The reflection equation: from algebra to application

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General philosophy

Develop an algebraic approach to the reflection equation (boundary Yang-Baxter equation) which:

- is *universal* and *uniform*;
- is useful for quantum integrable systems;
- extends the approach to the Yang-Baxter equation in terms of quasitriangular Hopf algebras.

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We will consider a *pair* of symmetry algebras:

$$(U_q(\widehat{\mathfrak{g}}), B_\theta) \qquad \text{quantum affine symmetric pair}$$

where $\mathfrak{g}$ is a fin.dim. simple Lie algebra, $\widehat{\mathfrak{g}}$ the 1-dim. central extension of $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ and $U_q(\widehat{\mathfrak{g}})$ the corresponding quantum affine algebra.

The "new" ingredient is a suitable involutive automorphism $\theta : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$, using which one defines a coideal subalgebra $B_\theta \subset U_q(\widehat{\mathfrak{g}})$, also known as affine $\imath$-quantum group, q-deforming the Hopf subalgebra $U(\widehat{\mathfrak{g}}^\theta) \subset U(\widehat{\mathfrak{g}})$.

1. Cylindrical structures on quasitriangular bialgebras
2. Trigonometric K-matrices from quantum affine symmetric pairs
3. Tensor K-matrices and their applications

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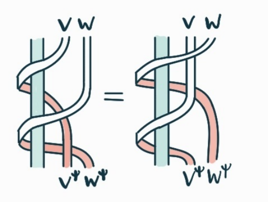
Key reference:
[AV20] A. Appel & B. Vlaar, *Universal K-matrices for quantum Kac-Moody algebras*. *Representation Theory of the American Mathematical Society* **26** (2022) and at arXiv:2007.09218.



R-matrices and the Yang-Baxter equation

Recall, a bialgebra A with coproduct $\Delta : A \to A \otimes A$ is called quasitriangular if there exists $R \in (A \otimes A)^{\times}$ (universal R-matrix) satisfying

$$R \cdot \Delta(a) = \Delta^{\mathrm{op}}(a) \cdot R \quad \text{for all } a \in A,$$

$$(\Delta \otimes \mathrm{id})(R) = R_{13} \cdot R_{23}, \qquad (\mathrm{id} \otimes \Delta)(R) = R_{13} \cdot R_{12}.$$

$\Rightarrow$ universal Yang-Baxter equation (YBE)
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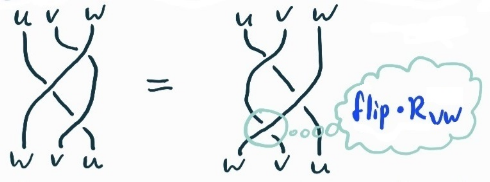
$\in A \otimes A \otimes A$.



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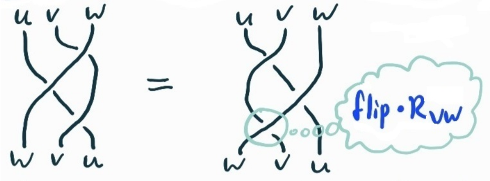

In many cases, R arises as canonical element of nondegenerate bialgebra pairing between two "halves" of A. Example: Drinfeld-Jimbo quantum group $A = U_q(\mathfrak{L})$ where $\mathfrak{L}$ is a Kac-Moody Lie algebra,

$$R \in \text{completion of } U_q(\mathfrak{L})^{\leq 0} \otimes U_q(\mathfrak{L})^{\geq 0}$$

acting on tensor products of suitable modules.

Yang-Baxter equation (YBE) with spectral parameter:

$$R(\tfrac{y}{x})_{12} \cdot R(\tfrac{z}{x})_{13} \cdot R(\tfrac{z}{y})_{23} = R(\tfrac{z}{y})_{23} \cdot R(\tfrac{z}{x})_{13} \cdot R(\tfrac{y}{x})_{12}$$

where $R(z) \in \mathrm{End}(V \otimes V)$ for some finite-dimensional $\mathbb{C}$-linear space V, depending on a parameter z.

The universal R-matrix of Drinfeld-Jimbo quantum groups of affine type lies at the origin of a range of solutions $R(z)$ with rational dependence on z, called *trigonometric R-matrices*.

Example (XXZ/6-vertex R-matrix, associated to $U_q(\widehat{\mathfrak{sl}}_2)$, $V = \mathbb{C}^2$)

$$R(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{q(1-z)}{q^2-z} & \frac{(q^2-1)z}{q^2-z} & 0 \\ 0 & \frac{q^2-1}{q^2-z} & \frac{q(1-z)}{q^2-z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sin(u)}{\sin(u+\eta)} & \frac{\sin(\eta)e^{\sqrt{-1}u}}{\sin(u+\eta)} & 0 \\ 0 & \frac{\sin(\eta)e^{-\sqrt{-1}u}}{\sin(u+\eta)} & \frac{\sin(u)}{\sin(u+\eta)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $z = e^{2\sqrt{-1}u}$, $q = e^{-\sqrt{-1}\eta}$.

