Exact loop densities in the O(1) dense loop model on an infinite cylinder of odd circumference

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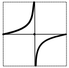
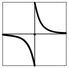
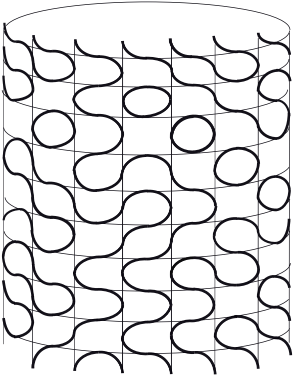
Outline

1. Problem
2. Motivation
3. Results
4. Solution

$O(n)$ dense loop model

Consider a two-dimensional square lattice. In every vertex put one out of two local vertex configurations equiprobably

 or .



Configuration of the model is a set of paths distributed according to a unnormalized measure in a finite domain

$$weight(C) = n^{\#\text{loops}}.$$

$$Z^{(DLM)} = \sum_{\eta \in \Omega} n^{\#loops(\eta)}$$

What is the average density of loops $\nu(L)$ per lattice size?

Outline

Motivation: study of a **phase transition**

- Critical Edge-percolation on **2d-square lattice** Potts Model at Q $= 1$

- $O(1)$ dense loop model XXZ quantum spin chain at $\Delta = \frac{1}{2}$

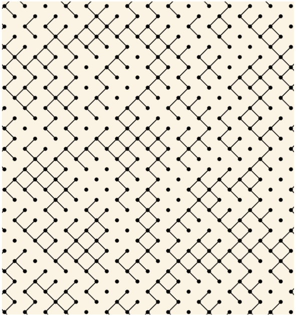
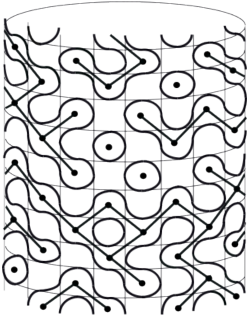


Figure: $p_c = \frac{1}{2}$

Observable: average number of connected clusters ρ
Boundary conditions: infinite cylinder of arbitrary circumference L

There is a one-to-one correspondence between configurations of

$O(1)$ DLM on a cylinder of size L



critical percolation configurations on square lattice rotated 45° on a cylinder of size $\frac{L}{2}$.

Even case $L = 2N$: $O(1)$ DLM $\longleftrightarrow$ critical percolation

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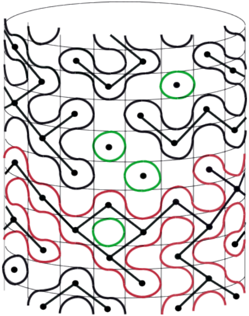


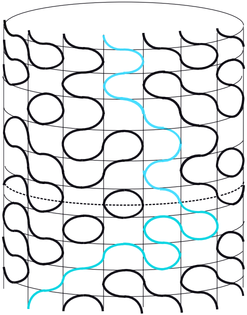
critical percolation configurations on square lattice rotated 45° on a cylinder of size $\frac{L}{2}$.

The density of loops $\nu(L)$ and of finite percolation clusters are equal.

$$\nu(L) = \nu_c(L) + \nu_{nc}(L) = \nu_{inscribed}(L) + \nu_{circumscribed}(L) + \nu_{nc}(L)$$
$$= \rho_c(L) + \rho_{nc}(L)$$

Odd case: Critical percolation $\longleftrightarrow$ $O(1)$ DLM

$L = 2N + 1$. There is one path going to infinity called **a defect line**. Thus, there is no non-contractible loops.



Can we still map it to any version of percolation?

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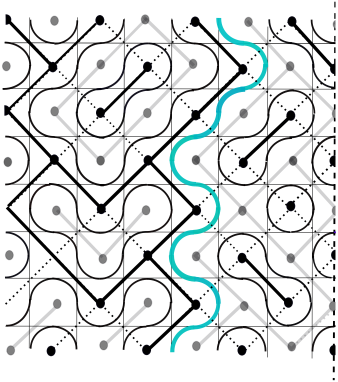

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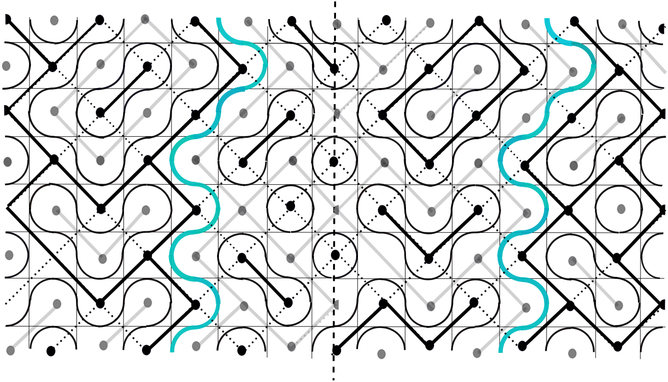

