_{s_2,s_1}(J = -jt\,|\,t)} \asymp e^{-t[I_{s_1,s_2}(j)-I_{s_2,s_1}(-j)]} = 1$$

*Relates fluctuations in two spatially reflected ratchets!*

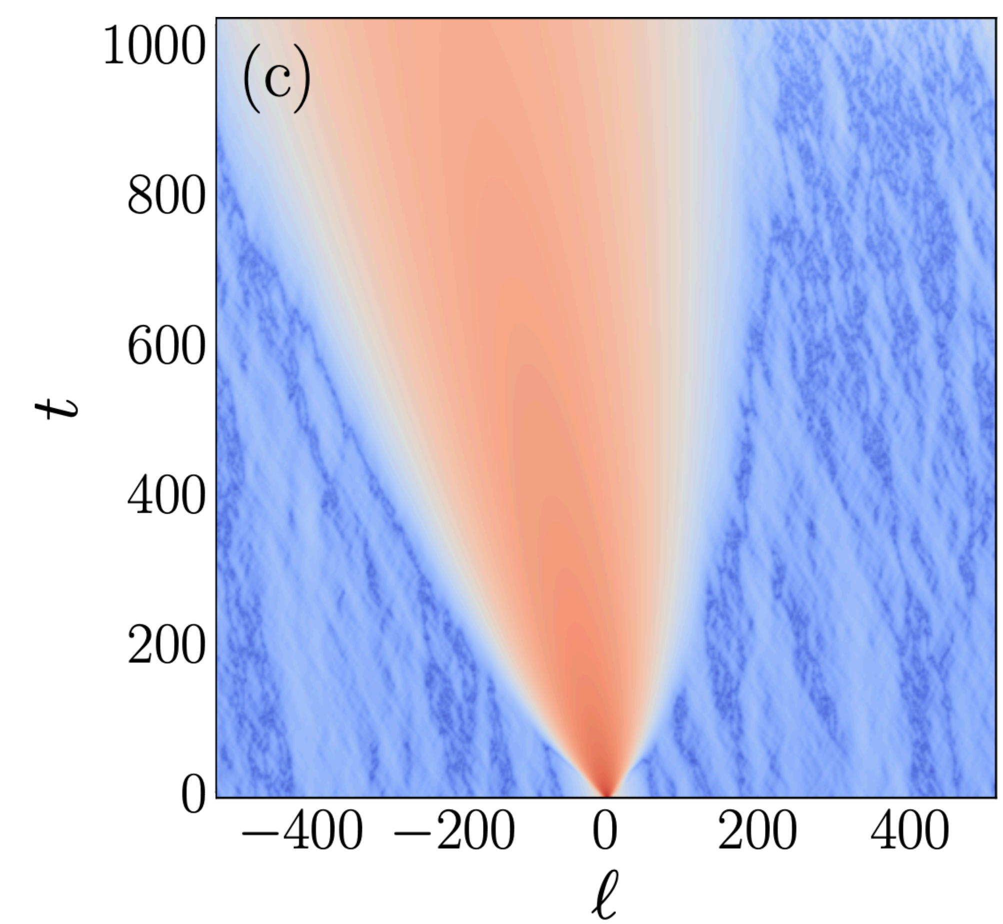


# Recap

1. An integrable alternating-spin cellular automaton that generalizes the Trotterization of the XXX model.
2. Simulable on qubits and/or qudits!
3. Chiral dynamics due to broken $P$: drift dependends only on local Casimirs and is stable under integrability breaking!
4. Broken $T$: no detailed balance in maximum-entropy stationary states!

# Outlook

1. Lattice discretizations of CFT/QFT …
   [D. Bernard, B. Doyon, J. Phys. A: Math. Theor. 45 362001 (2012)]
   [O. Castro-Alvaredo et al., JHEP 2020, 45 (2020)]
   [A. Roy, S. L. Lukyanov, Nat. Commun. 14, 7433 (2023)]
   [A. Roy et al., NPB 968, 115445 (2021)]
2. Classical models, same phenomenology
   [Ž. Krajnik et al., SciPost Phys. 11, 051 (2021)]
3. What is the best def. of equilibrium?




**THANK YOU!**