Reflection equation

(Left) reflection equation with spectral parameter:

$$R(\tfrac{z}{y})_{21} \cdot K(z)_2 \cdot R(yz) \cdot K(y)_1 = K(y)_1 \cdot R(yz)_{21} \cdot K(z)_2 \cdot R(\tfrac{z}{y})$$

where $K(z) \in \mathrm{End}(V)$ (Cherednik, '84) (Sklyanin, '88).

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Example (K-matrices for the XXZ/6-vertex R-matrix)

Classification, up to symmetries, of invertible *symmetrizable* solutions $K(z)$ (with $V = \mathbb{C}^2$ and $R(z)$ given by the XXZ R-matrix):

$$\frac{\mu_0\mu_1 z}{(\mu_0\mu_1 - z)(\mu_0 z - \mu_1)} \begin{pmatrix} \sigma_0 - \sigma_1 z & z^{-1} - z \\ z - z^{-1} & \sigma_0 - \sigma_1 z^{-1} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & \frac{\xi - z^{-1}}{\xi - z} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with free parameters $\sigma_i = \mu_i + \mu_i^{-1} \in \mathbb{C}$ and $\xi \in \mathbb{C}^\times$.

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Analogous classification for $U_q(\widehat{\mathfrak{sl}}_n)$ R-matrices, see (Regelskis & V., '18): many independent solutions ($\sim n^2$), no "most general" solution.

There are also independent solutions of triangular type, whose algebraic origin is far from clear.

Obstacle for universality & uniformity

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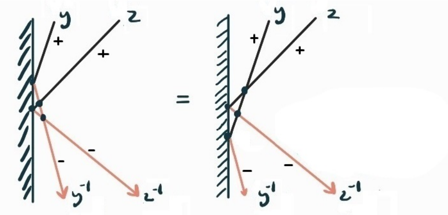
Generalized reflection equation (Cherednik,'92). Also see (Freidel & Maillet,'91) and (Kulish & Sklyanin,'92)

$$R^{--}(\tfrac{z}{y})_{21} \cdot K(z)_2 \cdot R^{-+}(yz) \cdot K(y)_1 = \\ = K(y)_1 \cdot R^{-+}(yz)_{21} \cdot K(z)_2 \cdot R^{++}(\tfrac{z}{y})$$



Special cases:

$R^{\pm\pm} = R$: ordinary RE

$R^{++} = R$, $R^{-+} = (R^{-1})^{t_1}$,
$R^{--} = R^{t_1,t_2}$: crossed RE

The three R-matrices satisfy mixed Yang-Baxter equations, e.g.

$$R^{--}(\tfrac{y}{x})_{12} \cdot R^{-+}(\tfrac{z}{x})_{13} \cdot R^{-+}(\tfrac{z}{y})_{23} = R^{-+}(\tfrac{z}{y})_{23} \cdot R^{-+}(\tfrac{z}{x})_{13} \cdot R^{--}(\tfrac{y}{x})_{12}.$$

Twisted modules

It looks like we have *different* representations of a symmetry algebra A on the *same* vector space V. We should think of a K-matrix as an invertible intertwiner: $V \to V^\psi$ where in V^ψ the action is preceded by an algebra automorphism ψ, called twist map.

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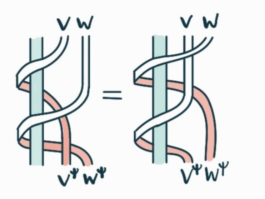
Goals

1. Identify what structure to add to a quasitriangular bialgebra (A, Δ, R) to produce a solution $K \in A^\times$ of the universal reflection equation

$$(R^{\psi\psi})_{21} \cdot K_2 \cdot R^\psi \cdot K_1 = \\ = K_1 \cdot (R^\psi)_{21} \cdot K_2 \cdot R \in A \otimes A$$

with $R^\psi := (\psi \otimes \text{id})(R)$, $R^{\psi\psi} := (\psi \otimes \psi)(R)$.



2. Find examples of such algebraic structures.
3. Explain how it leads to trigonometric matrix solutions (note: a nontrivial ψ is necessary to get parameter inversion $z \mapsto z^{-1}$).

Notation

For $a \in A$ and an A-module V, $a_V :=$ action of a on V.
E.g. if $\psi : A \to A$ is an algebra automorphism, $a_{V^\psi} = \psi(a)_V$.

For $X \in A \otimes A$ and A-modules V and W, $X_{V,W} :=$ action of X on $V \otimes W$. E.g. if $\Delta : A \to A \otimes A$ is a coproduct, $a_{V \otimes W} := \Delta(a)_{V,W}$.

How do these candidate intertwiners $K_V : V \to V^\psi$ act on tensor products? In other words, what should $\Delta(K)$ be?

