Lagrangian multiforms

paths from classical to quantum

Mats Vermeeren

RAQIS
Annecy

September 3, 2024

Lagrangian multiform theory

Variational principle for classical integrable systems, applicable to:

- Liouville-integrable ODEs
- Hierarchies of integrable PDEs

  E.g. KdV, AKNS, KP, ...
- Semi-discrete systems

  E.g. Toda lattice
- Fully discrete systems

  Integrable maps, partial difference equations

Main contributors:

- Frank Nijhoff, Vincent Caudrelier (University of Leeds)
- Yuri Suris (TU Berlin)

Contents

Contents

Liouville integrability

A Hamiltonian system with Hamilton function $H : T^*Q \cong \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \ldots H_N$ such that $\{H_i, H_j\} = 0$.

- Each H_i defines its own flow $\phi^t_{H_j}$: N dynamical systems
- Each H_i is a conserved quantity for all flows
- Each common level set (if compact and nondegenerate) is a torus
- The flows commute

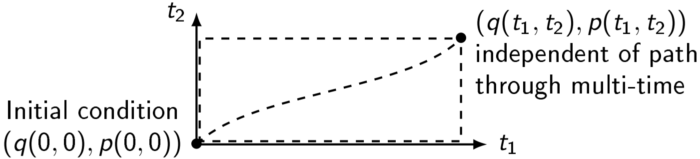
Liouville integrability

A Hamiltonian system with Hamilton function $H : T^*Q \cong \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \ldots H_N$ such that $\{H_i, H_j\} = 0$.

- Each H_i defines its own flow $\phi^t_{H_j}$: N dynamical systems
- Each H_i is a conserved quantity for all flows
- Each common level set (if compact and nondegenerate) is a torus
- The flows commute

We can consider (q, p) as a function of multi-time, $\mathbb{R}^N \to T^*Q$:

$$(t_1, \ldots, t_N) \mapsto (q(t_1, \ldots, t_N), p(t_1, \ldots, t_N))$$



Variational principle for commuting flows

Suppose we have Lagrange functions L_i associated to H_i. Consider

$$q : \mathbb{R}^N \to Q \quad \text{(multi-time to configuration space)}$$

Variational principle for commuting flows

Suppose we have Lagrange functions L_i associated to H_i. Consider

$$q : \mathbb{R}^N \to Q \quad \text{(multi-time to configuration space)}$$

Pluri-Lagrangian principle

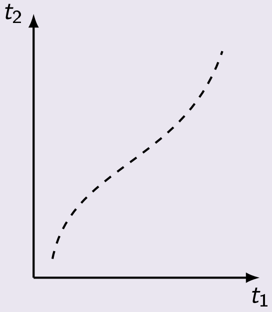
Combine the L_i into a 1-form

$$\mathcal{L}[q] = \sum_{i=1}^{N} L_i[q] \, \mathrm{d} t_i .$$

Look for dynamical variables $q(t_1, \ldots, t_N)$ such that the action

$$S_\Gamma = \int_\Gamma \mathcal{L}[q]$$

is critical w.r.t. variations of q, simultaneously over every curve Γ in multi-time $\mathbb{R}^N$



The Lagrangian multiform principle considers variations of the *curve* too

Variational principle for commuting flows

Suppose we have Lagrange functions L_i associated to H_i. Consider

$$q : \mathbb{R}^N \to Q \quad \text{(multi-time to configuration space)}$$

Pluri-Lagrangian principle

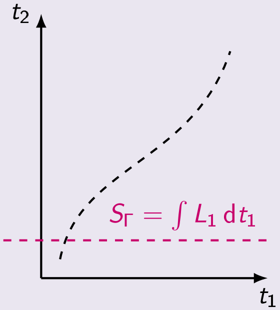
Combine the L_i into a 1-form

$$\mathcal{L}[q] = \sum_{i=1}^{N} L_i[q]\, \mathrm{d} t_i .$$

Look for dynamical variables $q(t_1, \ldots, t_N)$ such that the action

$$S_\Gamma = \int_\Gamma \mathcal{L}[q]$$

is critical w.r.t. variations of q, simultaneously over every curve Γ in multi-time $\mathbb{R}^N$



