On the nested algebraic Bethe ansatz for closed spin chains with simple $\mathfrak{g}$ symmetry

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On the problem of the ABA for simple $\mathfrak{g}$

Give me a rational, $\mathfrak{g}$-symmetric, "regular" R-matrix on $V \otimes V$. Then I can write down:

- a $\mathfrak{g}$-symmetric integrable spin chain Hamiltonian on $M = V \otimes \cdots \otimes V$ with nearest neighbour interactions,

$$H \propto \sum_{s \in \mathbb{Z}_L} \left(R_{s,s+1}(0)\right)^{-1} R'_{s,s+1}(0);$$

- the Bethe equations of the resulting integrable system,

$$\frac{P_i^M(v_k^{(i)} + \hbar d_i)}{P_i^M(v_k^{(i)})} = -\prod_{j=1}^{r} \prod_{l=1}^{m^{(j)}} \frac{v_k^{(i)} - v_l^{(j)} + \frac{\hbar}{2}(\alpha_i, \alpha_j)}{v_k^{(i)} - v_l^{(j)} - \frac{\hbar}{2}(\alpha_i, \alpha_j)};$$

[Ogievetsky and Wiegmann 1986]

- and the generators and relations of the associated quantum group, the Yangian, in Drinfel'd's current presentation.

On the problem of the ABA for simple $\mathfrak{g}$

What about the Bethe eigenvector?

- Well-understood for $\mathfrak{sl}_n$ cases; less and less is known as we deviate from these.
- No method (as far as I am aware) for the exceptional cases.

Today: attempt to develop the nested algebraic Bethe ansatz for simple $\mathfrak{g}$.

Steps of the NABA

Start: monodromy matrix $T(u) \in \mathrm{End}(V \otimes M)$ satisfying RTT.

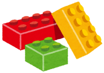
Decompose the auxiliary space V, cutting the monodromy matrix

$$T(u) = \left(\begin{array}{c|c} A(u) & B(u) \\ \hline C(u) & D(u) \end{array}\right).$$

$\downarrow$

Obtain the block relations between B operators and the A operators

$$A(u)B(v) = \ldots$$



$\downarrow$

Using B operators as creation operators, build the Bethe vector

$$\Phi(\{v\}) = B(v_1)\cdots B(v_m)\Phi'.$$

$\downarrow$

Act with A blocks and obtain the nested monodromy matrix

Non-universality of the NABA

$$T(u) = \left(\begin{array}{c|c} & \\ \hline & \end{array}\right) \qquad \text{type } \mathfrak{a}$$

Kulish and Reshetikhin (1981); de Vega and Gonzalez-Ruiz (1993); Galleas and Martins (2005); Belliard and Ragoucy (2008); Belliard and Ragoucy (2009)

$$T(u) = \left(\begin{array}{c|c} & \\ \hline & \end{array}\right) \qquad \text{type } \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$$

Reshetikhin (1985); Reshetikhin (1988); de Vega and Karowski (1987); Gombor and Palla (2016); G., MacKay and Regelskis (2019); G. and Regelskis (2020)

$$T(u) = \left(\begin{array}{c|c|c} & & \\ \hline & & \\ \hline & & \end{array}\right) \qquad \text{type } \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$$

Martins and Ramos (1997); Galleas and Martins (2004); Li, Shi and Yue (2004); Li, Shi and Yue (2005); Babujian, Foerster and Karowski (2012); Gombor (2018)

Barriers to universality

The key to the algebraic Bethe ansatz is that all relations stem from the RTT relation.

$$R(u) = \begin{pmatrix} R^{00}_{00}(u) & R^{00}_{01}(u) & \dots \\ R^{01}_{00}(u) & R^{01}_{01}(u) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

studied cases $\longrightarrow$ The R-matrix consists of familiar P and P^t matrices

general case $\longrightarrow$ we cannot hope for the R-matrix to be made from "familiar" matrices

decomposition possible by hand but rather tedious!