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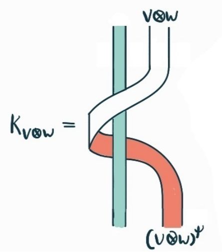

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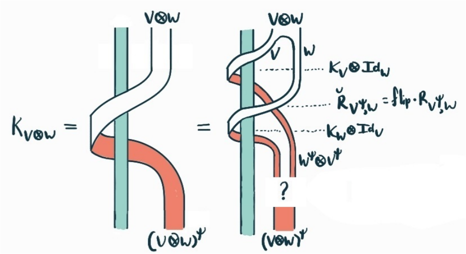

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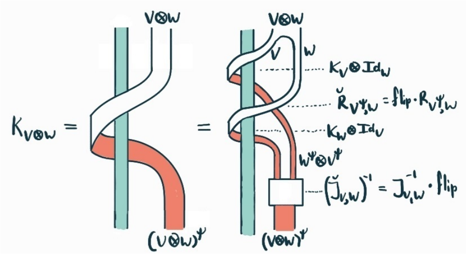

Wanted: an invertible A-intertwiner $(V \otimes W)^\psi \xrightarrow{\check{J}_{V,W}} W^\psi \otimes V^\psi$. Similar to $\check{R}_{V,W}$, assume that $\check{J}_{V,W} = \text{flip} \circ J_{V,W}$ for some $J \in (A \otimes A)^\times$.

The intertwining condition on $\check{J}$ means

$$\check{J}_{V,W} \cdot \Delta(\psi(a))_{V,W} = (\psi \otimes \psi)(\Delta(a))_{W,V} \cdot \check{J}_{V,W} \qquad \forall a \in A,$$

which is guaranteed if we assume

$$\big((\psi \otimes \psi) \circ \Delta^{\mathrm{op}} \circ \psi^{-1}\big)(a) = J \cdot \Delta(a) \cdot J^{-1} \qquad \forall a \in A.$$

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We conclude that $\mathrm{Ad}(J) \circ \Delta$ defines a coproduct on A. This requires that J is a Drinfeld twist, i.e. a normalized solution of

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This constraint on (ψ, J) naturally extends to the universal R-matrix. We call (ψ, J) a twist pair on (A, Δ, R) if

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Lemma

Suppose $K \in A^{\times}$ satisfies $\Delta(K) = J^{-1} \cdot K_2 \cdot R^{\psi} \cdot K_1$ and that (ψ, J) is a twist pair. Then the universal RE holds.

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$$\begin{aligned}(R^{\psi\psi})_{21} \cdot K_2 \cdot R^{\psi} \cdot K_1 &= (R^{\psi\psi})_{21} \cdot J \cdot \Delta(K) &&= J_{21} \cdot R \cdot \Delta(K) \\ &= J_{21} \cdot \Delta^{\mathrm{op}}(K) \cdot R &&= K_1 \cdot (R^{\psi})_{21} \cdot K_2 \cdot R.\end{aligned}$$

Cylindrical structures on quasitriangular bialgebra

Recall that $R \in (A \otimes A)^{\times}$ satisfies a *linear relation*:

$$R \cdot \Delta(a) = \Delta^{\mathrm{op}}(a) \cdot R \quad \text{for all } a \in A$$

In other words, $$R \cdot b = \mathrm{flip}(b) \cdot R \quad \text{for all } b \in \Delta(A) \subset A \otimes A.$$

Let us generalize this to the K-matrix and require that there exists a subalgebra $B \subseteq A$ such that

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Definition [AV20]

We call (ψ, J, K) a cylindrical structure on a quasitriangular bialgebra (A, Δ, R) with respect to a subalgebra $B \subset A$ if (ψ, J) is a twist pair and $K \in A^\times$, called basic universal K-matrix, satisfies

$$K \cdot b = \psi(b) \cdot K \qquad \text{for all } b \in B,$$
$$\Delta(K) = J^{-1} \cdot K_2 \cdot R^\psi \cdot K_1.$$

Gauge transformations

Before we discuss the existence of cylindrical structures for Drinfeld-Jimbo quantum groups, we make a useful observation.

Given a quasitriangular bialgebra (A, Δ, R) and a subalgebra $B \subset A$, there is a natural group action of $A^\times$ on cylindrical structures on (A, Δ, R) with respect to B, called gauge transformation.

Namely for $g \in A^\times$ set

$$g \cdot (\psi, J, K) = \Big((\text{conjugate by } g) \circ \psi, (g \otimes g) \cdot J \cdot \Delta(g)^{-1}, g \cdot K \Big).$$

We can use gauge transformations to replace a cylindrical structure by another one with a particularly nice automorphism ψ, or by one with a universal K-matrix in a particularly nice subalgebra, etc.