The Lagrangian multiform principle considers variations of the *curve* too

Variational principle for commuting flows

Suppose we have Lagrange functions L_i associated to H_i. Consider

$$q : \mathbb{R}^N \to Q \quad \text{(multi-time to configuration space)}$$

Pluri-Lagrangian principle

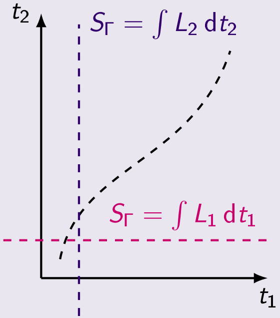
Combine the L_i into a 1-form

$$\mathcal{L}[q] = \sum_{i=1}^{N} L_i[q]\,\mathrm{d}t_i.$$

Look for dynamical variables $q(t_1, \ldots, t_N)$ such that the action

$$S_\Gamma = \int_\Gamma \mathcal{L}[q]$$

is critical w.r.t. variations of q, simultaneously over every curve Γ in multi-time $\mathbb{R}^N$



The Lagrangian multiform principle considers variations of the *curve* too

Multi-time Euler-Lagrange equations

Assume that

$$L_1[q] = L_1(q, q_{t_1}),$$
$$L_i[q] = L_i(q, q_{t_1}, q_{t_i}), \quad i \neq 1$$

Then the multi-time Euler-Lagrange equations for

$$\mathcal{L} = \sum_i L_i[q]\, \mathrm{d}t_i$$

are:

Usual Euler-Lagrange equations: $\dfrac{\partial L_i}{\partial q} - \dfrac{\mathrm{d}}{\mathrm{d}t_i}\dfrac{\partial L_i}{\partial q_{t_i}} = 0$

? : $\dfrac{\partial L_i}{\partial q_{t_1}} = 0, \qquad i \neq 1$

Compatibility conditions: $\dfrac{\partial L_i}{\partial q_{t_i}} = \dfrac{\partial L_j}{\partial q_{t_j}}$

Example: Kepler Problem

Take

$$L_1 = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

$$L_2 = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot \hat{v} \qquad (\hat{v} \text{ fixed unit vector})$$

In general, for systems of Newtonian type, $L_i = q_{t_1} q_{t_i} - H_i(q, q_{t_1})$

Example: Kepler Problem

Take

$$L_1 = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

$$L_2 = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot \hat{v} \qquad (\hat{v} \text{ fixed unit vector})$$

In general, for systems of Newtonian type, $L_i = q_{t_1} q_{t_i} - H_i(q, q_{t_1})$

Multi-time Euler-Lagrange equations of $\mathcal{L} = L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$

$$\frac{\partial L_1}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_1} \frac{\partial L_1}{\partial q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_1 t_1} = -\frac{q}{|q|^3} \qquad \text{(Keplerian motion)}$$

$$\frac{\partial L_2}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_2} \frac{\partial L_2}{\partial q_{t_2}} = 0 \quad \Rightarrow \quad q_{t_1 t_2} = \hat{v} \times q_{t_1}$$

$$\frac{\partial L_2}{\partial q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_2} = \hat{v} \times q \qquad \text{(Rotation)}$$

$$\frac{\partial L_1}{\partial q_{t_1}} = \frac{\partial L_2}{\partial q_{t_2}} \quad \Rightarrow \quad q_{t_1} = q_{t_1}$$

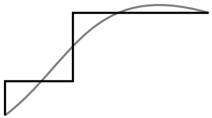
Derivation of the multi-time Euler-Lagrange equations

Consider a Lagrangian one-form $\mathcal{L} = \sum_i L_i[q] \, \mathrm{d}t_i$, with

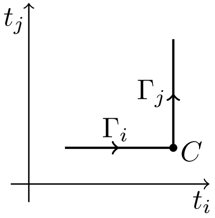
$$L_1[q] = L_1(q, q_{t_1}),$$
$$L_i[q] = L_i(q, q_{t_1}, q_{t_i}), \quad i \neq 1$$

Lemma

If the action $\int_\Gamma \mathcal{L}$ is critical on all stepped curves Γ in $\mathbb{R}^N$, then it is critical on all smooth curves.