Can we still map it to any version of percolation? Yes! Place two copies!

$$\nu(L) = \nu_{inscribed}(L) + \nu_{circumscribed}(L) = \rho_{clusters}(L)$$

History of study $\nu(L)$

- approximate methods gave exact infinite plane limits of $\nu(L)$

  (Sykes, Essam, 1964; Baxter, Temperley and Ashley, 1978)

- exact methods at p_c (Baxter, 2016), in particular gave exact infinite plane limit

$$\lim_{L\to\infty} \nu(L) \approx 0.098076211$$

  (Lieb, Baxter, 1971, Baxter, Temperley, Ashley 1978)

- Coulomb gas theory predicts the universality in **the finite-size corrections** (depending on BC)

$$\nu(L) = \frac{3\sqrt{3}-5}{2} + \mathbf{C_1} L^{-2} + O(L^{-4}).$$

  (Ziff, Finch, 1997; Adamchik, Kleban, Ziff, 1998)

- The ground state of Markov chain associated with the $O(1)$ DLM has a remarkable combinatorial structure discovered by Razumov and Stroganov.

  (L. Cantini, A. Sportiello, J.de Gier, P. Zinn-Justin, P. di Francesco)

Outline

Results

Odd case $L = 2N + 1$.

$$\nu(2N+1) = \frac{1}{1+2N}\left(\frac{\Gamma(\frac{N}{2})\Gamma(\frac{3}{2}+\frac{3N}{2})}{\Gamma(\frac{3N}{2})\Gamma(\frac{1}{2}+\frac{N}{2})} + \frac{\Gamma(\frac{1}{2}+\frac{N}{2})\Gamma(2+\frac{3N}{2})}{\Gamma(1+\frac{N}{2})\Gamma(\frac{1}{2}+\frac{3N}{2})}\right) - \frac{5}{2}$$
$$= \frac{1}{12}, \frac{37}{400}, \frac{597}{6272}, \frac{2441}{25344}, \frac{78035}{805376}, \ldots.$$

Even case $L = 2N$. The density of loops/percolation clusters is given by the following formula (Povolotsky, 2021)

$$\nu_c(2N) = \frac{3\Gamma(\frac{N}{2})\Gamma(\frac{1}{2}+\frac{3N}{2})}{4\Gamma(\frac{3N}{2})\Gamma(\frac{1}{2}+\frac{N}{2})} + \frac{9\Gamma(\frac{1}{2}+\frac{N}{2})\Gamma(\frac{3N}{2})}{4\Gamma(\frac{N}{2})\Gamma(\frac{1}{2}+\frac{3N}{2})} - \frac{5}{2}.$$

$$\nu_{nc}(2N) = \frac{2^{2(N-2)}\Gamma(N)}{N\pi^2\Gamma(3N)}\left(3^{3N}\Gamma\left(\frac{N}{2}+\frac{1}{6}\right)^2\Gamma\left(\frac{N}{2}+\frac{5}{6}\right)^2 - \frac{12\pi^2\Gamma\left(\frac{3N}{2}\right)^2}{\Gamma\left(\frac{N}{2}\right)^2}\right)$$

Results

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$$= \frac{1}{12}, \frac{37}{400}, \frac{597}{6272}, \frac{2441}{25344}, \frac{78035}{805376}, \ldots.$$

This representation is suitable for an asymptotic expansion where the application of Stirling formula for the gamma functions gives

$$\nu(L) = \frac{3\sqrt{3}-5}{2} - \frac{1}{4\sqrt{3}}L^{-2} + \frac{35}{144\sqrt{3}}L^{-4} + O(L^{-6}).$$

comparing to

$$\nu_c(L) = \frac{3\sqrt{3}-5}{2} + \frac{1}{4\sqrt{3}}(2N)^{-2} - \frac{23}{48\sqrt{3}}(2N)^{-4} + O((2N)^{-6})$$

$$\nu_{nc}(L) = \frac{1}{\sqrt{3}}(2N)^{-2} - \frac{17}{18\sqrt{3}}(2N)^{-4} + O((2N)^{-6})$$

(consistent with the results for continuous-time DLM obtained and Raise and Peel model)