Drinfeld-Jimbo quantum groups $U_q(\mathfrak{L})$

Fix a symmetrizable generalized Cartan matrix $C = (c_{ij})_{i,j \in I}$ with I a finite index set (Dynkin diagram). Kac-Moody Lie algebra:

$$\mathfrak{L} = \mathbb{C}\langle \{e_i, f_i\}_{i \in I}, \mathfrak{L}^0 \,|\, \text{relations defined by } C \rangle$$

where $\mathfrak{L}^0$ is an abelian subalgebra of dimension $2|I| - \mathrm{rk}(C)$.
Let $\mathfrak{L}^- = \langle \{f_i\}_{i \in I} \rangle$, $\mathfrak{L}^+ = \langle \{e_i\}_{i \in I} \rangle$. Then $\mathfrak{L} = \mathfrak{L}^- \oplus \mathfrak{L}^0 \oplus \mathfrak{L}^+$ and there is a nondegenerate invariant symmetric bilinear form which pairs

$$\mathfrak{L}^{\geq 0} := \mathfrak{L}^0 \oplus \mathfrak{L}^+ \quad \text{and} \quad \mathfrak{L}^{\leq 0} := \mathfrak{L}^- \oplus \mathfrak{L}^0.$$

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$$\mathfrak{L}^{\geq 0} := \mathfrak{L}^0 \oplus \mathfrak{L}^+ \quad \text{and} \quad \mathfrak{L}^{\leq 0} := \mathfrak{L}^- \oplus \mathfrak{L}^0.$$

Let $q \in \mathbb{C}^\times$, not root of unity. There is a bialgebra

$$U_q(\mathfrak{L}) = \mathbb{C}\langle \{E_i, F_i\}_{i \in I}, \{t_h\}_{h \in \text{lattice} \subset \mathfrak{L}^0} \,|\, \text{relations}_q \text{ defined by } C \rangle$$

such that $U_q(\mathfrak{L}) \xrightarrow{q \to 1} U(\mathfrak{L})$. Then $U_q(\mathfrak{L}) = U_q(\mathfrak{L}^-) \cdot U_q(\mathfrak{L}^0) \cdot U_q(\mathfrak{L}^+)$.

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$$\mathfrak{L} = \mathbb{C}\langle \{e_i, f_i\}_{i \in I}, \mathfrak{L}^0 \,|\, \text{relations defined by } C \rangle$$

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Let $\mathfrak{L}^- = \langle \{f_i\}_{i \in I} \rangle$, $\mathfrak{L}^+ = \langle \{e_i\}_{i \in I} \rangle$. Then $\mathfrak{L} = \mathfrak{L}^- \oplus \mathfrak{L}^0 \oplus \mathfrak{L}^+$ and there is a nondegenerate invariant symmetric bilinear form which pairs

$$\mathfrak{L}^{\geq 0} := \mathfrak{L}^0 \oplus \mathfrak{L}^+ \quad \text{and} \quad \mathfrak{L}^{\leq 0} := \mathfrak{L}^- \oplus \mathfrak{L}^0.$$

Let $q \in \mathbb{C}^\times$, not root of unity. There is a bialgebra

$$U_q(\mathfrak{L}) = \mathbb{C}\langle \{E_i, F_i\}_{i \in I}, \{t_h\}_{h \in \text{lattice} \subset \mathfrak{L}^0} \,|\, \text{relations}_q \text{ defined by } C \rangle$$

such that $U_q(\mathfrak{L}) \xrightarrow{q \to 1} U(\mathfrak{L})$. Then $U_q(\mathfrak{L}) = U_q(\mathfrak{L}^-) \cdot U_q(\mathfrak{L}^0) \cdot U_q(\mathfrak{L}^+)$.
The bilinear form induces a pairing between $U_q(\mathfrak{L}^{\geq 0})$ and $U_q(\mathfrak{L}^{\leq 0})$.
The universal R-matrix is the canonical element of this pairing (hence, in completion of $U_q(\mathfrak{L}^{\leq 0}) \otimes U_q(\mathfrak{L}^{\geq 0})$). It acts on tensor products in category $\mathcal{O}$ (e.g. h.w. modules with h.w. vector annihilated by all E_i).

Example ($\mathfrak{L} = \widehat{\mathfrak{sl}}_2^{\text{ext}}$, $\quad C = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$)

$\widehat{\mathfrak{sl}}_2^{\text{ext}}$ is generated by e_i, f_i, h_i for $i = 0, 1$, and d subject to

$$[h_i, e_i] = 2e_i, \qquad [h_i, f_i] = -2f_i, \qquad [e_i, f_i] = h_i,$$

$$\left.\begin{array}{c} [h_i, e_j] = -2e_j, \quad [h_i, f_j] = 2f_j, \quad [e_i, f_j] = 0, \quad [h_i, h_j] = 0 \\ [e_i, [e_i, [e_i, e_j]]] = [f_i, [f_i, [f_i, f_j]]] = 0 \end{array}\right\} \text{ if } i \neq j,$$

$$[d, e_i] = \delta_{i,0} e_i, \qquad [d, f_i] = -\delta_{i,0} f_i, \qquad [d, h_i] = 0.$$