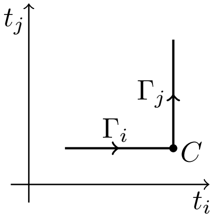
Variations are local, so it is sufficient to look at one corner $\Gamma = \Gamma_i \cup \Gamma_j$ at a time.



Derivation of the multi-time Euler-Lagrange equations

On one of the straight pieces, Γ_i $(i \neq 1)$, we get

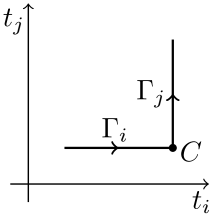
$$\delta \int_{\Gamma_i} L_i \, \mathrm{d}t_i = \int_{\Gamma_i} \left(\frac{\partial L_i}{\partial q} \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} + \frac{\partial L_i}{\partial q_{t_i}} \delta q_{t_i} \right) \mathrm{d}t_i$$



Derivation of the multi-time Euler-Lagrange equations

On one of the straight pieces, Γ_i $(i \neq 1)$, we get

$$\delta \int_{\Gamma_i} L_i \, \mathrm{d}t_i = \int_{\Gamma_i} \left(\frac{\partial L_i}{\partial q} \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} + \frac{\partial L_i}{\partial q_{t_i}} \delta q_{t_i} \right) \mathrm{d}t_i$$



Integration by parts (wrt t_i only) yields

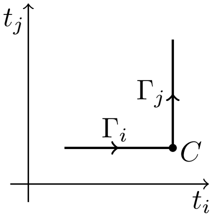
$$\delta \int_{\Gamma_i} L_i \, \mathrm{d}t_i = \int_{\Gamma_i} \left(\left(\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_i}} \right) \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} \right) \mathrm{d}t_i + \frac{\partial L_i}{\partial q_{t_i}} \delta q \bigg|_C$$

Since p is an interior point of the curve, we cannot set $\delta q(C) = 0$!

Derivation of the multi-time Euler-Lagrange equations

On one of the straight pieces, Γ_i ($i \neq 1$), we get

$$\delta \int_{\Gamma_i} L_i \, \mathrm{d}t_i = \int_{\Gamma_i} \left(\frac{\partial L_i}{\partial q} \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} + \frac{\partial L_i}{\partial q_{t_i}} \delta q_{t_i} \right) \mathrm{d}t_i$$



Integration by parts (wrt t_i only) yields

$$\delta \int_{\Gamma_i} L_i \, \mathrm{d}t_i = \int_{\Gamma_i} \left(\left(\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_i}} \right) \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} \right) \mathrm{d}t_i + \left. \frac{\partial L_i}{\partial q_{t_i}} \delta q \right|_C$$

Since p is an interior point of the curve, we cannot set $\delta q(C) = 0$!

Arbitrary δq and δq_{t_1}, so we find:

Multi-time Euler-Lagrange equations

$$\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_i}} = 0, \qquad \frac{\partial L_i}{\partial q_{t_1}} = 0, \qquad \frac{\partial L_i}{\partial q_{t_i}} = \frac{\partial L_j}{\partial q_{t_j}}$$

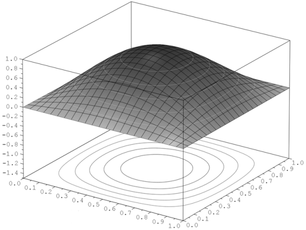
Variational principle for PDEs ($d = 2$)

