Generalizations of spin Sutherland models from Hamiltonian reductions of Heisenberg doubles

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To begin, recall that the classical Sutherland Hamiltonian,

$$H_{\text{trig-Suth}}(q,p) \equiv \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{8}\sum_{j\neq k} \frac{x^2}{\sin^2((q_j-q_k)/2)},$$

admits two kinds of spin extensions. The first one contains Lie algebraic ('collective') spin variables,

$$H_{\text{spin-Suth}}(q,p,\xi) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{8}\sum_{j\neq k} \frac{|\xi_{jk}|^2}{\sin^2((q_j-q_k)/2)},$$

where $\xi \in \mathfrak{u}(n)^*$, with zero diagonal part. These models exist for all simple Lie algebras,

$$H_{\text{spin-Suth}}(q,p,\xi) = \frac{1}{2}\langle p,p\rangle + \frac{1}{8}\sum_{\alpha\in\Delta} \frac{2}{|\alpha|^2}\frac{|\xi_\alpha|^2}{\sin^2(\alpha(q)/2)},$$

and arise from Hamiltonian reduction of the cotangent bundle T^*G of the corresponding compact Lie group. (Here, we use the Killing form and the set of roots $\Delta = \{\alpha\}$ of the complexified Lie algebra $\mathcal{G}^{\mathbb{C}}$.) The spin variables matter up to gauge transformation by the maximal torus $G_0 < G$.

The second kind of generalization is the Gibbons–Hermsen (1984) model

$$H_{\text{G-H}} = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \frac{1}{8}\sum_{j\neq k} \frac{|(S_j S_k^\dagger)|^2}{\sin^2((q_j - q_k)/2)}.$$

The complex row-vector $S_j := [S_{j1}, \ldots, S_{jd}] \in \mathbb{C}^d$, $d \geq 2$, is attached to the particle with coordinate q_j, representing internal degrees of freedom. The overall phases of the spin vectors S_j can be changed by gauge transformations. This model descends from the extended cotangent bundle $T^*U(n) \times \mathbb{C}^{n\times d}$.

The purpose of the talk is to explain that generalizations of these models arise if one replaces the cotangent bundles by the so-called Heisenberg doubles, which are their Poisson–Lie analogues. We shall mainly focus on the first kind of models.

The talk is based on the following papers:

- LF, *Poisson–Lie analogues of spin Sutherland models*, Nucl. Phys. B 949, 114807 (2019)
- LF, *Poisson–Lie analogues of spin Sutherland models revisited*, J. Phys A: Math. Theor. 57, 205202 (2024)
- Fairon, L.F. and Marshall, *Trigonometric real form of the spin RS model of Krichever and Zabrodin*, Ann. Henri Poincaré 22, 615-675 (2021)

Let G be a (connected and simply connected) compact Lie group with simple Lie algebra $\mathcal{G}$. Denote $\mathcal{G}^{\mathbb{C}}$ and $G^{\mathbb{C}}$ the complexifications, and define $\mathfrak{P} := \exp(\mathrm{i}\mathcal{G}) \subset G^{\mathbb{C}}$. Example: $G = SU(n)$, $G^{\mathbb{C}} = SL(n,\mathbb{C})$, $\mathfrak{P} = \{X \in SL(n,\mathbb{C}) \mid X^\dagger = X,\ X \text{ positive}\}$.

One has the following 3 'classical doubles' of G:

$$\text{Cotangent bundle} \quad T^*G \simeq G \times \mathcal{G}^* \simeq G \times \mathcal{G} =: \mathcal{M}_1$$

$$\text{Heisenberg double} \quad G^{\mathbb{C}}_{\mathbb{R}} \simeq G \times G^* \simeq G \times \mathfrak{P} =: \mathcal{M}_2$$

$$\text{Internally fused quasi-Poisson double} \quad G \times G =: \mathcal{M}_3$$

The pull-backs of the relevant rings of invariants

$$C^\infty(G)^G, \quad C^\infty(\mathcal{G})^G, \quad C^\infty(\mathfrak{P})^G$$

give rise to two 'master integrable systems' on each double.

The group G acts on these phase spaces by 'diagonal conjugations', i.e., by the diffeomorphisms

$$A^i_\eta : (x,y) \mapsto (\eta x \eta^{-1}, \eta y \eta^{-1}), \quad \forall (x,y) \in \mathcal{M}_i \ (i = 1,2,3),\ \eta \in G.$$

The G-invariant functions form closed Poisson algebras, and thus the quotient space $\mathcal{M}_i^{\text{red}} \equiv \mathcal{M}_i/G$ becomes a (singular) Poisson space, which carries the corresponding reduced integrable systems.