Identification with central extension of $\mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}]$, with $\mathrm{ad}(d) = t\frac{\mathrm{d}}{\mathrm{d}t}$:

$$e_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes t, \qquad f_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes t^{-1}, \qquad h_0 = c - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1,$$

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes 1, \qquad f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes 1, \qquad h_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1.$$

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$U_q(\widehat{\mathfrak{sl}}_2^{\mathrm{ext}})$ is generated by E_i, F_i, $t_i^{\pm 1}$ for $i = 0, 1$, and $t_d^{\pm 1}$, subject to

$$t_i E_i = q^2 E_i t_i, \qquad t_i F_i = q^{-2} F_i t_i, \qquad [E_i, F_i] = \frac{t_i - t_i^{-1}}{q - q^{-1}},$$

$$\left. \begin{array}{c} t_i E_j = q^{-2} E_j t_i, \quad t_i F_j = q^2 F_j t_i, \quad [E_i, F_j] = 0, \quad [t_i, t_j] = 0 \\ [E_i, [E_i, [E_i, E_j]_{q^2}]_1]_{q^{-2}} = [F_i, [F_i, [F_i, F_j]_{q^2}]_1]_{q^{-2}} = 0 \end{array} \right\} \text{ if } i \neq j,$$

$$t_d E_i = q^{\delta_{i,0}} E_i t_d, \qquad t_d F_i = q^{-\delta_{i,0}} F_i t_d, \qquad t_d t_i = t_i t_d.$$

Quantum symmetric Kac-Moody pair $(U_q(\mathfrak{L}), B_\theta)$

Let θ be any Lie algebra involution of $\mathfrak{L}$ of *the second kind*, viz.

$$\dim(\mathfrak{L}^+ \cap \theta(\mathfrak{L}^+)) < \infty.$$

Consider the fixed-point Lie subalgebra

$$\mathfrak{L}^\theta = \{X \in \mathfrak{L} \,|\, \theta(X) = X\}.$$

These maps and subalgebras can be explicitly prescribed by Satake diagrams (decorated Dynkin diagrams).
Example: Chevalley involution $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$, $\omega|_{\mathfrak{L}^0} = -\mathrm{id}_{\mathfrak{L}^0}$

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$$\mathfrak{L}^\theta = \mathbb{C}\langle \{f_i - e_i\}_{i \in I} \rangle.$$

The study of q-analogues of $U(\mathfrak{L}^\theta) \subseteq U(\mathfrak{L})$ in the case $\dim(\mathfrak{L}) < \infty$ started in the 1990s (A. Gavrilik & A. Klimyk; T. Koornwinder; M. Noumi et al.; G. Letzter). We follow the approach by (S. Kolb, '14) who defined subalgebras $B_\theta \subseteq U_q(\mathfrak{L})$ with common properties:

- Right coideal: $\Delta(B_\theta) \subset B_\theta \otimes U_q(\mathfrak{L})$;
- Maximal subspace of $U_q(\mathfrak{L})$ such that $\lim_{q \to 1} B_\theta = U(\mathfrak{L}^\theta)$;
- Finite generating set, containing F_i + element of $U_q(\mathfrak{L}^{\geq 0})$, $\forall i \in I$.

- For any $\mathfrak{L}$, one can take $\theta = \omega$, yielding

$$B_\omega = \mathbb{C}\langle \{ F_i - q^{\text{integer}} E_i t_i^{-1} + (\text{scalar}) t_i^{-1} \}_{i \in I} \rangle$$

It is straightforward to check that B_ω is a right coideal, noting that in our conventions Δ satisfies

$$\Delta(E_i) = E_i \otimes 1 + t_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes t_i^{-1} + 1 \otimes F_i, \quad \Delta(t_i^{\pm 1}) = t_i^{\pm 1} \otimes t_i^{\pm 1}.$$

When $\mathfrak{L}$ is of affine type, B_ω is also called *(embedded) generalized q-Onsager algebra* (P. Baseilhac & S. Belliard, '10).

Example (*q-Onsager algebra*, cf. (P. Terwilliger, '93))

$$B_\omega = \mathbb{C}\langle \{ F_i - q^{-1} E_i t_i^{-1} - (q - q^{-1}) \sigma_i t_i^{-1} \}_{i=0,1} \rangle \subset U_q(\widehat{\mathfrak{sl}}_2^{\text{ext}}).$$

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- Twisted q-Yangians/twisted quantum loop algebras: subalgebras of $U_q(\widehat{\mathfrak{gl}}_N)$ defined via a boundary analogue of the R-matrix realization of quantum groups, see (A. Molev, E. Ragoucy & P. Sorba, '03) and (H. Chen, N. Guay & X. Ma, '14).

Theorem 1 [AV20]

The following defines a cylindrical structure on $U_q(\mathfrak{L})$ w.r.t. B_θ.