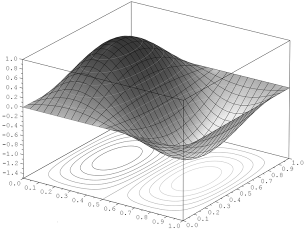
Pluri-Lagrangian principle

Given a 2-form $\mathcal{L}[q] = \sum_{i,j} L_{ij}[q] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$, find $q : \mathbb{R}^N \to \mathbb{R}$, such that

$\int_\Gamma \mathcal{L}[q]$ is critical on all surfaces Γ in multi-time $\mathbb{R}^N$,

with respect to variations of q.





Example: potential KdV hierarchy. We obtain evolutionary equations:

$$q_{t_2} = q_{xxx} + 3q_x^2,$$

$$q_{t_3} = q_{xxxxx} + 10 q_x q_{xxx} + 5 q_{xx}^2 + 10 q_x^3, \qquad \dots$$

Exterior derivative of $\mathcal{L}$

Revisit the Kepler problem: $\mathcal{L} = L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$ with

$$L_1[q] = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

$$L_2[q] = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot \hat{v} \qquad (\hat{v} \text{ fixed unit vector})$$

Multi-time Euler-Lagrange equations:

$$q_{t_1 t_1} = -\frac{q}{|q|^3}$$

$$q_{t_2} = \hat{v} \times q$$

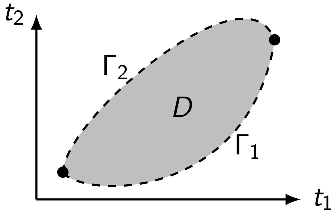
Coefficient of $\mathrm{d}\mathcal{L}$

$$\frac{\mathrm{d}L_2}{\mathrm{d}t_1} - \frac{\mathrm{d}L_1}{\mathrm{d}t_2} = \left(q_{t_1 t_1} + \frac{q}{|q|^3} \right) (q_{t_2} - \hat{v} \times q)$$

$\mathrm{d}\mathcal{L}$ has a double zero on solutions.

Interpretation of closedness condition

If $\mathrm{d}\mathcal{L} = 0$, the action is invariant wrt variations in geometry



$$\int_{\Gamma_1} \mathcal{L} - \int_{\Gamma_2} \mathcal{L} = \int_D \mathrm{d}\mathcal{L} = 0$$

Lagrangian multiform principle

Require that

- pluri-Lagrangian principle holds (action critical wrt variations of q),
- deforming the curve of integration leaves action invariant.

Interpretation of closedness condition

$\mathrm{d}\mathcal{L}$ provides an alternative derivation of the EL equations:

WLOG, we can restrict the variational principle to simple closed curves, i.e. boundaries of a surface D.

Then

$$\delta \int_{\partial D} \mathcal{L} = - \int_D \delta \mathrm{d}\mathcal{L},$$

hence the pluri-Lagrangian principle is equivalent to $\delta \mathrm{d}\mathcal{L} = 0$.

Interpretation of closedness condition

$\mathrm{d}\mathcal{L}$ provides an alternative derivation of the EL equations:

WLOG, we can restrict the variational principle to simple closed curves, i.e. boundaries of a surface D.

Then

$$\delta \int_{\partial D} \mathcal{L} = - \int_D \delta \mathrm{d}\mathcal{L},$$

hence the pluri-Lagrangian principle is equivalent to $\delta \mathrm{d}\mathcal{L} = 0$.

$$\updownarrow$$

Double zero property:

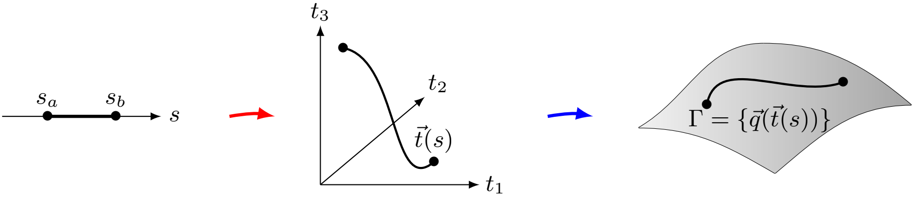
$$\mathrm{d}\mathcal{L} = \sum (\ldots)(\ldots)$$

