On correlation functions for open XXZ and XYZ spin chains

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based on joined works with G. Niccoli (XXZ and XYZ with non-diagonal b.c., $T = 0$), and K. K. Kozlowski (XXZ with diagonal b.c. $T > 0$)

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The open XXZ/XYZ chain with boundary fields

$$H_{\rm XXZ}^{\rm open} = \sum_{m=1}^{L-1} \left[\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta\, \sigma_m^z \sigma_{m+1}^z\right] + \sum_{a\in\{x,y,z\}} \left[h_+^a \sigma_1^a + h_-^a \sigma_L^a\right]$$

- space of states: $\mathcal{H} = \otimes_{n=1}^{L} \mathcal{H}_n$ with $\mathcal{H}_n \simeq \mathbb{C}^2$
- $\sigma_m^{x,y,z} \in \mathrm{End}(\mathcal{H}_n)$: local spin-1/2 operators (Pauli matrices) at site m
- anisotropy parameter $\Delta = \cosh\eta$
- boundary fields $h_\pm^{x,y,z}$ parametrised in terms of 6 boundary parameters $\varsigma_\pm, \kappa_\pm, \tau_\pm$, or alternatively $\varphi_\pm, \psi_\pm, \tau_\pm$:

$$h_\pm^x = 2\kappa_\pm \sinh\eta \frac{\cosh\tau_\pm}{\sinh\varsigma_\pm}, \quad h_\pm^y = 2i\kappa_\pm \sinh\eta \frac{\sinh\tau_\pm}{\sinh\varsigma_\pm}, \quad h_\pm^z = \sinh\eta \coth\varsigma_\pm$$

$$\sinh\varphi_\pm \cosh\psi_\pm = \frac{\sinh\varsigma_\pm}{2\kappa_\pm}, \quad \cosh\varphi_\pm \sinh\psi_\pm = \frac{\cosh\varsigma_\pm}{2\kappa_\pm}$$

Question: Correlation functions $\langle \prod_{j=1}^{m} \sigma_{i_j}^{\alpha_j} \rangle$?
Previous works: [Jimbo et al. 95] from q-vertex operators, [Kitanine et al 07] from ABA ($h_\pm^x = h_\pm^y = 0,\ T = 0$)

The open XXZ/XYZ chain with boundary fields

$$H_{\rm XYZ}^{\rm open} = \sum_{a\in\{x,y,z\}} \left[\sum_{n=1}^{L} J_a \sigma_n^a \sigma_{n+1}^a + h_+^a \sigma_1^a + h_-^a \sigma_L^a \right]$$

boundary fields parametrised in terms of 6 boundary parameters $c_\pm^a$, $a = x, y, z$, or alternatively $\alpha_\ell^\pm$, $\ell = 1, 2, 3$:

$$J_x = \frac{\theta_4(\eta)}{\theta_4(0)}, \qquad h_\pm^x = c_\pm^x \frac{\theta_1(\eta)}{\theta_4(0)} = \frac{\theta_1(\eta)}{\theta_4(0)} \prod_{\ell=1}^{3} \frac{\theta_4(\alpha_\ell^\pm)}{\theta_1(\alpha_\ell^\pm)},$$

$$J_y = \frac{\theta_3(\eta)}{\theta_3(0)}, \qquad h_\pm^y = i c_\pm^y \frac{\theta_1(\eta)}{\theta_3(0)} = -i \frac{\theta_1(\eta)}{\theta_3(0)} \prod_{\ell=1}^{3} \frac{\theta_3(\alpha_\ell^\pm)}{\theta_1(\alpha_\ell^\pm)},$$

$$J_z = \frac{\theta_2(\eta)}{\theta_2(0)}, \qquad h_\pm^z = c_\pm^z \frac{\theta_1(\eta)}{\theta_2(0)} = \frac{\theta_1(\eta)}{\theta_2(0)} \prod_{\ell=1}^{3} \frac{\theta_2(\alpha_\ell^\pm)}{\theta_1(\alpha_\ell^\pm)}.$$

with $\theta_i(u) \equiv \theta_i(u|\omega)$ $(\Im(\omega) > 0)$

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Previous works: [Hara 00] from q-vertex operators

A brief reminder of the XXZ periodic case

Correlation functions of the XXZ periodic chain at $T = 0$ can be computed (among other methods) within ABA

→ numerical results [Caux et al. 05. . .]

→ analytical derivation of the large distance asymptotic behavior at the thermodynamic limit. . . [Kitanine, Kozlowski, Maillet, Slavnov, VT 08, 11. . .]

Both approaches are based

- on the form factor decomposition of the correlation functions:

$$\langle \psi_g | \sigma_n^\alpha \, \sigma_{n'}^\beta | \psi_g \rangle = \sum_{\substack{\text{eigenstates} \\ |\psi_i\rangle}} \langle \psi_g | \sigma_n^\alpha | \psi_i \rangle \cdot \langle \psi_i | \sigma_{n'}^\beta | \psi_g \rangle$$

- on the exact determinant representations for the form factors $\langle \psi_i | \sigma_n^\alpha | \psi_j \rangle$ in finite volume [Kitanine, Maillet, VT 1999] , obtained from
  - the action of local operators on Bethe states (using the solution of the quantum inverse problem, e.g. $\sigma_n^- = t(0)^{n-1} \, B(0) \, t(0)^{-n}$)
  - the use of Slavnov's determinant representation for the scalar products of Bethe states [Slavnov 89]

$$\langle \{\mu\}_{\text{off-shell}} | \{\lambda\}_{\text{on-shell}} \rangle \propto \det_{1 \le j,k \le n} \left[\frac{\partial \tau(\mu_j | \{\lambda\})}{\partial \lambda_k} \right]$$

where $t(\mu_j) \, |\{\lambda\}\rangle = \tau(\mu_j | \{\lambda\}) \, |\{\lambda\}\rangle$