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decomposition possible by hand but rather tedious!

$\rightarrow$ alternative method which does not use the blocks directly!

Nesting

Decomposition of V

Nesting

$$\begin{tikzcd} \boxed{\text{Subalgebra } \mathfrak{k} \leq \mathfrak{g}} \arrow[r] & \boxed{\text{Decomposition of } V} \end{tikzcd}$$

Nesting

$$\begin{tikzcd} \boxed{\textbf{Removing one simple root}} \arrow[r] & \boxed{\text{Subalgebra } \mathfrak{k} \leq \mathfrak{g}} \arrow[r] & \boxed{\text{Decomposition of } V} \end{tikzcd}$$

The removed simple root

Removing one simple root → Subalgebra $\mathfrak{k} \leq \mathfrak{g}$ → Decomposition of V

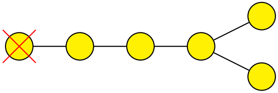
- Simple roots correspond to sets of Bethe roots $\left\{v^{(p)}\right\} \leftrightarrow \alpha_p$. If we remove α_p, schematically we will have Bethe vector

$$\Phi\left(\{v\}\right) = B(v_1^{(p)}) \cdots B(v_{m_p}^{(p)}) \Phi'\left(\{v\}\right).$$

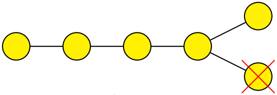
- It seems that any one simple root may be removed (cf. [Kosmakov and Tarasov (2024)]), but for simplicity we consider only the case where $\mathfrak{k}$ is simple (so consider only roots on the ends of the Dynkin diagram).

Removing a simple root

Example: consider the case $\mathfrak{g} \cong \mathfrak{so}_{12}$ with auxiliary space $V \cong \mathbf{12}$, the vector representation.



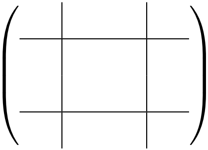
$\mathfrak{so}_{10} < \mathfrak{so}_{12}$

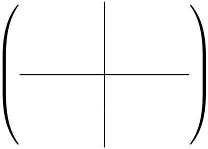


$\mathfrak{sl}_6 < \mathfrak{so}_{12}$

$$\mathbf{12}_{\mathfrak{so}_{12}} \cong \mathbf{1}_{\mathfrak{so}_{10}} \oplus \mathbf{10}_{\mathfrak{so}_{10}} \oplus \mathbf{1}_{\mathfrak{so}_{10}}$$

$$\mathbf{12}_{\mathfrak{so}_{12}} \cong \mathbf{6}_{\mathfrak{sl}_6} \oplus \overline{\mathbf{6}}_{\mathfrak{sl}_6}$$

$T(u) =$ 

$T(u) =$ 

The $U(1)$ charge

Removing a simple root means specifically

$$\mathfrak{k} \cong \langle h_i | i \neq p \rangle + \left\langle e_{\pm\alpha} | (\alpha, \varpi_p^\vee) = 0 \right\rangle .$$

The following basis vector of the Cartan subalgebra of $\mathfrak{g}$ commutes with $\mathfrak{k}$:

$$\left[h_{\varpi_p^\vee}, \mathfrak{k} \right] = 0.$$

This will be referred to as the **charge**, and it institutes a $\mathbb{Z}$-grading on $\mathfrak{g}$ and its representations.
For example, the adjoint representation

$$\mathfrak{g} \cong \mathfrak{g}^{(-n_p)} \oplus \cdots \oplus \left(\mathfrak{k} \oplus \mathbb{C} h_{\varpi_p^\vee} \right) \oplus \cdots \oplus \mathfrak{g}^{(n_p)}$$