Contents

Path integrals in multitime and configuration space

Nijhoff (INI meeting 2013) proposed a multi-time propagator of the form

$$K(\vec{q}_b, \vec{t}_b, s_b; \vec{q}_a, \vec{t}_a, s_a) = \int_{\vec{t}(s_a)=\vec{t}_a}^{\vec{t}(s_b)=\vec{t}_b} [\mathcal{D}\vec{t}(s)] \int_{\vec{q}(\vec{t}_a)=\vec{q}_a}^{\vec{q}(\vec{t}_b)=\vec{q}_b} [\mathcal{D}\vec{q}(\vec{t})] \exp\left(\frac{i}{\hbar} \int_\Gamma \mathcal{L}[\vec{q}] \right)$$

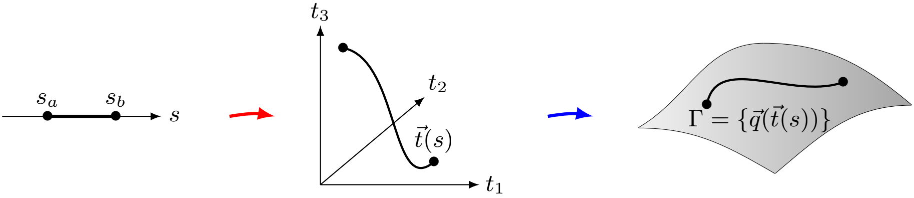


Path in multi-time   Path in configuration space

Path integrals in multitime and configuration space

Nijhoff (INI meeting 2013) proposed a multi-time propagator of the form

$$K(\vec{q}_b, \vec{t}_b, s_b; \vec{q}_a, \vec{t}_a, s_a) = \int_{\vec{t}(s_a)=\vec{t}_a}^{\vec{t}(s_b)=\vec{t}_b} [\mathcal{D}\vec{t}(s)] \int_{\vec{q}(\vec{t}_a)=\vec{q}_a}^{\vec{q}(\vec{t}_b)=\vec{q}_b} [\mathcal{D}\vec{q}(\vec{t})] \exp\left(\frac{i}{\hbar} \int_\Gamma \mathcal{L}[\vec{q}] \right)$$

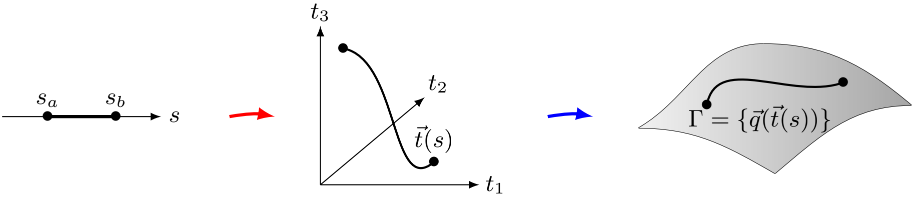


Path in multi-time   Path in configuration space

- Treats dependent variables and independent variables the same way
- What is the physical meaning of such a propagator?

Multiform propagators for harmonic oscillators

[King & Nijhoff. Quantum variational principle and quantum multiform structure: the case of quadratic Lagrangians. Nuclear Physics B, 2019]



For quadratic Lagrangians, the closure property (path independence) does not need equations of motion, so all paths Γ in multi-time have the same action:

$$\int_{\vec{t}(s_a)=\vec{t}_a}^{\vec{t}(s_b)=\vec{t}_b} \left[\mathcal{D}\vec{t}(s)\right] = \mathrm{Id}$$