- $\psi =$ lift of θ to algebra automorphism of $U_q(\mathfrak{L})$,
- $J =$ R-matrix of maximal subbialgebra fixed pointwise by θ,
- $K = (K_V)_{V \in \mathcal{O}}$ in a completion of $U_q(\mathfrak{L}^{\geq 0})$, essentially uniquely defined by imposing
$$K \cdot b = \psi(b) \cdot K \qquad \text{for all } b \in B_\theta.$$

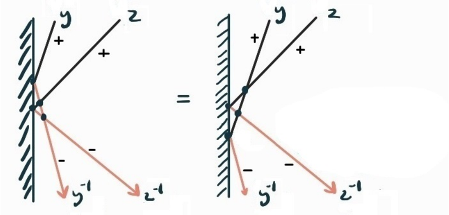
- We call this cylindrical structure the standard one.
- The resulting linear maps K_V satisfy the *constant* generalized RE.
- The proof is uniform. No explicit formula for K is obtained, just existence, analogous to Lusztig's proof of the existence of R.
- Originally due to (H. Bao & W. Wang, '18) for certain subalgebras of $U_q(\mathfrak{sl}_n)$. This was generalized by (M. Balagović & S. Kolb, '19) to all quantum symmetric pairs $(U_q(\mathfrak{L}), B_\theta)$ with $\dim(\mathfrak{L}) < \infty$, who used gauge transformations to produce a cylindrical structure with ψ a diagram automorphism, and K acting on fin.dim. $U_q(\mathfrak{L})$-modules.

1 Cylindrical structures on quasitriangular bialgebras

2 Trigonometric K-matrices from quantum affine symmetric pairs

3 Tensor K-matrices and their applications

Key reference:
[AV22] A. Appel & B. Vlaar, *Trigonometric K-matrices for finite-dimensional representations of quantum affine algebras*. Preprint at arXiv:2203.16503.



Let $\mathfrak{g}$ be a simple fin.dim. Lie algebra of rank r, with Dynkin diagram labelled by $\{1, \ldots, r\}$. Let $\mathfrak{L} = \widehat{\mathfrak{g}}^{\mathrm{ext}}$ the corresponding Kac-Moody algebra of affine type with affine Dynkin diagram $\{0, 1, \ldots, r\}$. Consider any quantum affine symmetric pair $(U_q(\mathfrak{L}), B_\theta)$.

Outstanding task

Explain how the standard cylindrical structure leads to trigonometric matrix solutions of the generalized RE.

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Problem

$U_q(\widehat{\mathfrak{g}}^{\text{ext}})$ has no interesting finite-dimensional modules.

The subalgebra $U_q(\widehat{\mathfrak{g}}) \subset U_q(\widehat{\mathfrak{g}}^{\text{ext}})$ (remove the generators $t_d^{\pm 1}=q^{\pm d}$) has many fin.dim. modules[a], e.g. evaluation modules. How can we make sense of an action of K on such modules?

[a]We will restrict our attention to so-called type-1 modules, which is standard.

Formal spectral parameter

Let $s_0, s_1, \ldots, s_r$ be nonnegative integers, not all zero. Consider the bialgebra homomorphism, called grading shift:

$$\Sigma_z : U_q(\widehat{\mathfrak{g}}^{\text{ext}}) \to U_q(\widehat{\mathfrak{g}}^{\text{ext}}) \otimes \mathbb{C}[z, z^{-1}]$$

$$\Sigma_z(E_i) = E_i \otimes z^{s_i}, \qquad \Sigma_z(F_i) = F_i \otimes z^{-s_i}, \qquad \Sigma_z(t_h) = t_h \otimes 1.$$

Examples: homogeneous grading: $s_0 = 1$, $s_1 = \ldots = s_r = 0$,
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By (Drinfeld, '86), the coefficients of the formal power series

$$R(z) := (\text{id} \otimes \Sigma_z)(R)$$

act on any tensor product $V \otimes W$ of fin.dim. $U_q(\widehat{\mathfrak{g}})$-modules V, W.
The resulting matrix-valued formal power series $R_{V,W}(z)$ satisfies YBE.

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Consider the ∞-dim. module $W((z)) := W \otimes \mathbb{C}((z))$ with $a \in U_q(\widehat{\mathfrak{g}})$ acting as $\Sigma_z(a)$. We obtain a $U_q(\widehat{\mathfrak{g}})$-intertwiner

$$\check{R}_{V,W}(z) := \text{flip} \cdot R_{V,W}(z) : V \otimes W((z)) \to W((z)) \otimes V.$$

K-matrices as formal series

Theorem 2 [AV22]

For every quantum affine symmetric pair $(U_q(\widehat{\mathfrak{g}}^{\text{ext}}), B_\theta)$ consider the standard cylindrical structure (ψ, J, K).