Outline

To calculate $\nu(L)$ define the free energy of DLM model

$$\boldsymbol{f_L(n)} = \lim_{H\to\infty} \frac{1}{HL} \sum_{\eta\in\Omega} n^{\# loops(\eta)}.$$

Then,

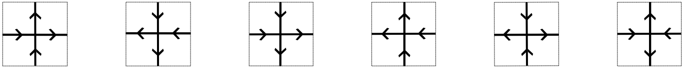
$$\nu(L) = \left(n \frac{d}{dn} \boldsymbol{f_L(n)} \right) \Big|_{n=1}.$$

With our observable and BC

Step 1: one can use $f_L^{(6V)}(n)$ instead due to relation to 6V model
Step 2: (algebraic) Bethe ansatz
Step 3: FSZ's solution of Baxter T-Q equation

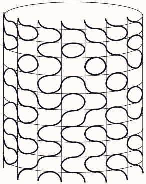
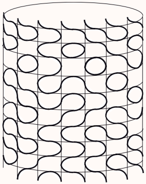
Step 1: $O(1)$ DLM $\Longleftrightarrow$ six-vertex model

Six-vertex model is a family of ice-type models on a sq. lattice with local vertex weights



a_1 a_2 b_1 b_2 c_1 c_2

Idea:

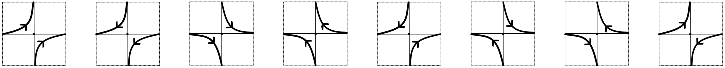
weight  = weight  + . . .

sum over all orientations of loops and a defect

The weight of an undirected loop $n = q + q^{-1}$ takes $n = 1$ at **the stochastic point** $q = e^{\frac{i\pi}{3}}$.

Step 1: $O(1)$ DLM $\Longleftrightarrow$ six-vertex model

The oriented configurations can be constructed of local or. vertices



Can we prescribe them local weights?

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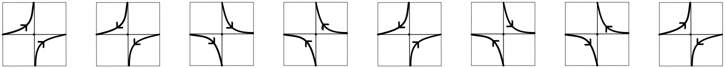
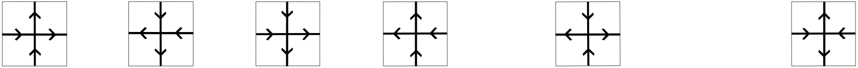


z z 1 1 $\mathrm{zq}^{\frac{1}{2}}$ $\mathrm{q}^{\frac{1}{2}}$ $\mathrm{q}^{\frac{1}{2}}$ $\mathrm{zq}^{-\frac{1}{2}}$

Can we prescribe them local weights? Yes!

If the weights $q^{\pm 1/4}$ are prescribed to every arc and the parameter z is auxiliary we have **asymmetric 6V model**

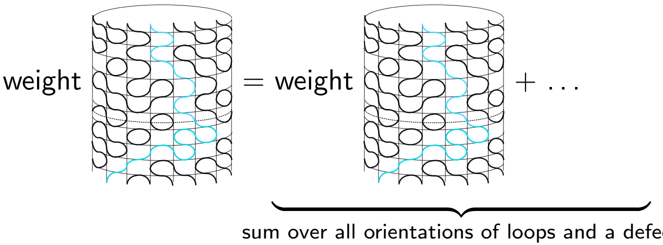


$a = z$ $a = z$ $b = 1$ $b = 1$ $c_1 = \mathrm{zq}^{\frac{1}{2}} + q^{-\frac{1}{2}}$ $c_2 = \mathrm{q}^{\frac{1}{2}} + zq^{-\frac{1}{2}}$

$\Rightarrow$ Attach the orientation of arcs to the bonds

$\Leftarrow$ Interpret the local bond directions as the ones of the local paths, which can be uncoupled

Remark on Step 1: $O(1)$ DLM $\Longleftrightarrow$ six-vertex model



The weights of the defects

were set equal to one in $O(1)$ DLM. Now it is **two**.