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At $T > 0$, correlation functions as sum over thermal form factors within the QTM approach ([Dugave, Göhmann, Kozlowski 12] and further works...)
$\rightsquigarrow$ asymptotic behaviour at low-T

The reflection algebra for the XXZ/XYZ open spin chain

The open spin chains are solvable in the framework of the representation theory of the reflection algebra (or boundary Yang-Baxter algebra) [Sklyanin 88]

- generators $\mathcal{U}_{ij}(\lambda)$, $1 \leq i, j \leq 2$ $\quad \leftarrow$ elements of the boundary monodromy matrix $\mathcal{U}(\lambda)$
- commutation relations given by the reflection equation:

$$R_{12}(\lambda-\mu)\,\mathcal{U}_1(\lambda)\,R_{12}(\lambda+\mu-\eta)\,\mathcal{U}_2(\mu) = \mathcal{U}_2(\mu)\,R_{12}(\lambda+\mu-\eta)\,\mathcal{U}_1(\lambda)\,R_{12}(\lambda-\mu)$$

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$\hookrightarrow$ most general 2×2 trigonometric solution of the refl. eq [de Vega, Gonzalez-Ruiz; Ghoshal, Zamolodchikov 93] :

$$K(\lambda; \varsigma, \kappa, \tau) = \frac{1}{\sinh \varsigma} \begin{pmatrix} \sinh(\lambda - \frac{\eta}{2} + \varsigma) & \kappa\, e^{\tau} \sinh(2\lambda - \eta) \\ \kappa\, e^{-\tau} \sinh(2\lambda - \eta) & \sinh(\varsigma - \lambda + \frac{\eta}{2}) \end{pmatrix}$$

$\rightsquigarrow$ boundary matrices $K^+(\lambda) \equiv K(\lambda + \eta/2; \varsigma_+, \kappa_+, \tau_+)$ and $K^-(\lambda) \equiv K(\lambda - \eta/2; \varsigma_-, \kappa_-, \tau_-)$ describing left/right boundary fields:

$$h^x_\pm = 2\kappa_\pm \sinh\eta \,\frac{\cosh \tau_\pm}{\sinh \varsigma_\pm}, \quad h^y_\pm = 2i\kappa_\pm \sinh\eta\, \frac{\sinh \tau_\pm}{\sinh \varsigma_\pm}, \quad h^z_\pm = \sinh\eta \coth \varsigma_\pm$$

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$\hookrightarrow$ most general 2×2 elliptic solution of the refl. eq [Inami, Konno 94; Hou, Shi, Fan,Yang 95] :

$$K(\lambda)\equiv K(\lambda;\alpha_1,\alpha_2,\alpha_3)=\frac{\theta_1(2\lambda-\eta)}{2\theta_1(\lambda-\frac{\eta}{2})}\left[\mathbb{I}+c^x\frac{\theta_1(\lambda-\frac{\eta}{2})}{\theta_4(\lambda-\frac{\eta}{2})}\,\sigma^x+ic^y\frac{\theta_1(\lambda-\frac{\eta}{2})}{\theta_3(\lambda-\frac{\eta}{2})}\,\sigma^y+c^z\frac{\theta_1(\lambda-\frac{\eta}{2})}{\theta_2(\lambda-\frac{\eta}{2})}\,\sigma^z\right],$$

with coefficients c^x, c^y, c^z given in terms of three boundary parameters $\alpha_1, \alpha_2, \alpha_3$ as

$$c^x=\prod_{\ell=1}^{3}\frac{\theta_4(\alpha_\ell)}{\theta_1(\alpha_\ell)},\qquad c^y=-\prod_{\ell=1}^{3}\frac{\theta_3(\alpha_\ell)}{\theta_1(\alpha_\ell)},\qquad c^z=\prod_{\ell=1}^{3}\frac{\theta_2(\alpha_\ell)}{\theta_1(\alpha_\ell)}.$$

$\rightsquigarrow$ boundary matrices $K^+(\lambda)\equiv K(\lambda+\eta;\{\alpha_\ell^+\})$ and $K^-(\lambda)\equiv K(\lambda;\{\alpha_\ell^-\})$ describing left/right boundary fields:

$$h_\pm^x=\frac{\theta_1(\eta)}{\theta_4(0)}\prod_{\ell=1}^{3}\frac{\theta_4(\alpha_\ell^\pm)}{\theta_1(\alpha_\ell^\pm)},\qquad h_\pm^y=-i\frac{\theta_1(\eta)}{\theta_3(0)}\prod_{\ell=1}^{3}\frac{\theta_3(\alpha_\ell^\pm)}{\theta_1(\alpha_\ell^\pm)},\qquad h_\pm^z=\frac{\theta_1(\eta)}{\theta_2(0)}\prod_{\ell=1}^{3}\frac{\theta_2(\alpha_\ell^\pm)}{\theta_1(\alpha_\ell^\pm)}$$

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$\rightsquigarrow$ $\mathcal{U}(\lambda) = T(\lambda)\,K^-(\lambda)\,\hat{T}(\lambda) = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}$ with $\hat{T}(\lambda) \propto \sigma^y\, T^t(-\lambda)\,\sigma^y$

$\rightsquigarrow$ transfer matrix: $t(\lambda) = \mathrm{tr}\{K^+(\lambda)\,\mathcal{U}(\lambda)\}$ $\quad [t(\lambda), t(\mu)] = 0$

$$H^{\text{open}} \propto \frac{d}{d\lambda} \log t(\lambda)\Big|_{\lambda=\eta/2}$$

Solution by ABA in the XXZ diagonal case

When both boundary matrices $K^{\pm}$ are diagonal ($\kappa_{\pm} = 0$, i.e. boundary fields along σ_1^z and σ_N^z only):

- the bulk reference state $|\,0\,\rangle = |\uparrow\uparrow\ldots\uparrow\rangle$ can still be used to construct the eigenstates as Bethe states in the ABA framework [Sklyanin 88]

$$|\,\{\lambda\}\,\rangle = \prod_{k=1}^{n} \mathcal{B}(\lambda_k)|\,0\,\rangle \in \mathcal{H}, \quad \langle\,\{\lambda\}\,| = \langle\,0\,|\prod_{k=1}^{n} \mathcal{C}(\lambda_k) \in \mathcal{H}^*$$

