From Set-Theoretical Solutions of the Braid Equation to Left Shelves.

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Credits

A. Doikou, BR, P. Stefanelli,
Quandles as pre-Lie skew braces, set-theoretic Hopf algebras & universal R-matrices

Braid equation

Definition

Let X be a set, a **set-theoretic solution of braid equation** is a map $r : X \times X \to X \times X$ such that

$$(r \times id)(id \times r)(r \times id) = (id \times r)(r \times id)(id \times r).$$

We say that $r(a, b) := (\sigma_a(b), \tau_b(a))$, for $a, b \in X$, is **left non-degenerate** if σ_a is a bijection for all $a \in X$.

What we call set-theoretic solution of braid equation is also called set-theoretic solution of Yang-Baxter equation or third Reidemeister move.

Simplified history

- V.G. Drinfel'd, On some unsolved problems in quantum group theory, in: Quantum groups (Leningrad, 1990), vol. 1510 of Lecture Notes in Math., Springer, Berlin, (1992), pp. 1–8.
- Groups and YBE (90's), P. Etingof, T. Schedler, A. Soloviev, T. Gateva-Ivanova, S. Majid, J.-H. Lu, M. Yan, Y.-C. Zhu and more
- Braces and Skew braces 2007 W. Rump, 2017 Guarnieri & Vendramin, and many more.
- 1940's 50's 80's Third Reidemeister move and invariants of knots (Quandles), M. Takasaki, J. Conway, D. Joyce, S. Matveev.

Shelves

Definition

Let X be a non-empty set and $\triangleright$ a binary operation on X. Then, the pair $(X, \triangleright)$ is said to be a *left shelf* if $\triangleright$ is left self-distributive, namely, the identity

$$a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c) \tag{1}$$

is satisfied, for all $a, b, c \in X$. Moreover, a left shelf $(X, \triangleright)$ is called

1. a *left spindle* if $a \triangleright a = a$, for all $a \in X$;
2. a *left rack* if $(X, \triangleright)$ if for every $a, b \in X$ exists $c \in X$ such that $c \triangleright a = b$.
3. a *quandle* if $(X, \triangleright)$ is both a left spindle and a left rack.

Definition

If $(X, \triangleright)$ and $(Y, \blacktriangleright)$ are left shelves, a map $f : X \to Y$ is said to be a *shelf homomorphism* if $f(a \triangleright b) = f(a) \blacktriangleright f(b)$, for all $a, b \in X$.

Derived solutions

$$(X,\, \triangleright) - \text{left shelf} \longrightarrow r_{\triangleright}(a,b) = (b,\ b \triangleright a) - \text{l.n-d.s}$$

$$(X,r) - \text{l.n-d.s} \longrightarrow (X,\, \triangleright_r) - \text{left shelf}$$

$$a \triangleright_r b := \sigma_a \tau_{\sigma_b^{-1}(a)}(b)\,, \ \text{for all } a,b \in X.$$

A left non-degenerate solution (X,r) is bijective if and only if $(X,\triangleright_r)$ is a left rack.

D-homomorphisms

Definition

Let (X, r) and (Y, s) be solutions. Then we say that a map $\varphi : X \times X \to Y \times Y$ is a *Drinfel'd homomorphism* or in short *D-homomorphism* if

$$\varphi\, r = s\, \varphi.$$

If φ is a bijection, we call φ a *D-isomorphism* and we say that (X, r) and (Y, s) are *D-isomorphic (via φ)*, and we denote it by $r \cong_D s$.

Lemma

Let (X, r) be a left non-degenerate solution and $(X, r_{\triangleright_r})$ be the derived solution of (X, r). Then r is D-isomorphic to $r_{\triangleright_r}$ with $\varphi(a, b) = (a, \sigma_a(b))$

Twists

Definition

Let $(X, \triangleright)$ be a left shelf. We say that $\varphi : X \to \mathrm{Aut(X, \triangleright)}$, $a \mapsto \varphi_a$ is a *twist* if for all $a, b \in X$,

$$\varphi_a \varphi_b = \varphi_{\varphi_a(b)} \varphi_{\varphi^{-1}_{\varphi_a(b)}(\varphi_a(b) \triangleright (a))}. \tag{2}$$

Theorem

Let $(X, \triangleright)$ be a left shelf and $\varphi : X \to \mathrm{Sym}_X$, $a \mapsto \varphi_a$. Then, the function $r_\varphi : X \times X \to X \times X$ defined by

$$r_\varphi(a, b) = \left(\varphi_a(b), \varphi^{-1}_{\varphi_a(b)}(\varphi_a(b) \triangleright a)\right), \tag{3}$$

for all $a, b \in X$, is a solution if and only if φ is a twist. Moreover, any left non-degenerate solution can be obtained that way.

Skew braces

Definition (W. Rump, L. Guarnieri & L. Vendramin)

A *left skew brace* is a set B together with two group operations $+, \circ : B \times B \to B$, the first is called addition and the second is called multiplication, such that for all $a, b, c \in B$,

$$a \circ (b + c) = a \circ b - a + a \circ c. \tag{4}$$

If $+$ is an abelian group operation B is called a *left brace*. Moreover, if B is a left skew brace and for all $a, b, c \in B$ $(b + c) \circ a = b \circ a - a + c \circ a$, then B is called a *skew brace*. Analogously if $+$ is abelian and B is a skew brace, then B is called a *brace*.