Charge decomposition

We have decomposition of the auxiliary space into charge eigenspaces, which are also $\mathfrak{k}$-reps:

$$V \cong V^0 \oplus V^1 \oplus \cdots \oplus V^N.$$

The tensor product representation may be similarly partitioned according to total charge

$$V \otimes V \cong \underbrace{\boxed{V^0 \otimes V^0}}_{\text{charge } 0} \oplus \underbrace{\boxed{\begin{matrix} V^1 \otimes V^0 \\ \oplus V^0 \otimes V^1 \end{matrix}}}_{\text{charge } 1} \oplus \underbrace{\boxed{\begin{matrix} V^2 \otimes V^0 \\ \oplus V^1 \otimes V^1 \\ \oplus V^0 \otimes V^2 \end{matrix}}}_{\text{charge } 2} \oplus \cdots$$

The R matrix preserves the *total charge*, giving it a block diagonal form.

The R-matrix and the block relations

The R matrix preserves the *total charge*, giving it a block diagonal form.

$$R(u) = \left(\begin{array}{c|cc|c} R^{00}_{00}(u) & & & \\ \hline & R^{01}_{01}(u) & R^{01}_{10}(u) & \\ & R^{10}_{01}(u) & R^{10}_{10}(u) & \\ \hline & & & \ddots \end{array}\right)$$

So now we can write down the block relations, right?

$$R_{ab}(u-v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{ab}(u-v)$$

$$T(u) = \begin{pmatrix} A^0_0(u) & B^0_1(u) & \cdots & B^0_N(u) \\ C^1_0(u) & A^1_1(u) & \cdots & B^1_N(u) \\ \vdots & \vdots & \ddots & \vdots \\ C^N_0(u) & C^N_1(u) & \cdots & A^N_N(u) \end{pmatrix}$$

The block relations

Sure, we can write down the block relations.