1. There exists a grading shift Σ_z^θ such that $\Sigma_z^\theta \circ \psi = \psi \circ \Sigma_{z^{-1}}^\theta$.
2. The coefficients of the formal Laurent series

$$K(z) := \Sigma_z^\theta(K)$$

   have a well-defined action on any fin.dim. $U_q(\widehat{\mathfrak{g}})$-module V. The resulting matrix-valued formal power series $K_V(z)$ is a B_θ-intertwiner: $V((z)) \to V((z))^\psi = V^\psi((z^{-1}))$ and satisfies the generalized RE.

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Towards a representation-theoretic approach for open Q-operators

At least for some B_θ, the coefficients of the series $K(z)$ actually act on any $U_q(\mathfrak{L}^{\geq 0})$-module, including those associated to Baxter Q-operators. In (A. Cooper, BV & R. Weston, '24) we develop this for the case $\theta = \omega \circ$ (diagram automorphism) for $U_q(\widehat{\mathfrak{sl}}_2)$.

Origin of trigonometric K-matrices

The B_θ intertwining condition means:

$$K_V(z) \cdot \Sigma^\theta_z(b)_V = \Sigma^\theta_{z^{-1}}(b)_{V^\psi} \cdot K_V(z) \qquad \text{for all } b \in B_\theta \qquad (*)$$

This is a consistent finite linear system defined over $\mathbb{C}(z) \subset \mathbb{C}((z))$.
Hence $\exists$ solution of (*) defined over $\mathbb{C}(z)$. Let's call it $K_V^{\text{trig}}(z)$.

Theorem 3 [AV22]

Let V be any *irreducible* finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module.

1. V is irreducible as a module over $U_q(\mathfrak{L}^-) = \mathbb{C}\langle F_0, F_1, \ldots, F_r\rangle$, cf. (Hernandez & Jimbo, '12).
2. $V \otimes \mathbb{C}((z))$ is irreducible as a B_θ-module for the *principal grading*.
3. The solution space of (*) is one-dimensional.
4. $K_V(z) =$ (Laurent series scalar) $\cdot K_V^{\text{trig}}(z)$ and hence $K_V^{\text{trig}}(z)$ satisfies the generalized RE.
5. Can make ψ involutive by gauge transforming. After rescaling,

$$K_V^{\text{uni}}(z)^{-1} = K_{V^\psi}^{\text{uni}}(z^{-1}) : V^\psi(z^{-1}) \to V(z).$$

Comments

- The 1-dimensionality of the solution space of (*) gives an effective method for computing trigonometric solutions of generalized REs as twisted B_θ-intertwiners for any irreducible fin.dim. $U_q(\widehat{\mathfrak{g}})$-module, cf. (L. Mezincescu & R. Nepomechie, '98) (G. Delius & A. George, '02) (G. Delius & N. Mackay, '03) (V. Regelskis & BV, '16). The RE automatically holds and the description via Satake diagrams can be used to methodically cover several cases.
- By a gauge transformation, solutions of the "original" RE are obtained for so-called *Kirillov-Reshetikhin modules*, subject to a combinatorial condition on the Satake diagram. See (H. Kusano, M. Okado & H. Watanabe, '24) for an alternative approach.

Example

In this way one obtains, for the q-Onsager algebra $B_\omega \subset U_q(\widehat{\mathfrak{sl}}_2)$,

$$K_V^{\text{trig}}(z) = \frac{\mu_0\mu_1 z}{(\mu_0\mu_1 - z)(\mu_0 z - \mu_1)} \begin{pmatrix} \sigma_0 - \sigma_1 z & z^{-1} - z \\ z - z^{-1} & \sigma_0 - \sigma_1 z^{-1} \end{pmatrix}$$

Open problem 1: classification of solutions of the reflection equation

Quantum affine *pseudo*-symmetric pairs are more general pairs $(U_q(\widehat{\mathfrak{g}}^{\text{ext}}), B_\theta)$, see [V. Regelskis & BV '21]. All results generalize.

Conjecture: given a trigonometric R-matrix for a quantum untwisted-affine algebra, all invertible *symmetrizable* solutions of the ordinary and crossed RE arise from the universal K-matrix associated to some quantum affine pseudo-symmetric pair.

Open problem 2: meromorphicity

The action of $R(z)$ on tensor product of fin.dim. modules is the series expansion of a meromorphic linear map (I. Frenkel & N. Reshetikhin '92) (P. Etingof & A. Moura '02). What about $K(z)$?

Open problem 3: infinite product

Universal R-matrices for quantum affine algebras have an infinite product factorization (V. Tolstoy & S. Khoroshkin, '92). What about universal K-matrices for quantum affine symmetric pairs?

1 Cylindrical structures on quasitriangular bialgebras

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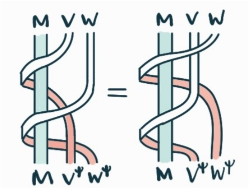

What do we want?