- $\exists$ generalization of Slavnov's determinant representation for the scalar products of Bethe states $\langle\{\mu\}_{\text{off-shell}}|\{\lambda\}_{\text{on-shell}}\rangle$ [Tsuchiya 98; Wang 02]

- but a simple generalization of the quantum inverse problem to the boundary case (i.e. expressions of σ_n^{α} in terms of elements of the boundary monodromy matrix) is missing (except at site 1)

  $\rightsquigarrow$ no simple closed formula for the form factors $\langle\,\{\mu\}\,|\,\sigma_m^{\alpha}\,|\,\{\lambda\}\,\rangle$

- correlation functions in the ABA framework ? [Kitanine et al. 07]
  - decompose boundary Bethe states into bulk Bethe states
  - use the bulk inverse problem to compute the action of local operators
  - reconstruct the result in terms of boundary Bethe states

  $\rightsquigarrow$ multiple sums over scalar products

  $\rightsquigarrow$ multiple integrals in the half-infinite chain limit (recovering the results of [Jimbo et al. 95] from q-vertex operators)

Questions

- more explicit representations for correlation functions at $T = 0$?

  $\rightsquigarrow$ magnetization at distance m from the boundary (explicit dependance on m) ?

- temperature case ? (with K. Kozlowski)

- case of non-longitudinal boundary fields (non-diagonal K matrices) ? (with G. Niccoli)

- XYZ case ? (in progress with G. Niccoli)

The temperature case ? [Kozlowski, V.T. 23]

Consider the XXZ chain with longitudinal boundary fields in a uniform external magnetic field h:

$$H_h = H - \frac{h}{2} \sum_{k=1}^{L} \sigma_k^z$$

with

$$H = \sum_{m=1}^{L-1} \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta\, \sigma_m^z \sigma_{m+1}^z \right\} + h_-^z \sigma_1^z + h_+^z \sigma_L^z$$

$$\Delta = \cos\zeta \qquad h_\pm^z = \sinh(-i\zeta) \coth \xi_\pm$$

Given r local operators $\mathcal{O}^{(1)}_{m_1+1}, \ldots, \mathcal{O}^{(r)}_{m_r+1}$ acting on sites $m_1+1, \ldots, m_r+1$, we want to compute the thermal average

$$\mathbb{E}_{L;T}\left[\mathcal{O}^{(1)}_{m_1+1} \ldots \mathcal{O}^{(r)}_{m_r+1}\right] = \frac{\operatorname{tr}_{1,\ldots,L}\left[\mathcal{O}^{(1)}_{m_1+1} \ldots \mathcal{O}^{(r)}_{m_r+1}\, e^{-\frac{H_h}{T}}\right]}{\operatorname{tr}_{1,\ldots,L}\left[e^{-\frac{H_h}{T}}\right]}$$

and its thermodynamic limit:

$$\langle \mathcal{O}^{(1)}_{m_1+1} \ldots \mathcal{O}^{(r)}_{m_r+1} \rangle_T = \lim_{L\to+\infty} \mathbb{E}_{L;T}\left[\mathcal{O}^{(1)}_{m_1+1} \ldots \mathcal{O}^{(r)}_{m_r+1}\right]$$

$\rightarrow$ use of the Quantum Transfer Matrix approach (cf Wuppertal group works...)

The QTM approach for the open spin chain

Adaptation of the method to the open case to compute the surface free energy of the XXZ chain

- Göhmann, Bortz and Frahm (2005) : expression of the surface free energy for the XXZ chain in the thermodynamic limit as a Trotter limit of the expectation value, in the dominant eigenstate of the quantum transfer matrix, of a certain (non-local) 'finite temperature boundary operator'
- Kozlowski, Pozsgay (2012) : interpret the above mean value as a product of two specific cases of partition functions of the six-vertex model with reflecting ends

  $\rightarrow$ expression in terms of Tsuchiya's determinant representation

  $\rightarrow$ possibility to take the Trotter limit in the formula

  $\rightarrow$ simple integral representation for the boundary magnetization

  $\rightarrow$ possibility to study the low-T limit
- Pozsgay, Rakos (2018) : generalisation to arbitrary boundary conditions ($h = 0$)

Correlation functions ?

A Trotter approximant for multi-point functions

Using

$$\left(t(-\frac{\beta}{N}) \cdot t^{-1}(0)\right)^N = e^{-\frac{H}{T}} \cdot (1 + O(N^{-1}))$$

with

$$\beta = \frac{\sinh(-i\zeta)}{T}, \qquad \Delta = \cos\zeta$$

we have

$$\mathbb{E}_{L;T}\left[\mathcal{O}^{(1)}_{m_1+1} \cdots \mathcal{O}^{(r)}_{m_r+1}\right] = \lim_{N\to+\infty} \frac{\mathrm{tr}_{1,\ldots,L}\left[\mathcal{O}^{(1)}_{m_1+1} \cdots \mathcal{O}^{(r)}_{m_r+1} \cdot t^N(-\frac{\beta}{N}) \cdot t^{-N}(0) \cdot \prod_{n=1}^{L} e^{\frac{h}{2T}\sigma_n^z}\right]}{\mathrm{tr}_{1,\ldots,L}\left[t^N(-\frac{\beta}{N}) \cdot t^{-N}(0) \cdot \prod_{n=1}^{L} e^{\frac{h}{2T}\sigma_n^z}\right]}$$