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'ableForm=
-(a[0, 0]_2 · a[0, 0]_1 · r0000[u]_1,2) + r0000[u]_1,2 · a[0, 0]_1 · a[0, 0]_2
-(a[0, 0]_2 · b[0, 1]_1 · r1001[u]_1,2) - b[0, 1]_2 · a[0, 0]_1 · r0101[u]_1,2 + r0000[u]_1,2 · a[0, 0]_1 · b[0, 1
-(a[0, 0]_2 · b[0, 1]_1 · r1010[u]_1,2) - b[0, 1]_2 · a[0, 0]_1 · r0110[u]_1,2 + r0000[u]_1,2 · b[0, 1]_1 · a[0, 0
-(a[0, 0]_2 · b[0, 2]_1 · r2002[u]_1,2) - b[0, 1]_2 · b[0, 1]_1 · r1102[u]_1,2 - b[0, 2]_2 · a[0, 0]_1 · r0202[u]
-(a[0, 0]_2 · b[0, 2]_1 · r2011[u]_1,2) - b[0, 1]_2 · b[0, 1]_1 · r1111[u]_1,2 - b[0, 2]_2 · a[0, 0]_1 · r0211[u]
-(a[0, 0]_2 · b[0, 2]_1 · r2020[u]_1,2) - b[0, 1]_2 · b[0, 1]_1 · r1120[u]_1,2 - b[0, 2]_2 · a[0, 0]_1 · u0220[u]
-(b[0, 1]_2 · b[0, 2]_1 · r2112[u]_1,2) - b[0, 2]_2 · b[0, 1]_1 · r1212[u]_1,2 + r0000[u]_1,2 · b[0, 1]_1 · b[0, 2
-(b[0, 1]_2 · b[0, 2]_1 · r2121[u]_1,2) - b[0, 2]_2 · b[0, 1]_1 · u1221[u]_1,2 + r0000[u]_1,2 · b[0, 2]_1 · b[0, 1
-(b[0, 2]_2 · b[0, 2]_1 · r2222[u]_1,2) + r0000[u]_1,2 · b[0, 2]_1 · b[0, 2]_2
-(c[1, 0]_2 · a[0, 0]_1 · r0000[u]_1,2) + r0101[u]_1,2 · a[0, 0]_1 · c[1, 0]_2 + r0110[u]_1,2 · c[1, 0]_1 · a[0, 0
-(a[0, 0]_2 · c[1, 0]_1 · r0000[u]_1,2) + r1001[u]_1,2 · a[0, 0]_1 · c[1, 0]_2 + r1010[u]_1,2 · c[1, 0]_1 · a[0, 0
-(a[1, 1]_2 · a[0, 0]_1 · r0101[u]_1,2) - c[1, 0]_2 · b[0, 1]_1 · r1001[u]_1,2 + r0101[u]_1,2 · a[0, 0]_1 · a[1, 1
```

But they're not really useful in this form, because we don't know what $R^{ij}_{kl}(u)$ are in general.

The block relations

Hint: if we consider $R_{00}^{00}(u)$, it satisfies the Yang-Baxter equation (see [Chari-Pressley 1991]). This is easy to deduce from:

$$(V \otimes V \otimes V)^0 = V^0 \otimes V^0 \otimes V^0,$$

with the YBE for $R(u)$. So, we can use the uniqueness of R-matrices to work out $R_{00}^{00}(u)$.

We can use a similar idea to get some more info about $R(u)$.

The block Gauss decomposition

Consider the block Gauss decomposition of the R-matrix:

$$R(u) = U(u)D(u)L(u),$$

where

$$L(u) = \left(\begin{array}{c|cc|c} I & & & \\ \hline & I & & \\ & L^{10}_{01}(u) & I & \\ \hline & & & \ddots \end{array}\right) \qquad U(u) = \left(\begin{array}{c|cc|c} I & & & \\ \hline & I & U^{01}_{10}(u) & \\ & & I & \\ \hline & & & \ddots \end{array}\right)$$

$$D(u) = \left(\begin{array}{c|cc|c} D^{00}_{00}(u) & & & \\ \hline & D^{01}_{01}(u) & & \\ & & D^{10}_{10}(u) & \\ \hline & & & \ddots \end{array}\right).$$

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$$D(u) = \left(\begin{array}{c|cc|c} D_{00}^{00}(u) & & & \\ \hline & D_{01}^{01}(u) & & \\ & & D_{10}^{10}(u) & \\ \hline & & & \ddots \end{array}\right).$$

Proposition

The $D_{IJ}^{IJ}(u)$ satisfy the Yang-Baxter equations:

$$D_{IJ}^{IJ}(u-v)D_{IK}^{IK}(u-w)D_{JK}^{JK}(v-w) = D_{JK}^{JK}(v-w)D_{IK}^{IK}(u-w)D_{IJ}^{IJ}(u-v).$$

Identifying the D-matrices