Plan of the rest of the talk

- The (well known) case of the cotangent bundle T^*G
- Spin RS type models from Heisenberg doubles
- Krichever–Zabrodin type generalizations of the Gibbons–Hermsen model (if time permits)
- Conclusion

The example of the cotangent bundle

The canonical Poisson bracket on the cotangent bundle

$$\mathcal{M} := G \times \mathcal{G} = \{(g, J) \mid g \in G,\ J \in \mathcal{G}\} \text{ has the form}$$

$$\{\mathcal{F}, \mathcal{H}\}(g, J) = \langle \nabla_1 \mathcal{F}, d_2\mathcal{H}\rangle - \langle \nabla_1 \mathcal{H}, d_2\mathcal{F}\rangle + \langle J, [d_2\mathcal{F}, d_2\mathcal{H}]\rangle,$$

where the $\mathcal{G}$-valued derivatives are taken at (g, J). Here, $\langle X, Y\rangle$ is the Cartan-Killing inner product on $\mathcal{G}$. The derivative $d_2\mathcal{F} \in \mathcal{G}$ w.r.t. the second variable $J \in \mathcal{G}$ is the usual gradient, while the derivative $\nabla_1\mathcal{F} \in \mathcal{G}$ w.r.t. first variable $g \in G$ is defined by

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{F}(e^{tX}g, J) =: \langle X, \nabla_1\mathcal{F}(g, J)\rangle, \quad \forall X \in \mathcal{G}.$$

The equations of motion generated by the Hamiltonians $\mathcal{H}$ of the form $\mathcal{H}(g, J) = \varphi(J)$ with $\varphi \in C^\infty(\mathcal{G})^G$ read

$$\dot{g} = (d\varphi(J))g,\ \dot{J} = 0 \implies (g(t), J(t)) = (\exp(td\varphi(J(0)))g(0), J(0)).$$

The constants of motions are arbitrary functions of J and $g^{-1}Jg$.

We reduce by going to the orbit space of $\mathcal{M}$ w.r.t. the conjugation action of G.

We characterize the reduced system using a partial gauge fixing. Define

$$\mathcal{M}^{\text{reg}} := \{(g,J) \in \mathcal{M} \mid g \in G^{\text{reg}}\}, \quad \mathcal{M}_0^{\text{reg}} := \{(Q,J) \in \mathcal{M} \mid Q \in G_0^{\text{reg}}\}.$$

Here, G^{reg} contains the group elements whose centralizer is a maximal torus, and G_0 is a fixed maximal torus. Let $\mathfrak{N}$ denote the normalizer of $G_0 < G$, which is the 'group of residual gauge transformations'.

Then, $\mathcal{M}^{\text{reg}}/G \equiv \mathcal{M}_0^{\text{reg}}/\mathfrak{N}$, and the restriction of functions yields the isomorphism

$$C^\infty(\mathcal{M}^{\text{reg}})^G \Longleftrightarrow C^\infty(\mathcal{M}_0^{\text{reg}})^{\mathfrak{N}},$$

By transferring the Poisson bracket from $C^\infty(\mathcal{M}^{\text{reg}})^G$ to $C^\infty(\mathcal{M}_0^{\text{reg}})^{\mathfrak{N}}$, we get

$$\{F,H\}_{\text{red}}(Q,J) = \langle \nabla_1 F, d_2 H\rangle - \langle \nabla_1 H, d_2 F\rangle + \langle J, [d_2 F, d_2 H]_{\mathcal{R}(Q)}\rangle,$$

with $[X,Y]_{\mathcal{R}} \equiv [\mathcal{R}X, Y] + [X, \mathcal{R}Y]$. The 'reduced evolution equations' generated by the invariant functions $\varphi \in C^\infty(\mathcal{G})^G$ can be written on $\mathcal{M}_0^{\text{reg}}$ as

$$\dot{Q} = (d\varphi(J))_0 Q, \qquad \dot{J} = [\mathcal{R}(Q) d\varphi(J), J].$$

Here, the subscript zero refers to the decomposition $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_0^\perp$, and $\mathcal{R}(Q) \in \text{End}(\mathcal{G})$ is the basic trigonometric solution of the modified classical dynamical Yang–Baxter equation. $\mathcal{R}(Q)$ vanishes on $\mathcal{G}_0$ and, writing $Q = \exp(\text{i}q)$ with $\text{i}q \in \mathcal{G}_0$, is given on $\mathcal{G}_0^\perp$ by $\mathcal{R}(Q) = \frac{1}{2}\coth(\frac{\text{i}}{2}\text{ad}_q)$.