Hence:

$$K(\vec{q}_b, \vec{t}_b, s_b; \vec{q}_a, \vec{t}_a, s_a) = \int_{\vec{q}(\vec{t}_a)=\vec{q}_a}^{\vec{q}(\vec{t}_b)=\vec{q}_b} \left[\mathcal{D}\vec{q}(\vec{t})\right] \exp\left(\frac{i}{\hbar}\int_\Gamma \mathcal{L}[\vec{q}]\right)$$

Semi-classical approximation to avoid path-dependence

[Kongkoom Yoo-kong. Quantum integrability: Lagrangian 1-form case. Nuclear Physics B, 2023]

Semi-classical approximation: only count contributions of classical solutions toward the path integral.

Use the property that $\mathrm{d}\mathcal{L} = 0$ on classical solutions to remove the integral

$$\int_{\vec{t}(s_a)=\vec{t}_a}^{\vec{t}(s_b)=\vec{t}_b} \left[\mathcal{D}\vec{t}(s)\right]$$

This also leads to the propagator

$$K(\vec{q}_b, \vec{t}_b, s_b; \vec{q}_a, \vec{t}_a, s_a) = \int_{\vec{q}(\vec{t}_a)=\vec{q}_a}^{\vec{q}(\vec{t}_b)=\vec{q}_b} \left[\mathcal{D}\vec{q}(\vec{t})\right] \exp\left(\frac{i}{\hbar}\int_\Gamma \mathcal{L}[\vec{q}]\right)$$

Can you really study integrable systems in semi-classical approximation?

When is semiclassical approximation exact?

- Geodesics on SO(3)

  [Schulman. A path integral for spin. Physical Review. 1968]

- Geodesics on Lie groups

  [Dowker. When is the 'sum over classical paths' exact? Journal of Physics A: General Physics. 1970]

Is exactness of semiclassical approximation an attribute of integrability?

Or is this too restrictive?

Quantum system with classical symmetries

In the classical setting, $\mathrm{d}\mathcal{L}$ has a double zero on solutions:
$\mathrm{d}\mathcal{L} = \sum(\ldots)(\ldots)$ where each $(\ldots)$ vanishes on solutions.

Hence $\mathrm{d}\mathcal{L} = 0$ and the action S_Γ is independent of the path Γ in multi-time as soon as all but one of the equations of motion are satisfied.

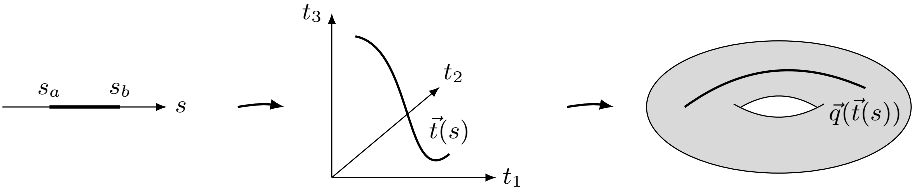
Quantum system with classical symmetries

In the classical setting, $d\mathcal{L}$ has a double zero on solutions:
$d\mathcal{L} = \sum(\ldots)(\ldots)$ where each $(\ldots)$ vanishes on solutions.

Hence $d\mathcal{L} = 0$ and the action S_Γ is independent of the path Γ in multi-time as soon as all but one of the equations of motion are satisfied.

One quantum Hamiltonian with N classical symmetries

- Classical symmetries $\Rightarrow d\mathcal{L} = 0$
  $\Rightarrow$ all (homotopy-equiv) paths on a Liouville torus have same action.
  $\Rightarrow$ Multi-time propagator independent of path through multi-time.
- Pick the "easiest" path to evaluate the physical t_1-propagator.