So the matrix $D^{IJ}_{IJ}(u) \in \mathrm{End}(V^I \otimes V^J)$ is an R-matrix, which is something we know a lot about.

If we assume that each V^I is an irreducible representation of the Yangian (quite restrictive, but still includes all as-yet studied cases, as well as many more), then we have, by uniqueness:

$$D^{IJ}_{IJ}(u) = d^{IJ}(u) R^{V^I V^J}(u - w_{IJ}).$$

Actually, from the Yang-Baxter equation we have $w_{IJ} = w_I - w_J$ for each I, J.

Bonus: we can determine the shift w_I associated to each space V^J using the Baxter polynomials of the representation V.

What does this give us?

We are ready to state the block relations:

$$T(u) = \begin{pmatrix} A_0^0(u) & B_1^0(u) & \cdots & B_N^0(u) \\ C_0^1(u) & A_1^1(u) & \cdots & B_N^1(u) \\ \vdots & \vdots & \ddots & \vdots \\ C_0^N(u) & C_1^N(u) & \cdots & A_N^N(u) \end{pmatrix}$$

$C_J^I(u)$ for each I, J are annihilation operators, so will consider terms $\mod C$, where C consists of terms ending in matrix elements of annihilation operators.

The RTT relation is (we omit the spectral parameter for clarity)

$$[UDL]T_aT_b = T_bT_a[UDL].$$

Now we can do some trickery:

$$\underbrace{DLT_aT_bL^{-1}}_{\text{lower triangular} \mod C} = \underbrace{U^{-1}T_bT_aUD}_{\text{upper triangular} \mod C}.$$

Result I: RAA relations

$$\underbrace{DLT_a T_b L^{-1}}_{\text{lower triangular} \mod C} = \underbrace{U^{-1} T_b T_a UD}_{\text{upper triangular} \mod C} .$$

Looking at the diagonal blocks of this relation, we obtain

$$D^{IJ}(u-v)\, A^I_I(u)_a\, A^J_J(v)_b = A^J_J(v)_b\, A^I_I(u)_a\, D^{IJ}(u-v) \mod C$$

This is the RTT relation for the diagonal blocks of the monodromy matrix!

- It confirms that the nested system will have a Yangian underlying algebra.
- It also confirms that each of the nested transfer matrices commute.

Result II: The AB relations

We can use the same technique to show:

$$
\begin{aligned}
(A_I^I)_a(B_{J+1}^J)_b = (D^{I,J})^{-1}(B_{J+1}^J)_b(A_I^I)_a D^{I,J+1} \\
- L_{I-1,J+1}^{I,J}(B_I^{I-1})_a(A_{J+1}^{J+1})_b \\
+ (B_{I+1}^I)_a(A_J^J)_b L_{I,J+1}^{I+1,J} \mod C
\end{aligned}
$$

Practitioners of the Bethe ansatz will recognise the “wanted” and “unwanted” terms.

Very important caveat: This technique works for the first excitation only. After the first one, we can't count on C terms vanishing. This means that, in general, the higher level terms need to be generated from a relation like $RRRTTT = TTTRRR$.

The nested Bethe vector

Recall the schematic (M is the spin chain state space):

$$\Phi\left(\{v\}\right) = B(v_1^{(p)}) \cdots B(v_{m_p}^{(p)})\Phi'\left(\{v\}\right) \in M.$$

What exactly are $\Phi'\left(\{v\}\right)$ and $B(v)$ here?

- We should have some *auxiliary site* V^{aux} for each excitation:

$$B(v) \in \mathrm{Hom}(V^{aux}, \mathbb{C}) \otimes \mathrm{End}(M).$$

- We also need, in order to use the previous results,