The (well known) spin Sutherland interpretation

Parametrize $J \in \mathcal{G}$ according to

$$J = -\mathrm{i}p + \sum_{\alpha\in\Delta_+}\left(\frac{\xi_\alpha}{e^{-\mathrm{i}\alpha(q)}-1}E_\alpha - \frac{\xi_\alpha^*}{e^{\mathrm{i}\alpha(q)}-1}E_{-\alpha}\right), \quad p \in \mathrm{i}\mathcal{G}_0, \,,$$

and take $\varphi(J) = -\frac{1}{2}\langle J, J\rangle$. Then (using $\langle E_\alpha, E_{-\alpha}\rangle = 2/|\alpha|^2$) we get

$$-\frac{1}{2}\langle J, J\rangle = \frac{1}{2}\langle p, p\rangle + \frac{1}{8}\sum_{\alpha\in\Delta}\frac{2}{|\alpha|^2}\frac{|\xi_\alpha|^2}{\sin^2(\alpha(q)/2)},$$

which is a standard spin Sutherland Hamiltonian $H_{\text{spin-Suth}}(q,p,\xi)$. Here, we use the Killing form and the root space decomposition of the complexified Lie algebra $\mathcal{G}^{\mathbb{C}}$, with the set of roots $\Delta = \{\alpha\}$ and corresponding root vectors E_α.

The 'spin variable' $\xi = \sum_{\alpha\in\Delta_+}(\xi_\alpha E_\alpha - \xi_\alpha^* E_{-\alpha}) \in \mathcal{G}_0^\perp$ matters up to conjugations by the maximal torus G_0. After dividing by G_0, there remains a residual gauge symmetry under the Weyl group $W = \mathfrak{N}/G_0$, and the pertinent dense open subset of the reduced phase space can be identified as $\left(T^*G_0^{\text{reg}} \times (\mathcal{G}^*//_0 G_0)\right)/W$.

Spin RS type models from the Heisenberg double $\mathcal{M} := G \times \mathfrak{P}$

Let us realize $\mathcal{G}$ as a compact real form of a complex simple Lie algebra $\mathcal{G}^{\mathbb{C}}$. Using positive roots associated with the Cartan subalgebra $\mathcal{G}_0^{\mathbb{C}}$, consider the triangular decompositions $\mathcal{G}^{\mathbb{C}} = \mathcal{G}_<^{\mathbb{C}} + \mathcal{G}_0^{\mathbb{C}} + \mathcal{G}_>^{\mathbb{C}}$. We also consider a corresponding connected and simply connected Lie group $G^{\mathbb{C}}$.

The realification $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ of $\mathcal{G}^{\mathbb{C}}$ decomposes as the vector space direct sum

$$\mathcal{G}_{\mathbb{R}}^{\mathbb{C}} = \mathcal{G} + \mathcal{B} \quad \text{with} \quad \mathcal{B} := \mathrm{i}\mathcal{G}_0 + \mathcal{G}_>^{\mathbb{C}}.$$

$\mathcal{G}$ and $\mathcal{B}$ are isotropic subalgebras with respect to the invariant, symmetric, non-degenerate, real bilinear form $\langle -, - \rangle_{\mathbb{I}}$ on $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ defined by the imaginary part of the complex Killing form of $\mathcal{G}^{\mathbb{C}}$. If $\mathcal{G} = su(n)$, then $X \in \mathcal{B}$ is upper-triangular with real diagonal entries. For any $Z = Z_1 + \mathrm{i}Z_2$ in $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$, with $Z_1, Z_2 \in \mathcal{G}$, we let $Z^\dagger := -Z_1 + \mathrm{i}Z_2$.

For a real $f \in C^\infty(G)$ we define its $\mathcal{B}$-valued left- and right-derivatives by

$$\langle \mathcal{D}f(g), X \rangle_{\mathbb{I}} + \langle \mathcal{D}' f(g), Y \rangle_{\mathbb{I}} := \frac{d}{dt}\Big|_{t=0} f(e^{tX} g e^{tY}), \quad \forall X, Y \in \mathcal{G}.$$

For a real function $\phi \in C^\infty(\mathfrak{P})$ we define its $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$-valued derivative $\mathcal{D}\phi$ by

$$\langle X, \mathcal{D}\phi(L) \rangle_{\mathbb{I}} := \frac{d}{dt}\Big|_{t=0} \phi(e^{tX} L e^{tX^\dagger}) \quad \text{and} \quad \langle Y, \mathcal{D}\phi(L) \rangle_{\mathbb{I}} := \frac{d}{dt}\Big|_{t=0} \phi(e^{tY} L e^{-tY})$$

$\forall X \in \mathcal{B}$ and $\forall Y \in \mathcal{G}$.

The phase space $\mathcal{M} = G \times \mathfrak{P} = \{(g, L)\}$ carries the following Poisson structure:

$$\{\mathcal{F}, \mathcal{H}\}(g, L) = \langle \mathcal{D}_2\mathcal{F}, (\mathcal{D}_2\mathcal{H})_{\mathcal{G}}\rangle_{\mathbb{I}} - \left\langle g\mathcal{D}_1'\mathcal{F}g^{-1}, \mathcal{D}_1\mathcal{H}\right\rangle_{\mathbb{I}} + \langle \mathcal{D}_1\mathcal{F}, \mathcal{D}_2\mathcal{H}\rangle_{\mathbb{I}} - \langle \mathcal{D}_1\mathcal{H}, \mathcal{D}_2\mathcal{F}\rangle_{\mathbb{I}},$$

where the derivatives of $\mathcal{F}, \mathcal{H} \in C^\infty(\mathcal{M})$ are evaluated at $(g, L) \in \mathcal{M}$.