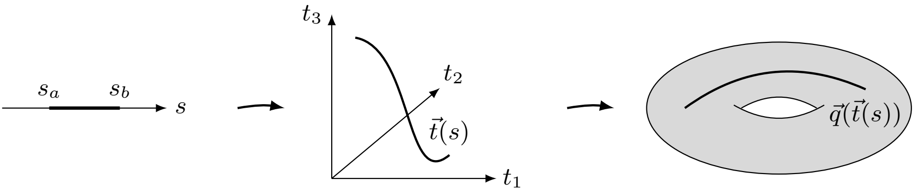


Path integral with fixed energies

All phase space paths that lie in a common level set of the classical integrals of motion ("energies") have the same action/propagator.

Maupertuis' principle: the classical trajectory is critical, among all trajectories of the same fixed energy, for the action $\int p \, \mathrm{d}q$

Is there a path integral formulation based on Maupertuis' principle?



Path in multi-time Path in configuration space

Contents

Summary

- Lagrangian multiform theory well-studied in classical domain (ODEs, PDEs, discrete systems)
- Initial attempts at quantum formulation (multi-time propagator) are of limited scope, their physical interpretation is not fully clear
- Idea: use multiforms as a tool to evaluate traditional path integrals

Summary

- Lagrangian multiform theory well-studied in classical domain (ODEs, PDEs, discrete systems)
- Initial attempts at quantum formulation (multi-time propagator) are of limited scope, their physical interpretation is not fully clear
- Idea: use multiforms as a tool to evaluate traditional path integrals

Questions for the audience:

- Exactness of semi-classical approximation vs integrability?
- Path integrals à la Maupertuis?

Summary

- Lagrangian multiform theory well-studied in classical domain (ODEs, PDEs, discrete systems)
- Initial attempts at quantum formulation (multi-time propagator) are of limited scope, their physical interpretation is not fully clear
- Idea: use multiforms as a tool to evaluate traditional path integrals

Questions for the audience:

- Exactness of semi-classical approximation vs integrability?
- Path integrals à la Maupertuis?

Thank you for your attention!

Selected references on Lagrangian multiform theory

Discrete Lagrangian 2-forms: Lobb, Nijhoff. Lagrangian multiforms and multidimensional consistency. J Phys A, 2009.

Discrete and continuous 1-forms: Suris. Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms. J Geom Mech, 2013

Continuous 2-forms: Suris, V. On the Lagrangian structure of integrable hierarchies. In: Advances in Discrete Differential Geometry, Springer, 2016.

Lax pairs: Sleigh, Nijhoff, Caudrelier. A variational approach to Lax representations. Journal of Geometry and Physics, 2019.

Hamiltonian vs Lagrangian perspectives: V. Hamiltonian structures for integrable hierarchies of Lagrangian PDEs. OCNMP, 2021.

Semi-discrete Lagrangian multiforms: Sleigh, V. Semi-discrete Lagrangian 2-forms and the Toda hierarchy. J Phys A, 2022.

Classical Gaudin models: Caudrelier, Dell'Atti, Singh. Lagrangian multiforms on coadjoint orbits for finite-dimensional integrable systems. LMP. 2024

Quantum multiforms: King, Nijhoff. Quantum variational principle and quantum multiform structure. Nuclear Physics B, 2019.

—— Kongkoom, Yoo-kong. Quantum integrability: Lagrangian 1-form case. Nuclear Physics B, 2023.

Higher order Lagranigans $L_i[q] = L_i(q, q_{t_i}, q_{t_i t_j}, \dots)$

For a string $I = t_{i_1} \dots t_{i_k}$ of time variables, denote the corresponding derivative by q_I.

If I is empty then $q_I = q$.

Denote by $\dfrac{\delta_i}{\delta q_I}$ the variational derivative in the direction of t_i wrt q_I:

$$\frac{\delta_i L_i}{\delta q_I} = \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\partial L_i}{\partial q_{I t_i^\alpha}}$$
$$= \frac{\partial L_i}{\partial q_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{I t_i}} + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial L_i}{\partial q_{I t_i^2}} - \dots$$

Multi-time Euler-Lagrange equations

Usual Euler-Lagrange equations: $\dfrac{\delta_i L_i}{\delta q_I} = 0 \qquad \forall I \not\ni t_i,$

Additional conditions: $\dfrac{\delta_i L_i}{\delta q_{I t_i}} = \dfrac{\delta_j L_j}{\delta q_{I t_j}} \qquad \forall I,$

Example: Potential KdV hierarchy

$$q_{t_2} = q_{xxx} + 3q_x^2,$$
$$q_{t_3} = q_{xxxxx} + 10q_x q_{xxx} + 5q_{xx}^2 + 10q_x^3,$$

where we identify $t_1 = x$.