The additive identity of a skew brace B will be denoted by 0 and the multiplicative identity by 1. In every skew brace $0 = 1$.

From skew braces to solutions

Theorem

Let B be a left skew brace and $z, z_1, z_2 \in B$ be such that for all $a, b \in B$

$$(a + b) \circ z_i = a \circ z_i - z_i + b \circ z_i$$

and that there exists $c_1, c_2 \in B$ such that

$$a \circ z_2 \circ z_1 - a \circ z = z_2 \circ z_1 - z = c_1 \quad \& \quad -a \circ z + a \circ z_1 \circ z_2 = -z + z_1 \circ z_2 = c_2$$

Then a map $r^p : B \times B \to B \times B$ defined for all $a, b \in B$ by

$$r^p(a, b) = (z_1 - a \circ z + a \circ b \circ z_2, (z_1 - a \circ z + a \circ b \circ z_2)^{-1} \circ a \circ b)$$

is a non-degenerate set-theoretic solution of the braid equation.

Corresponding shelf solution is

$$r_{\triangleright}(b, b \triangleright a) = (b, z_1 - b \circ z + a \circ z - z_1 + b) \quad \& \quad \varphi(a, b) = (a, z_1 - a \circ z + a \circ b \circ z_2)$$

In particular if $z_1 = z$ and $z_2 = 1$, we get that

$$r^p(a, b) = (z - a \circ z + a \circ b, (z - a \circ z + a \circ b)^{-1} \circ a \circ b)$$

Corresponding shelf solution is

$$r_{\triangleright}(b, b \triangleright a) = (b, z - b \circ z + a \circ z - z + b)$$

Examples-brace

Let us consider a brace $B = (U(\mathbb{Z}_8), +_1, \cdot)$, where $a +_1 b = a - 1 + b$. In this case $|B| = 4$ and $(B, \cdot)$ is Klein group.

If $z = 1$ then

$$r_1(a, b) = (ab - a + 1, (ab - a + 1)^{-1} ab).$$

$$r_{\triangleright_{r_1}}(a, b) = (b, -b + a + b) = (b, a)$$

If $z = 3$, then

$$r_1(a, b) = (ab - 3a + 3, (ab - 3a + 3)^{-1} ab)$$

$$r_{\triangleright_{r_3}}(a, b) = (b, -b - 3 + 3a - 3b + 3) = (b, 2b + 3a)$$

- Let V be a vector space over a field $\mathbb{F}$ and $\alpha \in \mathbb{F}$, then $Q = (V, \triangleright_\alpha)$ is a quandle, where $a \triangleright_\alpha b = \alpha b - \alpha a + a$.

Yang-Baxter algebra (Universal algebra sense)

Definition

Let (X, r) be a set-theoretic solution of the braid equation. We say that a pair (X, m), where $m : X \times X \to X$, is a Yang-Baxter (or braided) algebra, if for all $x, y \in X$, $m(x, y) = m(r(x, y))$.

Remark

Observe that we assume nothing about m, thus (X, m) is in general a magma.

Remark

If (X, m) is a Yang-Baxter algebra for some solution r and $\varphi : X \times X \to X \times X$ is a D-isomorphism, then $(X, m\varphi)$ is a Yang-Baxter algebra for a solution $\varphi^{-1} r \varphi$.

YBA

Lemma

Let (X, r) be a left non-degenerate solution and $(X, \triangleright_r)$ the shelf associated to r. Then, if $x \in X$, the binary operation $\bullet$ on X defined by

$$a \bullet b = \sigma_a(b) \triangleright_r (a \triangleright_r x).$$

makes $(X, \bullet)$ a Yang-Baxter algebra of (X, r).

Examples of Yang-Baxter algebras

- For skew brace $(B, +, \circ)$ and associated solution r^p, $(B, \circ)$ is a Yang-Baxter algebra i.e. $\bullet = \circ$.
- For an affine quandle given by a vector space V, $(V, \bullet)$ is Yang-Baxter algebra, where

$$a \bullet b = -\alpha^2 a + \alpha a - \alpha b + b$$

pre-Lie skew brace

Definition

Let (X,+) be a group and $\bullet : X \times X \to X$ be a binary operation. We say that the triple $(X, +, \bullet)$ is a **right pre-Lie skew brace** if for all $a, b, c \in X$ the following hold:

1. Distributivity

$$a \bullet (b + c) = a \bullet b - a \bullet 0 + a \bullet c \quad \& \quad (a + b) \bullet c = a \bullet c - 0 \bullet c + b \bullet c.$$

2. Right pre-Lie condition

$$(a \bullet b) \bullet c - a \bullet (b \bullet c) = (a \bullet c) \bullet b - a \bullet (c \bullet b)$$

All the examples given before satisfies conditions of pre-Lie skew brace and additionally are Abelian with $+$ structure.

To Lie rings

Proposition

Let $(P, +, \bullet)$ *be a pre-Lie brace. Then* $(P, +, [-, -])$ *is a Lie ring, where*

$$[a, b] = a \bullet b - b \bullet a + 0 \bullet a - a \bullet 0 + b \bullet 0 - 0 \bullet b,$$

for all $a, b \in P$.

Examples

- Observe that since Klein group is Abelian $(B, +, [-, -])$ is Lie ring with zero multiplication. If we take brace such that $(B, \circ)$ is not Abelian, then $[-, -]$ is just commutator.
- For affine quandles, from the example, we also acquire Lie ring with zero multiplication.

Thank You for Your Attention

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