$$C \cdot \Phi'\left(\{v\}\right) = 0;$$

  this can be achieved with

$$\boxed{\Phi'\left(\{v\}\right) \in \left(V^{aux}\right)^{\otimes m_p} \otimes M^0}$$

The B operator

Let us focus on just one excitation.
We have multiple *different* creation operators for one excitation:

$$B_1^0(v) \text{ or } B_2^1(v) \text{ or } B_3^2(v), \ldots$$

It is expected that they all produce equivalent excitations (see e.g. [Melo and Martins (2009)]). But in our case, they all have a different shape

$$B_{l+1}^l(v) \in \mathrm{Hom}(V^{l+1}, V^l) \otimes \mathrm{End}(M)$$

In fact we will use

$$B_{l+1,\bar{l}}(v) \in \mathrm{Hom}(V^{l+1} \otimes \overline{V^l}, \mathbb{C}) \otimes \mathrm{End}(M)$$

How can we square this with $B(v) \in \mathrm{Hom}(V^{aux}, \mathbb{C}) \otimes \mathrm{End}(M)$?

The auxiliary site

Assertion: The auxiliary site V^{aux} must be isomorphic to the rep $\mathfrak{g}^{(1)}$. That is, the vector space spanned by generators $\left\langle e_{-\alpha} | (\alpha, \omega_p^\vee) = 1 \right\rangle$.

Evidence I: For each I, there is an intertwiner of $\mathfrak{k}$ representations

$$\Gamma^{I+1,\bar{I}} : \mathfrak{g}^{(1)} \hookrightarrow V^{I+1} \otimes \overline{V^I}.$$

Indeed, $\mathfrak{g}^{(1)} \otimes V^I \to V^{I+1}$, simply from restricting the action of $\mathfrak{g}$ on V.

This means that we can act with any B operator on the nested Bethe vector.

$$\Phi'(\{v\}) \to \Gamma^{I+1,\bar{I}} \cdot \Phi'(\{v\}) \to B_{I+1,\bar{I}}(v) \cdot \Gamma^{I+1,\bar{I}} \cdot \Phi'(\{v\})$$

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Assertion: The auxiliary site V^{aux} must be isomorphic to the rep $\mathfrak{g}^{(1)}$. That is, the vector space spanned by generators $\langle e_{-\alpha} | (\alpha, \omega_p^\vee) = 1 \rangle$.

Evidence II: There is a way we could have predicted this from the Bethe equations. Assuming the NABA is successful, we must have the following equivalence:

Bethe equations for system on M

- Bethe equations for $v^{(p)}$ — Top level BEs
- Bethe equations for $v^{(p-1)}$
- $\vdots$
- Bethe equations for $v^{(1)}$

Bethe equations for *nested* system on $(V^{aux})^{\otimes m_p} \otimes M^0$

This is consistent if the highest weight of V^{aux} is $\pi(-\alpha_p)$, where π is the projector from the weight lattice of $\mathfrak{g}$ to that of $\mathfrak{k}$.

The one-excitation state

Finally, this puts us in a position to define the one-excitation state. Choose $B_{1,\bar{0}}(v)$ as the creation operator

$$\Phi(\{v\}) = B_{1,\bar{0}}(v) \cdot \Gamma^{1,\bar{0}} \cdot \Phi'(\{v\}).$$

Now act with the transfer matrix $t(u) = \mathrm{tr}_a\, T_a(u) = \sum_I \mathrm{tr}_I\, A_I^I(u)$:

$$t(u)\Phi(\{v\}) = \sum_I \mathrm{tr}_I(A_I^I(u))B_{1,\bar{0}}(v) \cdot \Gamma^{1,\bar{0}} \cdot \Phi'(\{v\}).$$