Noticing that

$$t(\lambda) = \mathrm{tr}_{a,b}\left[P_{a,b}(\lambda)\, T_b^{t_b}(\lambda)\, \hat{T}_a(\lambda)\right]$$

where $P_{a,b}$ is a one-dimensional projector:

$$\begin{aligned} P_{a,b}(\lambda) &= K_a^{+}(\lambda)\, \mathcal{P}_{ab}^{t_a}\, K_a^{-}(\lambda) \\ &= K_a^{+}(\lambda)\left(|+\rangle_a|+\rangle_b + |-\rangle_a|-\rangle_b\right)\left(\langle+|_a\langle+|_b + \langle-|_a\langle-|_b\right) K_a^{-}(\lambda), \end{aligned}$$

Göhmann, Bortz and Frahm have rewritten $t^N(-\frac{\beta}{N})$ in terms of the quantum monodromy matrix $T_{q;j}(\lambda)$ with 'quantum space' $q \equiv a_1, \ldots, a_{2N}$ and 'auxiliary space' j:

$$t^N(-\frac{\beta}{N}) \prod_{n=1}^{L} e^{\frac{h}{2T}\sigma_n^z} = \mathrm{tr}_q\left[\Pi_q(-\frac{\beta}{N})\, T_{q;1}(0) \ldots T_{q;L}(0)\right], \qquad q \equiv a_1 \ldots a_{2N},$$

with

$$\Pi_q(\varsigma) = P_{a_1 a_2}(\varsigma)\, P_{a_3 a_4}(\varsigma) \ldots P_{a_{2N-1} a_{2N}}(\varsigma)$$

$$\begin{aligned} T_{q;j}(\lambda) &= R_{a_{2N}j}^{t_{a_{2N}}}(-\frac{\beta}{N} - \lambda)\, R_{ja_{2N-1}}(\lambda - \frac{\beta}{N}) \ldots R_{a_2 j}^{t_{a_2}}(-\frac{\beta}{N} - \lambda)\, R_{ja_1}(\lambda - \frac{\beta}{N})\, e^{\frac{h}{2T}\sigma_j^z} \\ &= \begin{pmatrix} A_q(\lambda) & B_q(\lambda) \\ C_q(\lambda) & D_q(\lambda) \end{pmatrix}_{[j]} \end{aligned}$$

Finite-size multi-point function:

$$\mathbb{E}_{L;T}\left[\mathcal{O}^{(1)}_{m_1+1}\dots\mathcal{O}^{(r)}_{m_r+1}\right]$$
$$=\lim_{N\to\infty}\mathrm{tr}_{1,\dots,L}\ \mathrm{tr}_q\left\{\Pi_q(-\frac{\beta}{N})\,T_{q;1}(0)\dots T_{q;L}(0)\ \mathcal{O}^{(1)}_{m_1+1}\dots\mathcal{O}^{(r)}_{m_r+1}\right\}/Z_{N,L}$$
$$=\lim_{N\to\infty}\mathrm{tr}_q\left\{\Pi_q(-\frac{\beta}{N})\cdot[t_q(0)]^{m_1}\cdot\mathrm{tr}[T_q(0)\,\mathcal{O}^{(1)}]\cdot[t_q(0)]^{m_2-m_1-1}\right.$$
$$\left.\times\,\mathrm{tr}[T_q(0)\,\mathcal{O}^{(2)}]\cdot[t_q(0)]^{m_3-m_2-1}\dots\mathrm{tr}[T_q(0)\,\mathcal{O}^{(r)}]\,[t_q(0)]^{L-m_r-1}\right\}/Z_{N,L}$$

where

$$Z_{N,L}=\mathrm{tr}_{1,\dots,L}\ \mathrm{tr}_q\left\{\Pi_q(-\frac{\beta}{N})\ T_{q;1}(0)\dots T_{q;L}(0)\right\}$$
$$=\mathrm{tr}_q\left\{\Pi_q(-\frac{\beta}{N})\cdot[t_q(0)]^{L}\right\}$$

Remark. $t_q=\mathrm{tr}\,T_q$ is the same QTM as in the periodic case $\to$ use the results from the study of the periodic case (from the Wuppertal's group works)

Assuming

- that one can exchange the Trotter limit $N \to +\infty$ and thermodynamic limit $L \to +\infty$,
- that the QTM admits a non-degenerate, real and positive maximal eigenvalue $\hat{\Lambda}_0$ with corresponding eigenstate $|\,\Psi_0\,\rangle$

one obtains

$$\langle \mathcal{O}^{(1)}_{m_1+1} \dots \mathcal{O}^{(r)}_{m_r+1} \rangle_T = \lim_{N \to +\infty} \frac{\langle\, \Psi_0 \,|\, \Pi_q(-\frac{\beta}{N}) \cdot [t_q(0)]^{m_1} \cdot \Xi^{(1)} \cdot [t_q(0)]^{m_2-m_1-1} \cdot \Xi^{(2)} \dots \Xi^{(r)} |\, \Psi_0 \,\rangle}{\langle\, \Psi_0 \,|\, \Pi_q(-\frac{\beta}{N}) \,|\, \Psi_0 \,\rangle \cdot \hat{\Lambda}_0^{m_r+1}}$$

in which

$$\Xi^{(i)} = \mathrm{tr}[\, T_q(0)\, \mathcal{O}^{(i)}]$$

Thermal form factor expansion at finite Trotter number

Supposing that the quantum transfer matrix $t_q(0)$ is diagonalizable with eigenvectors $|\,\Psi_n\,\rangle$ and associated eigenvalues $\hat{\Lambda}_n$:

$$\langle \mathcal{O}^{(1)}_{m_1+1} \dots \mathcal{O}^{(r)}_{m_r+1} \rangle_T = \lim_{N \to +\infty} \sum_{k_1,\dots,k_r} \frac{\hat{\Lambda}_{k_1}^{m_1} \prod_{i=2}^r \hat{\Lambda}_{k_i}^{m_i - m_{i-1} - 1}}{\hat{\Lambda}_0^{m_r+1}} \times \underbrace{\frac{\langle\, \Psi_0 \,|\, \Pi_q(-\frac{\beta}{N}) \,|\, \Psi_{k_1} \,\rangle}{\langle\, \Psi_0 \,|\, \Pi_q(-\frac{\beta}{N}) |\, \Psi_0 \,\rangle}}_{\text{Boundary factor}} \cdot \prod_{i=1}^{r} \underbrace{\frac{\langle\, \Psi_{k_i} \,|\, \Xi^{(i)} \,|\, \Psi_{k_{i+1}} \,\rangle}{\langle \Psi_{k_i} \,|\, \Psi_{k_i} \rangle}}_{\text{QTM form factors}}$$