To allow nontrivial modules M of the boundary symmetry algebra B, which can form new B-modules by taking tensor products with A-modules. To have an enriched notion of a cylindrical structure: universal tensor K-matrix $\mathbb{K}$ acting on such tensor products.

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Axiomatics for quasitriangular bialgebras

The natural condition on B is the right coideal property $\Delta(B) \subset B \otimes A$. Then for a B-module M and an A-module V, can define action of $b \in B$ via $b_{M \otimes V} = \Delta(b)_{M,V}$.

Using graphical calculus again, now assigning B-modules to the "cylinder", can formulate following identities for $\mathbb{K} \in (B \otimes A)^{\times}$:

$$(\mathbb{K}1) \qquad \mathbb{K} \cdot \Delta(b) = (\mathrm{id} \otimes \psi)(\Delta(b)) \cdot \mathbb{K} \qquad \text{for all } b \in B,$$

$$(\mathbb{K}2) \qquad (\Delta \otimes \mathrm{id})(\mathbb{K}) = (R^{\psi})_{32} \cdot \mathbb{K}_{13} \cdot R_{23},$$

$$(\mathbb{K}3) \qquad (\mathrm{id} \otimes \Delta)(\mathbb{K}) = J_{23}^{-1} \cdot \mathbb{K}_{13} \cdot R_{23}^{\psi} \cdot \mathbb{K}_{12}.$$

To go "down" from a universal tensor K-matrix $\mathbb{K} \in B \otimes A$ to a basic universal K-matrix $K \in A$, force the trivial representation by acting with the counit map: $K = (\epsilon \otimes \mathrm{id})(\mathbb{K})$.

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Promoting K to $\mathbb{K}$

Conversely, a natural candidate for a universal tensor K-matrix is

$$\mathbb{K} := (R^{\psi})_{21} \cdot (1 \otimes K) \cdot R.$$

Automatically, it satisfies the three proposed axioms $(\mathbb{K}1) - (\mathbb{K}3)$. The only thing that needs to be checked is:

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Note: for cylindrical structures supported on B we obtain a second proof of the universal RE:

$$\begin{aligned} K_1 \cdot (R^{\psi})_{21} \cdot K_2 \cdot R &= (\psi \otimes \mathrm{id})\big((R^{\psi})_{21} \cdot K_2 \cdot R\big) \cdot K_1 \\ &= (R^{\psi\psi})_{21} \cdot K_2 \cdot R^{\psi} \cdot K_1. \end{aligned}$$

Theorem 4 [AV24]

1. For a quantum symmetric Kac-Moody pair $(U_q(\mathfrak{L}), B_\theta)$, the standard cylindrical structure (J, ψ, K) is supported on B_θ. Then $\mathbb{K} = (R^\psi)_{21} \cdot (1 \otimes K) \cdot R$ acts on tensor products of "weight B_θ-modules" and modules in $\mathcal{O}$ with a locally nilpotent action of those F_i fixed by θ.
2. In the affine case, the coefficients of the formal Laurent series

$$(\mathrm{id} \otimes \Sigma_z^\theta)(\mathbb{K}) = R(z)_{21}^\psi \cdot K(z)_2 \cdot R(z)$$

act on tensor products of weight B_θ-modules and fin.dim. $U_q(\widehat{\mathfrak{g}})$-modules.

- (S. Kolb, '20): tensor K for quantum symmetric pairs of finite type;
- (S. Kolb & M. Yakimov, '20): a very general approach for symmetric pairs based on Drinfeld doubles of Nichols algebras;
- (G. Lemarthe, P. Baseilhac & A. Gainutdinov, '23): the same axiomatic framework for comodule algebras, applied to an extension of the q-Onsager algebra. Also see G. Lemarthe's PhD thesis ('24).

A possible strand of future work

1. (in progress) In addition, assume A is a *balanced Hopf algebra* and consider a tensor K-matrix $\mathbb{K} \in B \otimes A$. For finite-dimensional A-modules we can define universal 2-boundary transfer matrices

$$\tau_V = \mathrm{Tr}_V(1 \otimes \widetilde{K}) \cdot \mathbb{K} \in B.$$

   Here $\widetilde{K} \in A^\times$ is a "dual basic universal K-matrix".
2. In the case $(A, B) = (U_q(\widehat{\mathfrak{g}}), B_\theta)$ we should get $\tau_V(z) \in B_\theta((z))$ and a boundary analogue of the q-character map from (E. Frenkel & N. Reshetikhin, '99) (E. Frenkel & E. Mukhin, '01), giving refined tools to study finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules. The missing piece is a Harish-Chandra-type map for B_θ, relying on its Drinfeld loop presentation, see (M. Lu, W. Wang & W. Zhang, '21-'23).
3. Then boundary analogues could be explored for the works (D. Hernandez & M. Jimbo, '12) (E. Frenkel, D. Hernandez, '15) on prefundamental representation theory for $U_q(\widehat{\mathfrak{g}})$, TQ relations and spectra; a major tool in this approach is the theory of q-characters.