$$= B_{1,\bar{0}}(v) \sum_I \underbrace{\mathrm{tr}_I \left[D^{I,1}(u-v)((D^{I,0}(u-v))^{-1})^{t_0}\, A_I^I(u) \right]}_{\text{nested transfer matrix on } V^{aux} \otimes M^0} \cdot \Gamma^{1,\bar{0}} \cdot \Phi'(\{v\}).$$

We have omitted unwanted terms.

Fusion

Expected nested transfer matrix expression:

$$\operatorname*{tr}_{l}\left[R^{l,V_{aux}}(u-v)A^{l}_{l}(u)\right] \text{ acting on } V^{aux}\otimes M^{0}$$

But what we got was:

$$\operatorname*{tr}_{l}\left[D^{l,1}(u-v)((D^{l,0}(u-v))^{-1})^{t_0}A^{l}_{l}(u)\right]$$

The missing piece of the puzzle must be fusion! This occurs if the above pair of D matrices are at the fusion point.

Conjecture: The matrix $[(D^{10}(0))^{-1}]^{t_0}$ is a projector to $V^{aux}\subset V^{1}\otimes\overline{V^{0}}$.

Note: this condition appears in Reshetikhin (1988), which considers the case $N=2$.

Conclusions

Here's a quick recap.

- The decomposition of the auxiliary space is induced from the removal of a single simple root.
- The subsequent block Gauss decomposition of the R-matrix reveals D-matrices which satisfy the Yang-Baxter equation; this is enough to prove the RTT relation for the nested system, as well as the wanted term.
- The auxiliary site appearing in the nested system has representation $\mathfrak{g}^{(1)}$, determined from the nesting. We conjecture that a particular D-matrix evaluated at 0 gives a projector to this representation.

This is just about enough to construct the Bethe vector for one excitation.

Discussion

There is a lot that has not been completed yet.

- We need to prove the conjecture.
- We need to show that the unwanted terms disappear if the Bethe equations are satisfied – need to understand the properties of $U(u)$ and $L(u)$ from the block Gauss decomposition.
- We need to generalise to multiple excitations – simple for $N = 2$, but difficult in general due to existence of $B_2^0(v)$ etc.

Thank you

Let's work together! Feel free to contact me on researchgate etc.

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email: `allan.j.gerrard⊗gmail.com`

Decomposition of some reps

$\mathfrak{g}$	α_p	$\mathfrak{k}$	V	V^0	V^1	V^2	V^3
$\mathfrak{a}_r$	α_1	$\mathfrak{a}_{r-1}$	$M^{\mathfrak{g}}(\omega_i)$	$M^{\mathfrak{k}}(\omega_i)$	$M^{\mathfrak{k}}(\omega_{i+1})_{\hbar/2}$		
$\mathfrak{b}_r$	α_1	$\mathfrak{b}_{r-1}$	$M^{\mathfrak{g}}(\omega_1)$	1	$M^{\mathfrak{k}}(\omega_1)_{\hbar}$	1	
			$M^{\mathfrak{g}}(\omega_r)$	$M^{\mathfrak{k}}(\omega_{r-1})$	$M^{\mathfrak{k}}(\omega_{r-1})_{2\hbar}$		
$\mathfrak{c}_r$	α_1	$\mathfrak{c}_{r-1}$	$M^{\mathfrak{g}}(\omega_1)$	1	$M^{\mathfrak{k}}(\omega_1)_{\hbar/2}$	1	
$\mathfrak{c}_r$	α_r	$\mathfrak{a}_{r-1}$	$M^{\mathfrak{g}}(\omega_1)$	$M^{\mathfrak{k}}(\omega_1)$	$M^{\mathfrak{k}}(\omega_{r-1})_{(r+2)\hbar/2}$		
$\mathfrak{d}_r$	α_1	$\mathfrak{d}_{r-1}$	$M^{\mathfrak{g}}(\omega_1)$	1	$M^{\mathfrak{k}}(\omega_1)_{\hbar/2}$	1	