- the QTM eigenstates for finite N can be constructed by Bethe ansatz and are described by solutions of Bethe equations
- the above sum runs over the same normalised QTM matrix elements as in the bulk case (given as ratios of Slavnov/Gaudin determinants)

  $\to$ we can directly use the study of [Dugave, Göhmann, Kozlowski 12] and further works...
- the whole dependence on the boundary is contained in the boundary factor, which can be reformulated, following [Kozlowski, Pozsgay 12] as a ratio of partition functions of the six-vertex model with reflecting ends ($\to$ ratio of Tsuchiya's determinants)

The boundary factor

Let $|\,\Psi_0\,\rangle \equiv |\,\Psi(\{\lambda_j\}_1^N)\,\rangle$ and $|\,\Psi_{k_1}\,\rangle \equiv |\,\Psi(\{\mu_j\}_1^M)\,\rangle$

Then, following [Kozlowski, Pozsgay 12] :

$$\langle\,\Psi_0\,|\,\Pi_q(-\frac{\beta}{N})\,|\,\Psi_{k_1}\,\rangle = \delta_{N,M}\,\mathcal{F}^{(+)}(\{\lambda_j\}_1^N)\cdot\mathcal{F}^{(-)}(\{\mu_j\}_1^N)$$

in which

$$\mathcal{F}^{(-)}(\{\mu_j\}_1^N) = e^{-\frac{Nh}{2T}}\,\mathcal{Z}_N(\{-\frac{\beta}{N}\}_1^N;\{\mu_j\}_1^N;\xi_-)$$

where $\mathcal{Z}_N(\{\xi_a\}_1^N;\{\mu_j\}_1^N;\xi_-)$ is the partition function of the six-vertex model with reflecting ends (given by a Tsuchiya determinant):

$$\mathcal{Z}_N\Big(\{\xi_a\}_1^N;\{\mu_a\}_1^N;\xi_-\Big) = \frac{\prod_{a,b=1}^{N}\prod_{\epsilon=\pm}\Big\{\sinh(\xi_a+\epsilon\mu_b)\sinh(\xi_a-i\zeta+\epsilon\mu_b)\Big\}}{\prod_{a<b}^{N}\Big\{\sinh(\xi_a-\xi_b)\sinh(\xi_a+\xi_b-i\zeta)\prod_{\epsilon=\pm}\sinh(\mu_b+\epsilon\mu_a)\Big\}}$$
$$\times\det_N\left[\frac{\sinh(-i\zeta)\sinh(\xi_-+\mu_b)\sinh(2\xi_a)}{\prod_{\epsilon=\pm}\sinh(\xi_a-i\zeta+\epsilon\mu_b)\sinh(\xi_a+\epsilon\mu_b)}\right]$$

so that $$\frac{\langle\,\Psi_0\,|\,\Pi_q(-\frac{\beta}{N})\,|\,\Psi_{k_1}\,\rangle}{\langle\,\Psi_0\,|\,\Pi_q(-\frac{\beta}{N})|\,\Psi_0\,\rangle} = \delta_{N,M}\,\frac{\mathcal{F}^{(-)}(\{\mu_j\}_1^N)}{\mathcal{F}^{(-)}(\{\lambda_j\}_1^N)}$$

Remark: depends only on ξ_- (and not on ξ_+)

Taking the Trotter limit

Can be done as usual:

- for a given solution $\{\mu_a\}_1^M$ of the Bethe equations, introduce the counting function

$$\hat{\mathfrak{a}}(\xi|\{\mu_a\}_1^M) = e^{-\frac{h}{T}}(-1)^s \prod_{k=1}^{M} \frac{\sinh(i\zeta - \xi + \mu_k)}{\sinh(i\zeta + \xi - \mu_k)} \left[\frac{\sinh(\xi - \frac{\beta}{N})\sinh(i\zeta + \xi + \frac{\beta}{N})}{\sinh(\xi + \frac{\beta}{N})\sinh(i\zeta - \xi + \frac{\beta}{N})}\right]^N$$

  with $s = N - M$, such that $\hat{\mathfrak{a}}(\mu_j|\{\mu_a\}_1^M) = -1$, $j = 1, \ldots, M$.
- fix a domain $\mathcal{D}$ with $\mathcal{C} = \partial\mathcal{D}$
  - which contains a neighbourhood of the origin ($\rightsquigarrow$ $\pm\frac{\beta}{N} \in \mathcal{D}$)
  - which contains all the Bethe roots $\{\lambda_a\}_1^N$ of the dominant state but no other roots of $1 + \hat{\mathfrak{a}}(\xi|\{\lambda_a\}_1^N)$
- characterize a sub-dominant eigenstate by
  - the set $\hat{\mathcal{Y}} = \{\hat{y}_j\}$ of particule roots (Bethe roots outsite of $\mathcal{D}$),
  - and the set $\hat{\mathcal{X}} = \{\hat{x}_j\}$ of holes (solutions of $\hat{\mathfrak{a}}(\xi|\{\mu_a\}_1^M) = -1$ which are not Bethe roots) inside $\mathcal{D}$

$\rightsquigarrow$ shortcut notation $\hat{\mathfrak{a}}_{\mathbb{Y}}$ for the counting function of a state with a given configuration $\mathbb{Y} = (\hat{\mathcal{X}}, \hat{\mathcal{Y}})$ of particles and holes