Example: Potential KdV hierarchy

$q_{t_2} = q_{xxx} + 3q_x^2,$

$q_{t_3} = q_{xxxxx} + 10q_x q_{xxx} + 5q_{xx}^2 + 10q_x^3,$

where we identify $t_1 = x$.

The differentiated equations $q_{xt_i} = \frac{\mathrm{d}}{\mathrm{d}x}(\cdots)$ are Lagrangian with

$$L_{12} = \frac{1}{2}q_x q_{t_2} - \frac{1}{2}q_x q_{xxx} - q_x^3,$$
$$L_{13} = \frac{1}{2}q_x q_{t_3} - \frac{1}{2}q_{xxx}^2 + 5q_x q_{xx}^2 - \frac{5}{2}q_x^4.$$

Example: Potential KdV hierarchy

$q_{t_2} = q_{xxx} + 3q_x^2,$
$q_{t_3} = q_{xxxxx} + 10q_x q_{xxx} + 5q_{xx}^2 + 10q_x^3,$
where we identify $t_1 = x$.

The differentiated equations $q_{xt_i} = \frac{\mathrm{d}}{\mathrm{d}x}(\cdots)$ are Lagrangian with

$$L_{12} = \frac{1}{2}q_x q_{t_2} - \frac{1}{2}q_x q_{xxx} - q_x^3,$$
$$L_{13} = \frac{1}{2}q_x q_{t_3} - \frac{1}{2}q_{xxx}^2 + 5q_x q_{xx}^2 - \frac{5}{2}q_x^4.$$

A suitable coefficient L_{23} of

$$\mathcal{L} = L_{12}\,\mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13}\,\mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23}\,\mathrm{d}t_2 \wedge \mathrm{d}t_3$$

can be found (nontrivial task!).

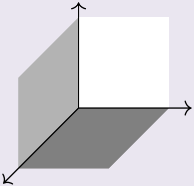
Multi-time EL equations

for $\mathcal{L}[q] = \sum_{i,j} L_{ij}[q] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$

$$\frac{\delta_{ij} L_{ij}}{\delta q_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta q_{It_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta q_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta q_{It_k t_i}} = 0 \qquad \forall I.$$



Where

$$\frac{\delta_{ij} L_{ij}}{\delta q_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial q_{It_i^\alpha t_j^\beta}}$$

Example: Potential KdV hierarchy

- The equations $\dfrac{\delta_{12} L_{12}}{\delta q} = 0$ and $\dfrac{\delta_{13} L_{13}}{\delta q} = 0$ yield

$$q_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(q_{xxx} + 3q_x^2\right),$$
$$q_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x}\left(q_{xxxxx} + 10 q_x q_{xxx} + 5q_{xx}^2 + 10q_x^3\right).$$

- The equations $\dfrac{\delta_{12} L_{12}}{\delta q_x} = \dfrac{\delta_{32} L_{32}}{\delta q_{t_3}}$ and $\dfrac{\delta_{13} L_{13}}{\delta q_x} = \dfrac{\delta_{23} L_{23}}{\delta q_{t_2}}$ yield

$$q_{t_2} = q_{xxx} + 3q_x^2,$$
$$q_{t_3} = q_{xxxxx} + 10 q_x q_{xxx} + 5q_{xx}^2 + 10q_x^3,$$

  the evolutionary equations!

- All other multi-time EL equations are corollaries of these.