The Hamiltonian $\mathcal{H}(g, L) = \phi(L)$, with $\phi \in C^\infty(\mathfrak{P})^G$, generates the evolution equation

$$\dot{g} = (\mathcal{D}\phi(L))g,\ \dot{L} = 0,\ \text{solved by } (g(t), L(t)) = (\exp\left(t\mathcal{D}\phi(L(0))\right)g(0), L(0)).$$

L and $g^{-1}Lg$ are constants of motion, and we obtained an integrable system on $\mathcal{M}$.

Similarly to the cotangent bundle case, we introduce

$$\mathcal{M}^{\text{reg}} := \{(g, L) \in \mathcal{M} \mid g \in G^{\text{reg}}\} \quad \text{and} \quad \mathcal{M}_0^{\text{reg}} := \{(Q, L) \in \mathcal{M} \mid Q \in G_0^{\text{reg}}\}.$$

The isomorphism $C^\infty(\mathcal{M}^{\text{reg}})^G \Longleftrightarrow C^\infty(\mathcal{M}_0^{\text{reg}})^{\mathfrak{N}}$ leads to the reduced Poisson bracket on $C^\infty(\mathcal{M}_0^{\text{reg}})^{\mathfrak{N}}$:

$$\{F, H\}_{\text{red}}(Q, L) = \langle \mathcal{D}_1 F, \mathcal{D}_2 H\rangle_{\mathbb{I}} - \langle \mathcal{D}_1 H, \mathcal{D}_2 F\rangle_{\mathbb{I}} + \langle \mathcal{R}(Q)(\mathcal{D}_2 H)_{\mathcal{G}}, \mathcal{D}_2 F\rangle_{\mathbb{I}} - \langle \mathcal{R}(Q)(\mathcal{D}_2 F)_{\mathcal{G}}, \mathcal{D}_2 H\rangle_{\mathbb{I}}$$

The derivatives $\mathcal{D}_1 F \in \mathcal{B}_0$ and $\mathcal{D}_2 F \in \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ are taken at (Q, L), and $\mathcal{R}(Q) \in \text{End}(\mathcal{G})$ is the standard dynamical r-matrix.

For $\phi \in C^\infty(\mathfrak{P})^G$ one has $\mathcal{D}\phi(L) \in \mathcal{G}$, and the 'reduced evolution equations' can be written on $\mathcal{M}_0^{\text{reg}}$ as

$$\dot{Q} = (\mathcal{D}\phi(L))_0 Q, \qquad \dot{L} = [\mathcal{R}(Q)\mathcal{D}\phi(L), L].$$

Canonically conjugate pairs and 'spin' variables. Let B_0 and B_+ be the subgroups of B associated with the subalgebras in $\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_>$. Any $b \in B$ is uniquely decomposed as $b = e^p b_+$ with $p \in \mathcal{B}_0$, $b_+ \in B_+$ and any $L \in \mathfrak{P}$ can be written as $L = bb^\dagger$.

Then, we introduce new variables by means of the map

$$\zeta : \mathcal{M}_0^{\text{reg}} \to G_0^{\text{reg}} \times \mathcal{B}_0 \times B_+$$

$$\zeta : (Q, L = e^p b_+ b_+^\dagger e^p) \mapsto (Q, p, \lambda) \quad \text{with} \quad \lambda := b_+^{-1} Q^{-1} b_+ Q.$$

The map ζ is a diffeomorphism.

In terms of the new variables introduced via the map ζ, the reduced Poisson bracket acquires the following 'decoupled form':

$$\{F, H\}^{\text{red}}(Q, p, \lambda) = \langle D_Q F, d_p H\rangle_{\mathbb{I}} - \langle D_Q H, d_p F\rangle_{\mathbb{I}} + \langle \lambda D'_\lambda F \lambda^{-1}, D_\lambda H\rangle_{\mathbb{I}},$$

where the derivatives of $F, H \in C^\infty(G_0^{\text{reg}} \times \mathcal{B}_0 \times B_+)^{\mathfrak{N}}$ are taken at (Q, p, λ).

Using the identification $(\mathcal{B}_+)^* \simeq \mathcal{G}_0^\perp$, the derivatives $D_\lambda F, D'_\lambda F \in \mathcal{G}_0^\perp$ are defined by

$$\langle X, D_\lambda F(Q, p, \lambda)\rangle_{\mathbb{I}} + \langle Y, D'_\lambda F(Q, p, \lambda)\rangle_{\mathbb{I}} = \frac{d}{dt}\bigg|_{t=0} F(Q, p, e^{tX} \lambda e^{tY}), \;\; \forall X, Y \in \mathcal{B}_>.$$

We obtained a non-linear analogue of the Poisson structure of the spin Sutherland model coming from T^*G. The term $\langle \lambda D'_\lambda F \lambda^{-1}, D_\lambda H\rangle_{\mathbb{I}}$ represents the reduction of the Poisson–Lie group $B = G^*$ with respect to $G_0 < G$, at the zero value of the moment map for the G_0-action on B (given by conjugations). One may restrict λ to a dressing orbit of G in B, and taking the minimal dressing orbit of $SU(n)$ results in the standard (spinless) real, trigonometric Ruijsenaars–Schneider model.