- rewrite the QTM spectrum in terms of non-linear integral equations [Klümper 92; Destri, de Vega. 92] satisfied by $\hat{\mathfrak{a}}_{\mathbb{Y}}(\xi) = \mathrm{e}^{\hat{\mathfrak{A}}_{\mathbb{Y}}(\xi)}$:

$$\hat{\mathfrak{A}}_{\mathbb{Y}}(\xi) = -\frac{h}{T} + \mathfrak{w}_N(\xi) - i\pi s + i \sum_{y \in \hat{\mathbb{Y}}} \theta(\xi - y) + \oint_{\mathcal{C}} K(\xi - u)\, \mathcal{L}\mathrm{n}\left[1 + e^{\hat{\mathfrak{A}}_{\mathbb{Y}}}\right](u) du$$

with

$$\mathfrak{w}_N(\xi) = N \ln \left(\frac{\sinh(\xi - \frac{\beta}{N}) \sinh(\xi + \frac{\beta}{N} - i\zeta)}{\sinh(\xi + \frac{\beta}{N}) \sinh(\xi - \frac{\beta}{N} - i\zeta)} \right)$$

$$\theta(\lambda) = i \ln \frac{\sinh(i\zeta + \lambda)}{\sinh(i\zeta - \lambda)} \qquad K(\lambda) = \frac{\theta'(\lambda)}{2\pi}$$

- rewrite the QTM form factors and boundary factors in terms of particles, holes, and appropriate contour integrals over $\mathcal{C}$ involving the counting function $\hat{\mathfrak{a}}_{\mathbb{Y}}(\xi)$
- assuming that $\hat{\mathfrak{A}}_{\mathbb{Y}} \underset{N \to +\infty}{\longrightarrow} \mathfrak{A}_{\mathbb{Y}}$ pointwise on $\mathcal{C}$, and the existence of the limit x_j and y_j of the particle and hole roots $\hat{x}_j$ and $\hat{y}_j$ (see [Göhmann, Goomanee, Kozlowski, Suzuki 20]), one obtains an integral equation for $\mathfrak{A}_{\mathbb{Y}}$, and one can express the Trotter limit of the TQM form factors and boundary factors in terms of $\mathfrak{A}_{\mathbb{Y}}$ and $\{x_j\}$ and $\{y_j\}$

Result for the one-point function

$$\langle \sigma^z_{m+1} \rangle_T = \lim_{N \to \infty} \left[2T \partial_{h'} D_m \mathcal{Q}_N(h', m) \right]_{h'=h}$$

with

$$D_m = u_{m+1} - u_m$$

$$\mathcal{Q}_N(h', m) = \sum_{\{\mu_a(h')\}_1^N} e^{\frac{N(h'-h)}{2T}} \left(\frac{\tau_{h'}(0|\{\mu_a(h')\}_1^N)}{\tau_h(0|\{\lambda_a(h)\}_1^N)} \right)^m \times \frac{\mathcal{F}^{(-)}(\{\mu_a(h')\}_1^N)}{\mathcal{F}^{(-)}(\{\lambda_a(h)\}_1^N)} \cdot \frac{\langle \Psi(\{\mu_a(h')\}_1^N) | \Psi(\{\lambda_a(h)\}_1^N) \rangle}{\langle \Psi(\{\mu_a(h')\}_1^N) | \Psi(\{\mu_a(h')\}_1^N) \rangle}$$

leads to the thermal form-factor expansion:

$$\langle \sigma^z_{m+1} \rangle_T = 2T \partial_{h'} D_m \mathcal{Q}(h', m) \Big|_{h'=h} \quad \text{with} \quad \mathcal{Q}(h', m) = \sum_{\substack{\text{particle/hole} \\ \text{configurations } \mathbb{Y}}} \left(\frac{\tau_{\mathbb{Y}}(0)}{\tau_{\emptyset}(0)} \right)^m \mathcal{A}^{(z)}_{h,h'}(\mathbb{Y})$$

and $\mathcal{A}^{(z)}_{h,h'}(\mathbb{Y})$ can be decomposed into

- a regular part (quite complicated, should have finite limit when $T \to 0^+$)
- a singular part (should give power law behaviour when $T \to 0^+$)

To do: study the low temperature limit

The non-diagonal case at T=0?

Description of the spectrum:

- It is possible to generalize usual Bethe ansatz equations to the open XXZ chain with non-longitudinal boundary fields with one constraint on the boundary parameters $\varphi_\pm, \psi_\pm, \tau_\pm$ [Nepomechie 03] :

$$\cosh(\tau_+ - \tau_-)$$
$$= \epsilon_{\varphi_+}\epsilon_{\varphi_-} \cosh(\epsilon_{\varphi_+}\varphi_+ + \epsilon_{\varphi_-}\varphi_- + \epsilon_{\varphi_+}\psi_+ - \epsilon_{\varphi_-}\psi_- + (L-2M-1)\eta)$$

  with $M \in \mathbb{N}$ (numbers of Bethe roots), $\epsilon_{\varphi_\pm} \in \{+,-\}$

  $\rightsquigarrow$ incomplete in general (except for $M = L$)

  $\rightsquigarrow$ Conjectures [Nepomechie, Ravanini 03] :
  - the Bethe equations yield the ground state for $M = \lfloor L/2 \rfloor$
  - the solutions for $(M, \epsilon_{\varphi_+}, \epsilon_{\varphi_-})$ together with the solutions for $(M' = L - M - 1, -\epsilon_{\varphi_+}, -\epsilon_{\varphi_-})$ produce the complete spectrum

- Similar kind of constraint also appear in the XYZ case [Yang, Zhang 06]

- most general boundaries ?

  $\exists$ description in terms of inhomogeneity parameters/discrete T-Q equations (for inhomogeneous models) but no known description in terms of usual Bethe equations

  Alternative proposals: Bethe equations with an additional term (Off-diagonal Bethe Ansatz. . .) [Cao et al 13. . .] , or use transfer matrix

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- Similar kind of constraint also appear in the XYZ case [Yang, Zhang 06] :

$$\sum_{\sigma=\pm} \sum_{i=1}^{3} \epsilon_i^\sigma \alpha_i^\sigma = (L-2M-1)\eta,$$

- most general boundaries ?