			$M^{\mathfrak{g}}(\omega_{r-1})$	$M^{\mathfrak{k}}(\omega_{r-2})$	$M^{\mathfrak{k}}(\omega_{r-1})_{\hbar}$		
$\mathfrak{d}_r$	α_r	$\mathfrak{a}_{r-1}$	$M^{\mathfrak{g}}(\omega_1)$	$M^{\mathfrak{k}}(\omega_1)$	$M^{\mathfrak{k}}(\omega_{r-1})_{(r-2)\hbar/2}$		
$\mathfrak{e}_6$	α_6	$\mathfrak{d}_5$	$M^{\mathfrak{g}}(\omega_1)$	$M^{\mathfrak{k}}(\omega_1)$	$M^{\mathfrak{k}}(\omega_5)_{3\hbar/2}$	1	
			$M^{\mathfrak{g}}(\omega_6)$	1	$M^{\mathfrak{k}}(\omega_4)_{\hbar/2}$	$M^{\mathfrak{k}}(\omega_1)_{2\hbar}$	
$\mathfrak{e}_6$	α_2	$\mathfrak{a}_5$	$M^{\mathfrak{g}}(\omega_1)$	$M^{\mathfrak{k}}(\omega_1)$	$M^{\mathfrak{k}}(\omega_4)_{3\hbar/2}$	$M^{\mathfrak{k}}(\omega_1)_{3\hbar}$	
$\mathfrak{e}_7$	α_7	$\mathfrak{e}_6$	$M^{\mathfrak{g}}(\omega_7)$	1	$M^{\mathfrak{k}}(\omega_6)_{\hbar/2}$	$M^{\mathfrak{k}}(\omega_1)_{5\hbar/2}$	1
$\mathfrak{e}_7$	α_1	$\mathfrak{d}_6$	$M^{\mathfrak{g}}(\omega_7)$	$M^{\mathfrak{k}}(\omega_1)$	$M^{\mathfrak{k}}(\omega_5)_{2\hbar}$	$M^{\mathfrak{k}}(\omega_1)_{4\hbar}$	
$\mathfrak{e}_7$	α_2	$\mathfrak{a}_6$	$M^{\mathfrak{g}}(\omega_7)$	$M^{\mathfrak{k}}(\omega_6)$	$M^{\mathfrak{k}}(\omega_2)_{2\hbar}$	$M^{\mathfrak{k}}(\omega_5)_{7\hbar/2}$	$M^{\mathfrak{k}}(\omega_1)_{11\hbar/2}$
$\mathfrak{f}_4$	α_1	$\mathfrak{c}_3$	$M^{\mathfrak{g}}(\omega_4)$	$M^{\mathfrak{k}}(\omega_1)$	$M^{\mathfrak{k}}(\omega_2)_{5\hbar/2}$	$M^{\mathfrak{k}}(\omega_1)_{5\hbar}$	
$\mathfrak{g}_2$	α_2	$\mathfrak{a}_1$	$M^{\mathfrak{g}}(\omega_1)$	$M^{\mathfrak{k}}(\omega)$	$M^{\mathfrak{k}}(2\omega)_{2\hbar}$	$M^{\mathfrak{k}}(\omega)_{5\hbar}$	

The nested Bethe equations

Explicitly, the Bethe equations are, for $1 \leq i \leq r$,

$$\frac{P_i(v_k^{(i)} + \hbar d_i)}{P_i(v_k^{(i)})} = -\prod_{j=1}^{r} \prod_{l=1}^{m^{(j)}} \frac{v_k^{(i)} - v_l^{(j)} + \frac{\hbar}{2}(\alpha_i, \alpha_j)}{v_k^{(i)} - v_l^{(j)} - \frac{\hbar}{2}(\alpha_i, \alpha_j)}.$$

Suppose α_p is removed. The Bethe equations, for $i \neq p$ are:

$$\frac{P_i(v_k^{(i)} + \hbar d_i)}{P_i(v_k^{(i)})} = -\prod_{l=1}^{m^{(p)}} \frac{v_k^{(i)} - v_l^{(p)} + \frac{\hbar}{2}(\alpha_i, \alpha_p)}{v_k^{(i)} - v_l^{(p)} - \frac{\hbar}{2}(\alpha_i, \alpha_p)} \prod_{\substack{j=1 \\ j \neq p}}^{r} \prod_{l=1}^{m^{(j)}} \frac{v_k^{(i)} - v_l^{(j)} + \frac{\hbar}{2}(\alpha_i, \alpha_j)}{v_k^{(i)} - v_l^{(j)} - \frac{\hbar}{2}(\alpha_i, \alpha_j)}.$$

These must be equivalent to the full Bethe equations for the nested system. That is:

$$\frac{P_i(v_k^{(i)} + \hbar d_i)}{P_i(v_k^{(i)})} \underbrace{\prod_{l=1}^{m^{(p)}} \frac{v_k^{(i)} - v_l^{(p)} - \frac{\hbar}{2}(\alpha_i, \alpha_p)}{v_k^{(i)} - v_l^{(p)} + \frac{\hbar}{2}(\alpha_i, \alpha_p)}}_{\text{Drinfel'd polys for aux. sites!}} = -\prod_{\substack{j=1 \\ j \neq p}}^{r} \prod_{l=1}^{m^{(j)}} \frac{v_k^{(i)} - v_l^{(j)} + \frac{\hbar}{2}(\alpha_i, \alpha_j)}{v_k^{(i)} - v_l^{(j)} - \frac{\hbar}{2}(\alpha_i, \alpha_j)}.$$