Interpretation as spin RS model: Consider the new variable $\lambda = b_+^{-1} Q^{-1} b_+ Q$ using

$$\lambda = e^{\sigma}, \quad b_+ = e^{\beta}, \quad \sigma = \sum_{\alpha>0} \sigma_\alpha E_\alpha, \quad \beta = \sum_{\alpha>0} \beta_\alpha E_\alpha, \quad Q = e^{\mathrm{i}q}.$$

We find β_α in terms of σ and $e^{\mathrm{i}q}$: $\beta_\alpha = \frac{\sigma_\alpha}{e^{-\mathrm{i}\alpha(q)}-1} + \sum_{k\geq 2} \sum_{\varphi_1,\dots,\varphi_k} f_{\varphi_1,\dots,\varphi_k}(e^{\mathrm{i}q}) \sigma_{\varphi_1} \dots \sigma_{\varphi_k}$, where $\alpha = \varphi_1 + \dots + \varphi_k$ and $f_{\varphi_1,\dots,\varphi_k}$ depends rationally on $e^{\mathrm{i}q}$.

Take any finite dimensional irreducible representation $\rho : G^{\mathbb{C}} \to \mathrm{SL}(V)$, with a G-invariant inner product on V. Then, the character $\phi^\rho(L) = \mathrm{tr}_\rho(L) := c_\rho \mathrm{tr} \rho(L)$ gives an element of $C^\infty(\mathfrak{P})^G$. (Here, c_ρ is a constant, so that $\mathrm{tr}_\rho(XY) := c_\rho \mathrm{tr}(\rho(X)\rho(Y)) = \langle X, Y\rangle$.)
Using the variables (Q, p, σ), $H^\rho(L) = \phi^\rho(L) = \mathrm{tr}_\rho(e^p b_+ b_+^\dagger e^p)$ can be expanded as

$$H^\rho(e^{\mathrm{i}q}, p, \sigma) = \mathrm{tr}_\rho\Big(e^{2p}\Big(\mathbf{1}_\rho + \frac{1}{4} \sum_{\alpha\in\Delta_+} \frac{|\sigma_\alpha|^2 E_\alpha E_{-\alpha}}{\sin^2(\alpha(q)/2)} + \mathrm{o}_2(\sigma, \sigma^*)\Big)\Big).$$

By expanding e^{2p}, we get

$$H^\rho(e^{\mathrm{i}q}, p, \sigma) = c_\rho \mathrm{dim}_\rho + 2\mathrm{tr}_\rho(p^2) + \frac{1}{2} \sum_{\alpha\in\Delta_+.} \frac{1}{|\alpha|^2} \frac{|\sigma_\alpha|^2}{\sin^2(\alpha(q)/2)} + \mathrm{o}_2(\sigma, \sigma^*, p).$$

The leading term matches the spin Sutherland Hamiltonian $H_{\text{spin-Suth}}$. The reduced Poisson bracket and the Lax matrix are also deformations of those pertaining to the spin Sutherland models. For example,

$$L(e^{\mathrm{i}q}, p, \sigma) = 1 + 2p + \sum_{\alpha\in\Delta_+} \left(\frac{\sigma_\alpha}{e^{-\mathrm{i}\alpha(q)} - 1} E_\alpha + \frac{\sigma_\alpha^*}{e^{\mathrm{i}\alpha(q)} - 1} E_{-\alpha} \right) + \mathrm{o}(\sigma, \sigma^*, p).$$

Our 'spin RS type models' turn into the spin Suthrerland models by a certain scaling limit, akin to the $c \to \infty$ limit.

Time permitting, we now sketch the idea of a generalization of the trigonometric Gibbons–Hermsen model. For this, recall the GH model is obtained by Hamiltonian reduction from

$$T^*U(n) \times \mathbb{C}^{n\times d}$$

The second factor encodes nd ($d \geq 2$) copies of the symplectic vector space $\mathbb{R}^2$. Denote the general element of $\mathbb{C}^{n\times d}$ as the matrix S_{aj}, and let (g, J) stand for the general element of the cotangent bundle, trivialized by right-translations. Then the following formula gives a Poisson map into $\mathfrak{u}(n) \simeq \mathfrak{u}(n)^*$,

$$\Phi(g, J, S) = J - g^{-1}Jg + \mathrm{i}SS^\dagger.$$

This is the moment map for the Hamiltonian action of $U(n)$ given by

$$A_\eta : (g, J, S) \mapsto (\eta g \eta^{-1}, \eta J \eta^{-1}, \eta S), \quad \forall \eta \in U(n).$$