  $\exists$ description in terms of inhomogeneity parameters/discrete T-Q equations (for inhomogeneous models) but no known description in terms

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  Alternative proposals: Bethe equations with an additional term (Off-diagonal Bethe Ansatz...) [Cao et al 13...] or use transfer matrix roots [Qiao et al 21...]

The non-diagonal case at T=0?

Construction of the transfer matrix eigenstates ?

- Under the constraint, construction of some Bethe states by means of a Vertex-IRF transformation [Fan et al. 96; Cao et al 03; Yang, Zhang 07; Filali, Kitanine 11] but problems in the ABA construction of "compatible" sets of Bethe states in $\mathcal{H}$ and $\mathcal{H}^*$

  $\rightsquigarrow$ scalar products and correlation functions could not be computed

- Alternative methods of construction for general boundaries:
  - Modified Bethe Ansatz [Belliard et al 13...]
  - Separation of Variables [Frahm et al 10, Niccoli 12, Faldella et al 13...]
    In particular : connexion to generalized Bethe Ansatz (states and T-Q/Bethe equations) under the constraint
    + computation of the scalar products

Solution by SoV in the general case

Goal: identify a basis $\{|\,\mathbf{h}\,\rangle\}_{\mathbf{h}\in\{0,1\}^L}$ of $\mathcal{H}$ and $\{\langle\,\mathbf{h}\,|\}_{\mathbf{h}\in\{0,1\}^L}$ of $\mathcal{H}^*$, with

$$\langle\,\mathbf{h}\,|\,\mathbf{k}\,\rangle \propto \frac{\delta_{\mathbf{h},\mathbf{k}}}{V_{\mathbf{h}}(\boldsymbol{\xi})}$$

which "separates the variables" for the transfer matrix spectral problem:

$$t(\lambda)\,|\,\Psi_\tau\,\rangle = \tau(\lambda)\,|\,\Psi_\tau\,\rangle \quad \text{with} \quad |\,\Psi_\tau\,\rangle = \sum_{\mathbf{h}\in\{0,1\}^L} \psi_\tau(\mathbf{h})\,|\,\mathbf{h}\,\rangle,$$

is solved by $\quad \psi_\tau(\mathbf{h}) = \prod_{n=1}^{L} Q_\tau(\xi_n^{(h_n)}) \cdot V_{\mathbf{h}}(\boldsymbol{\xi})$

where Q_τ and τ are solution of a discrete version of Baxter's T-Q equation:

$$\tau(x)\,Q_\tau(x) = \mathbf{A}(x)\,Q(x+\eta) + \mathbf{A}(-x)\,Q_\tau(x-\eta), \quad x \in \cup_{n=1}^{L}\{\xi_n^{(0)}, \xi_n^{(1)}\}$$

- $\rightarrow$ can be constructed by a generalisation of Sklyanin's method [Sklyanin 85,90] , see [Niccoli 12...] , using Baxter's vertex-IRF transformation (or by some new more general approach [Maillet, Niccoli 19])
- $\rightarrow$ works only on an inhomogeneous deformation of the model:

  $$T(\lambda) \longrightarrow T(\lambda;\xi_1,\ldots,\xi_L)$$

  such that the shifted inhomogeneity parameters $\xi_n^{(h_n)} = \xi_n + \eta/2 - h_n\eta$, $1 \le n \le L$, $h_n \in \{0,1\}$, are all pairwise distincts
- $\rightarrow$ completeness/works for any K-matrices (not both proportional to identity)

Solution by Sklyanin's SoV approach: more details

1. **simplify the expression of $t(\lambda) = \mathrm{tr}\{K^+(\lambda)\mathcal{U}(\lambda)\}$** : use (a trigonometric version of) Baxter's Vertex-IRF transformation to pseudo-diagonalize K^+

$$R_{12}(\lambda-\mu)\, S_1(\lambda|\alpha,\beta)\, S_2(\mu|\alpha,\beta+\sigma_1^z) = S_2(\mu|\alpha,\beta)\, S_1(\lambda|\alpha,\beta+\sigma_2^z)\, R_{12}^{\mathrm{SOS}}(\lambda-\mu|\beta)$$

with

$$S(\lambda|\alpha,\beta) = \begin{cases} \begin{pmatrix} e^{\lambda-\eta(\beta+\alpha)} & e^{\lambda+\eta(\beta-\alpha)} \\ 1 & 1 \end{pmatrix} & \text{(XXZ case)} \\ \begin{pmatrix} \vartheta_2(\lambda-(\alpha+\beta)\eta) & \vartheta_2(\lambda-(\alpha-\beta)\eta) \\ \vartheta_3(\lambda-(\alpha+\beta)\eta) & \vartheta_3(\lambda-(\alpha-\beta)\eta) \end{pmatrix} & \text{(XYZ case)} \quad \vartheta_i(\lambda) = \theta_i(\lambda|2\omega) \end{cases}$$

$\rightsquigarrow$ gauged transformed boundary monodromy matrix:

$$\mathcal{U}(\lambda|\alpha,\beta) = S^{-1}(\eta/2-\lambda|\alpha,\beta)\,\mathcal{U}(\lambda)\, S(\lambda-\eta/2|\alpha,\beta) = \begin{pmatrix} \mathcal{A}(\lambda|\alpha,\beta) & \mathcal{B}(\lambda|\alpha,\beta) \\ \mathcal{C}(\lambda|\alpha,\beta) & \mathcal{D}(\lambda|\alpha,\beta) \end{pmatrix}$$

$$\begin{cases} \beta : \text{dynamical parameter} \\ \alpha : \text{arbitrary shift} \end{cases}$$

$\rightsquigarrow$ fix α, β in terms of the '+'-boundary parameters ($\tau_+, \varphi_+, \psi_+$ or α_ℓ^+, $\ell = 1,2,3$) (up to some signs/periodicity) such that

$$t(\lambda) = \bar{\mathsf{a}}_+(\lambda)\,\mathcal{A}(\lambda|\alpha,\beta-1) + \bar{\mathsf{d}}_+(\lambda)\,\mathcal{D}(\lambda|\alpha,\beta+1)$$