This discrepancy does not affect the infinite cylinder limit of the average values of quantities like the loop densities, to which only the configurations with **a single defect** bring a non-vanishing contribution.

$$\tilde{Z}^{(DLM)}(n) = Z^{(6V)}(q, 1).$$

tilda stands for modified $O(n)$ model = defect weights 2

Step 2: Algebraic Bethe ansatz of equiv. **symmetric** 6V

Define a space $\mathcal{H} = (\mathbb{C}^2)^{\otimes L}$ that spans a spin basis $\{\uparrow, \downarrow\}^{\otimes L}$. BA constructs a row-to-row transfer matrix $\mathbb{T}_L^{6V}(z) : \mathcal{H} \to \mathcal{H}$

$$\mathbb{T}_L^{(6V)}(u) = tr_0 \left(R_{0L}(u) \dots R_{02}(u) R_{01}(u) \right)$$

in terms of

$$R(u) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \qquad c_1 c_2 = c^2, \Delta = -\frac{1}{2}$$

while the free energy

$$\boldsymbol{f_L(n)} = \lim_{H \to \infty} \frac{1}{LH} \log tr \left(\mathbb{T}_L^{(6V)}(n) \right)^H = \frac{1}{L} \log \Lambda_{max}^{(6V)}(1).$$

Bethe ansatz gives eigenvalues, but

Which eigenvalue is the dominant one ?

Conservation of spins in every horizontal row results in $\mathcal{H} = \oplus_M \mathcal{H}_M$

- Yang and Yang '66, Lieb'67; for $|\Delta| \leq 1$ $\Lambda^{(6V)}_{max}(u)$ **asymptotically** belongs to $\mathcal{H}_{L/2}$ **in the odd case $\Lambda^{(6V)}_{L/2}(u)$ is doubly degenerate**
- still no rigorous proofs for finite L
- at the stochastic point $\mathbb{T}^{(6V)}_L(u)$ can be reduced to $\mathbb{T}^{(DLM)}_L(u)$ (written in a basis of link patterns) which is the transition prob. matrix of a Markov chain

(Here, sp. par. is changed $z = \frac{u-q}{1-qu}$). The eigenvalues are

$$\Lambda_M(u) = \left(\frac{u-q}{1-qu}\right)^L \prod_{j=1}^{M} \frac{u_j - q^2 u}{q(u-u_j)} + \prod_{j=1}^{M} \frac{u - q^2 u_j}{q(u_j - u)},$$

in terms of $u_1, \ldots, u_M$, a solution of Bethe equations

$$\left(\frac{u_j - q}{1 - qu_j}\right)^L = (-)^{M-1} \prod_{k=1}^{M} \frac{u_j - q^2 u_k}{u_k - q^2 u_j}, \quad j = 1, \ldots M.$$

Step 3: Fridkin-Stroganov-Zagier's solution

The Bethe equatinns can be reformulated in terms of a Q-polynomial

$$Q(u) = \prod_{k=1}^{M} (u - u_k).$$

as one functional equation for polynomials $T(u)$ and $Q(u)$

$$T(u)Q(u) = \phi(q^{-1}u)Q(q^2u) + \phi(qu)Q(q^{-2}u)(-q)^{2M-L},$$

where $\phi(u) := (1-u)^L, \quad T(u) := \boldsymbol{\Lambda(u)}\phi(qu)(-q)^{M-L}.$

FSZ's solution at stochastic point $q^3 = 1$

T-Q eqn is equivalent to the homogenious system in Q_k. It has rank **one**.

$$T_k Q_k = \phi_k Q_{k+1} + \phi_{k+1} Q_{k-1} q^2, \quad k = 0, 1, 2.$$

$$T_k = q\phi_{k-1} \iff T(u) = q(1+u)^L,$$

while $Q(u)$ is found in terms of **gamma functions.**

Step 3: Fridkin-Stroganov-Zagier's solution

Introducing another polynomial $P(u)$ of degree $L-M$ one has

$$T(u)P(u) = (-q)^{2M-L}\phi(q^{-1}u)P(q^2u) + \phi(qu)P(q^{-2}u).$$

Quantum polynomial Wronskian relation

$$\phi(u) = \frac{Q(qu)P(q^{-1}u) - Q(q^{-1}u)P(qu)(-q)^{2M-L}}{(-q)^{2M-L}q - q^{-1}}.$$

Then,

$$T(u) = \frac{Q(q^2u)P(q^{-2}u) - Q(q^{-2}u)P(q^2u)(-q)^{4M-2L}}{(-q)^{2M-L}q - q^{-1}}.$$

(coincides with eqs for $\Delta = -\frac{1}{2}$ XXZ model found by Stroganov)

$$\nu = \frac{1}{2} + \frac{1}{2Lq(1+q)} + \frac{1}{L(1+q)}\left[\frac{d}{dq}\ln T(1)\right]\Big|_{q=e^{\frac{i\pi}{3}}}$$

($T(u)$ is substituted in terms of $Q(u)$ and $P(u)$ found in terms of **gamma functions**. Result follows.)

Thank You!