Now, reduce by imposing the moment map constraint $\Phi(g, J, S) = \mathrm{i}c\mathbf{1}_n$, with $c > 0$. On a dense open subset, one can employ the partial gauge fixing where $g = \exp(\mathrm{i}q) \in \mathbb{T}^n_{\mathrm{reg}}$ with the maximal torus $\mathbb{T}^n < U(n)$. Then, one gets

$$J_{ab} = \mathrm{i}p_a\delta_{ab} - \mathrm{i}(1 - \delta_{ab})\frac{S_a S_b^\dagger}{1 - \exp(\mathrm{i}(q_b - q_a))}, \quad \text{with arbitrary} \quad p_a \in \mathbb{R}.$$

In this gauge, the 'free' Hamiltonian gives $H = -\frac{1}{2}\mathrm{tr}(J^2) = \frac{1}{2}\sum_{a=1}^n p_a^2 + \frac{1}{8}\sum_{a\neq b}\frac{|S_a S_b^\dagger|^2}{\sin^2\frac{q_a - q_b}{2}}$, and the moment map constraint implies $S_a S_a^\dagger = c$. The residual gauge transformations are given by the torus $\mathbb{T}^n$ and by the permutation group S_n, and the pertinent open dense subset of the full reduced phase space can be identified as

$$\left(T^*\mathbb{T}^n_{\mathrm{reg}} \times (\mathbb{CP}^{d-1} \times \cdots \times \mathbb{CP}^{d-1})\right)/S_n,$$

with n-copies of the complex projective space. (If $d = 1$, then one gets the spinless Sutherland model.)

For generalization, take the unreduced phase space $\mathcal{M} := GL(n,\mathbb{C}) \times \mathbb{C}^{n\times d}$, where the real manifold $GL(n,\mathbb{C}) \simeq U(n) \times \mathfrak{P}(n)$ is the Heisenberg double of the Poisson–Lie group $U(n)$ and the d columns of $\mathbb{C}^{n\times d}$ carry a $U(n)$ covariant Poisson structure,

$$\{w_k, w_l\} = \mathrm{i}\,\mathrm{sgn}(k-l) w_k w_l, \qquad \forall 1 \le k, l \le n,$$
$$\{w_k, \overline{w}_l\} = \mathrm{i}\,\delta_{kl}(2 + |w|^2) + \mathrm{i} w_k \overline{w}_l + \mathrm{i}\,\delta_{kl} \sum_{r=1}^{n} \mathrm{sgn}(r-k)|w_r|^2 .$$

which is due to Zakrzewski (1996), and is actually symplectic. Consider the following Iwasawa decompositions of $g \in GL(n,\mathbb{C})$ and the factorization of $(\mathbf{1}_n + ww^\dagger) \in \mathfrak{P}(n)$:

$$g = k_L b_R^{-1} = b_L k_R^{-1}, \quad \mathbf{1}_n + ww^\dagger = \mathbf{b}(w)\mathbf{b}(w)^\dagger$$

where $k_L, k_R \in U(n)$ and $b_L, b_R, \mathbf{b}(w) \in B(n)$: the upper-triangular subgroup of $GL(n,\mathbb{C})$ with positive diagonal. Then, define the Poisson map $\Lambda : \mathcal{M} \to B(n) \equiv U(n)^*$ by

$$\Lambda(g, w^1, \dots, w^d) := b_L b_R \mathbf{b}(w^1)\mathbf{b}(w^2)\cdots\mathbf{b}(w^d), \quad \text{with} \quad (w^1, w^2, \dots, w^d) \in \mathbb{C}^{n\times d}.$$

This generates an action of the Poisson–Lie group $U(n)$ on $\mathcal{M}$, and we obtain the reduced phase space

$$\mathcal{M}_{\text{red}} = \Lambda^{-1}(e^\gamma \mathbf{1}_n)/U(n),$$

which is a smooth symplectic manifold for any $\gamma > 0$.

The unreduced phase space carries the commuting Hamiltonians

$$H_j := \mathrm{tr}(L^j) \quad \text{with} \quad L := b_R b_R^\dagger, \quad j = 1, \dots, n.$$

They have very simple flows and yield an integrable system on $\mathcal{M}$, quite similar to the cotangent bundle case.

We can go to the gauge slice where k_R becomes a diagonal matrix, $Q \in \mathbb{T}^n_{\text{reg}}$. Decomposing $b \in B(n)$ as $b = b_0 b_+$, with diagonal and unipotent factors, we write

$$b_R = b_0 b_+ \quad \text{and} \quad \mathbf{b}(w^1)\mathbf{b}(w^2)\cdots\mathbf{b}(w^d) =: S(W) =: S_0(W)S_+(W).$$

Then the moment map condition becomes equivalent to the following constraints:

$$S_0(W) = e^{\gamma}\mathbf{1}_n \quad \text{and} \quad b_+ S_+(W) = Q^{-1} b_+ Q.$$

The first equation constraints $W = (w^1, \dots, w^d)$ only, while the second one permits us to express b_+ in terms of $Q = e^{\mathrm{i}q} \in \mathbb{T}^n_{\text{reg}}$ and $S_+(W) \in \mathbb{C}^{n\times d}$. (Same eq. as $b_+\lambda = Q^{-1}b_+Q$.)

$Q \in \mathbb{T}^n_{\text{reg}}$ and $b_0 \equiv \exp(p)$, with $p = \text{diag}(p_1, \dots, p_n)$, are arbitrary, and a dense open subset of the reduced phase space is parametrized by Q, p and the constrained 'primary spins', W, up to the usual residual gauge transformations.

The reduction of the spectral invariants of $L = b_R b_R^\dagger$ yields an integrable system.

To connect our reduced system with the Gibbons–Hermsen model, we introduce a positive 'scaling parameter' ϵ and make the replacements

$$p \to \epsilon p, \quad W \to \epsilon^{\frac{1}{2}} W, \quad Q \to Q, \quad \Omega_{\mathcal{M}} \to \epsilon^{-1}\Omega_{\mathcal{M}}, \quad \gamma \to \epsilon\gamma,$$

where $\Omega_{\mathcal{M}}$ is the symplectic form on $\mathcal{M}$. With $L := b_R b_R^\dagger$ and $b_R = \exp(\epsilon p) b_+(Q, \epsilon^{\frac{1}{2}} W)$, writing $Q = \mathrm{diag}(e^{\mathrm{i}q_1}, \dots, e^{\mathrm{i}q_n})$ and letting w_i denote the i-th row of $W \in \mathbb{C}^{n\times d}$, we get

$$\lim_{\epsilon\to 0} \frac{1}{8\epsilon^2}(\mathrm{tr}(L) + \mathrm{tr}(L^{-1}) - 2n) = \frac{1}{2}\mathrm{tr}(p^2) + \frac{1}{32}\sum_{i\neq j} \frac{|w_i w_j^\dagger|^2}{\sin^2 \frac{q_i - q_j}{2}},$$

$$\lim_{\epsilon\to 0} \epsilon^{-1}(\Omega_{\mathrm{red}}) = \sum_{j=1}^n dp_j \wedge dq_j + \frac{\mathrm{i}}{2}\sum_{j=1}^n \sum_{\alpha=1}^d dw_j^\alpha \wedge d\overline{w}_j^\alpha,$$

reproducing the Hamiltonian and symplectic form of the Gibbons–Hermsen model.

Details are explained in Fairon, L.F. and Marshall: *Trigonometric real form of the spin RS model of Krichever and Zabrodin*, Ann. Henri Poincaré 22, 615-675 (2021)

Our construction is a 'real form' of earlier reduction treatments of the holomorphic spin RS models of Krichever–Zabrodin (1995), which are due to **Chalykh and Fairon** and to **Arutyunov and Olivucci**. (As far as I know, the connection to the Gibbons–Hermsen model was not analysed in those papers.)

Remark on multi-Hamiltonian structure. Let us consider the space of primary spins

$$\mathbb{C}^{n\times d} = \{(w^1, w^2, \dots, w^d)\},$$

and define on it the commuting vector fields V_j $(j = 1, \dots, d)$ that, as derivations of the evaluation functions, satisfy

$$V_j[w^k] = \mathrm{i}\delta_{j,k} w^k.$$

They are the infinitesimal generators of the natural $U(1)$ action on the d-copies of $\mathbb{C}^n$. They are naturally extended to $\mathcal{M} = GL(n,\mathbb{C}) \times \mathcal{W}$, and the previously introduced Poisson bivector $P_{\mathcal{M}}$ admits the modification

$$P_{\mathcal{M}} \to P_{\mathcal{M}} + \sum_{1\leq j<k\leq d} x_{jk} V_j \wedge V_k$$

with arbitrary real parameters x_{jk}. (The columns w^j and w^k no longer Poisson commute if $x_{jk} \neq 0$.) The modified Poisson structure remains symplectic. It admits the same Poisson–Lie moment map as for $x_{jk} \equiv 0$, generating the same $U(n)$ action, and the flows of the 'free Hamiltonians' do not change.

As a result, we obtain a multi-Hamiltonian structure for the reduced system on $\Lambda^{-1}(e^{\mathrm{i}\gamma}\mathbf{1}_n)/U(n)$. This procedure works for the Gibbons–Hermsen model as well.)

For a related system, this way of introducing a multi-Hamiltonian structure was used in Fairon and L. F.: *Integrable multi-Hamiltonian systems from reduction of an extended quasi-Poisson double of U(n)*, Ann. Henri Poincaré 24, 3461-3529 (2023)

Conclusion

1. I presented generalizations of the classical trigonometric spin Sutherland models built on Lie algebraic spin variables and mentioned also the analogous generalization of the classical spins Sutherland models of Gibbons–Hermsen type.

2. **Quantization of the novel spin RS type models? Via quantum Hamiltonian reduction or by other means? Any relation to other spin RS models?**

3. The old models can be recovered as scaling limits of the novel models, like the $c \to \infty$ limit connecting Ruijsenaars–Schneider models to Sutherland models.

4. The second kind of models are related to the $U(n)$ models of the first kind by means of a realization of the 'collective spin variable' $\lambda \in B_+$ in terms of 'spin vectors' attached to the particles, analogously to what happens in the linear case.

5. We note that the degenerate integrability of the first kind of models was shown after restriction on the dense open subset of the phase space corresponding to the principal orbit type for the G-action. Degenerate integrability of the trigonometric Krichever–Zabrodin models was also proved.

Appendix I: Recall of degenerate integrability on symplectic and Poisson manifolds

Definition 1. Suppose that $\mathcal{M}$ is a **symplectic** manifold of dimension $2m$ with associated Poisson bracket $\{-,-\}$ and two distinguished subrings $\mathfrak{H}$ and $\mathfrak{F}$ of $C^\infty(\mathcal{M})$ satisfying the following conditions:

1. The ring $\mathfrak{H}$ has functional dimension r and $\mathfrak{F}$ has functional dimension s such that $r+s=\dim(\mathcal{M})$ and $r<m$.
2. Both $\mathfrak{H}$ and $\mathfrak{F}$ form Poisson subalgebras of $C^\infty(\mathcal{M})$, satisfying $\mathfrak{H}\subset\mathfrak{F}$ and $\{\mathcal{F},\mathcal{H}\}=0$ for all $\mathcal{F}\in\mathfrak{F}$, $\mathcal{H}\in\mathfrak{H}$.
3. The Hamiltonian vector fields of the elements of $\mathfrak{H}$ are complete.

Then, $(\mathcal{M},\{-,-\},\mathfrak{H},\mathfrak{F})$ is called a **degenerate integrable system of rank** r. The rings $\mathfrak{H}$ and $\mathfrak{F}$ are referred to as the ring of Hamiltonians and constants of motion, respectively. (If $r=1$, then this is the same as 'maximal superintegrability' of a single Hamiltonian.)

Definition 2. Consider a **Poisson** manifold $(\mathcal{M},\{-,-\})$ whose Poisson tensor has maximal rank $2m\leq\dim(\mathcal{M})$ on a dense open subset. Then, $(\mathcal{M},\{-,-\},\mathfrak{H},\mathfrak{F})$ is called a degenerate integrable system of rank r if conditions (1), (2), (3) of Definition 1 hold, and the Hamiltonian vector fields of the elements of $\mathfrak{H}$ span an r-dimensional subspace of the tangent space over a dense open subset of $\mathcal{M}$.

Appendix II: Explicit formulas for $G^{\mathbb{C}} = SL(n, \mathbb{C})$: Now parametrize $b_+ \in B$ by its matrix elements. We have $b = e^p b_+$, and can find b_+ from the relation

$$Q^{-1} b_+ Q = b_+ \lambda,$$

where $Q = \mathrm{diag}(Q_1, \ldots, Q_n) \in G_0^{\mathrm{reg}}$, $\lambda \in B_+$ is the constrained 'spin' variable and b_+ is an upper triangular matrix with unit diagonal.

Denoting $\mathcal{I}_{a,a+j} := \frac{1}{Q_{a+j} Q_a^{-1} - 1}$, we have $(b_+)_{a,a+1} = \mathcal{I}_{a,a+1} \lambda_{a,a+1}$, and, for $k = 2, \ldots, n-a$, the matrix element $(b_+)_{a,a+k}$ equals

$$\mathcal{I}_{a,a+k} \lambda_{a,a+k} + \sum_{\substack{m=2,\ldots,k \\ (i_1,\ldots,i_m) \in \mathbb{N}^m \\ i_1+\cdots+i_m = k}} \prod_{\alpha=1}^{m} \mathcal{I}_{a,a+i_1+\cdots+i_\alpha} \lambda_{a+i_1+\cdots+i_{\alpha-1}, a+i_1+\cdots+i_\alpha}.$$

Then $H = \mathrm{tr}(bb^\dagger)$ gives

$$H(e^{\mathrm{i}q}, p, \lambda) = \sum_{a=1}^{n} e^{2p_a} + \frac{1}{4} \sum_{a=1}^{n-1} e^{2p_a} \sum_{k=1}^{n-a} \frac{|\lambda_{a,a+k}|^2}{\sin^2((q_{a+k} - q_a)/2)} + \mathrm{o}_2(\lambda, \lambda^\dagger).$$