2 **construct a SoV basis which pseudo-diagonalises $\mathcal{B}(\lambda|\alpha,\beta)$:**

$$|\,\mathbf{h}\,\rangle \equiv |\,\mathbf{h},\alpha,\beta+1\,\rangle_{\mathrm{Sk}} \text{ and } \langle\,\mathbf{h}\,| \equiv {}_{\mathrm{Sk}}\langle\,\alpha,\beta-1,\mathbf{h}\,|,$$

for $\mathbf{h} \equiv (h_1,\ldots,h_L) \in \{0,1\}^L$, such that

$$\mathcal{B}(\lambda|\alpha,\beta-1)\,|\,\mathbf{h},\alpha,\beta-1\,\rangle_{\mathrm{Sk}} = \mathsf{b}_R(\lambda|\alpha,\beta)\,a_{\mathbf{h}}(\lambda)\,a_{\mathbf{h}}(-\lambda)\,|\,\mathbf{h},\alpha,\beta+1\,\rangle_{\mathrm{Sk}},$$
$${}_{\mathrm{Sk}}\langle\,\alpha,\beta+1,\mathbf{h}\,|\,\mathcal{B}(\lambda|\alpha,\beta+1) = \mathsf{b}_L(\lambda|\alpha,\beta)\,a_{\mathbf{h}}(\lambda)\,a_{\mathbf{h}}(-\lambda)\,{}_{\mathrm{Sk}}\langle\,\alpha,\beta-1,\mathbf{h}\,|,$$

where $\quad a_{\mathbf{h}}(\lambda) = \prod_{n=1}^{L} \phi(\lambda - \xi_n^{(h_n)}) \qquad \phi(\lambda) = \begin{cases} \sinh(\lambda) & \text{(XXZ case)} \\ \theta_1(\lambda) & \text{(XYZ case)} \end{cases}$

with $\quad \xi_n^{(h_n)} = \xi_n + \eta/2 - h_n\eta,$

$+$ orthogonality condition:

$$ {}_{\mathrm{Sk}}\langle\,\alpha,\beta-1,\mathbf{h}\,|\,\mathbf{k},\alpha,\beta+1\,\rangle_{\mathrm{Sk}} \propto \frac{\delta_{\mathbf{h},\mathbf{k}}}{V_{\mathbf{h}}(\boldsymbol{\xi})}$$

with $V_{\mathbf{h}}(\boldsymbol{\xi}) = V(\xi_1^{(h_1)},\ldots,\xi_L^{(h_L)}) = \prod_{1\le i,j\le L} \phi(\xi_i^{(h_i)} - \xi_j^{(h_j)})\phi(\xi_i^{(h_i)} + \xi_j^{(h_j)})$

Remark: This construction needs $[K^-(\lambda|\alpha,\beta)]_{12} \neq 0$

Spectrum and eigenstates by SoV

- Eigenstates are special cases of separate states:

$$t(\lambda)\,|\,\Psi_\tau\,\rangle = \tau(\lambda)\,|\,\Psi_\tau\,\rangle \quad \text{with} \quad |\,\Psi_\tau\,\rangle = \sum_{\mathbf{h}\in\{0,1\}^L} \prod_{n=1}^{L} Q_\tau(\xi_n^{(h_n)})\, V_{\mathbf{h}}(\boldsymbol{\xi})\,|\,\mathbf{h}\,\rangle,$$

where Q_τ and τ are solution of a discrete T-Q equation:

$$\tau(x)\,Q_\tau(x) = \mathbf{A}(x)\,Q(x+\eta) + \mathbf{A}(-x)\,Q_\tau(x-\eta), \quad x \in \cup_{n=1}^{L}\{\xi_n^{(0)}, \xi_n^{(1)}\}$$

- can always be rewritten in terms of solutions of the form

$$Q(\lambda) = \prod_{j=1}^{L} \phi(\lambda-\lambda_j)\phi(\lambda+\lambda_j) \qquad \phi(\lambda) = \begin{cases} sinh(\lambda) & \text{(XXZ case)} \\ \theta_1(\lambda) & \text{(XYZ case)} \end{cases}$$

of a continuous T-Q equation with additional term (cf. off-diagonal BA):

$$\tau(\lambda)\,Q(\lambda) = \mathbf{A}(\lambda)\,Q(\lambda-\eta) + \mathbf{A}(-\lambda)\,Q(\lambda+\eta) + \mathbf{F}(\lambda),$$

with $\mathbf{F}(\xi_n^{(0)}) = \mathbf{F}(\xi_n^{(1)}) = 0$, $n = 1, \dots, N$ ($\rightarrow$ completeness)

- under the constraint (for a given $M \leq L$ & given signs), part of the spectrum/eigenstates can be rewritten in terms of solutions

$$Q(\lambda) = \prod_{j=1}^{M} \phi(\lambda-\lambda_j)\phi(\lambda+\lambda_j) \qquad \phi(\lambda) = \begin{cases} sinh(\lambda) & \text{(XXZ case)} \\ \theta_1(\lambda) & \text{(XYZ case)} \end{cases}$$

of the usual (i.e. continuous, without additional term) T-Q equation

$\rightsquigarrow$ in terms of usual Bethe equations

Computation of scalar products

- Eigenstates are special cases of separate states :

$$\langle\, P \,| = \sum_{\mathbf{h}} \prod_{n=1}^{L} [f(\xi_n)^{h_n} P(\xi_n^{(h_n)})]\; V_{1-\mathbf{h}}(\boldsymbol{\xi}) \langle\, \mathbf{h} \,|, \quad |\, Q \,\rangle = \sum_{\mathbf{h}} \prod_{n=1}^{L} Q(\xi_n^{(h_n)})\; V_{\mathbf{h}}(\boldsymbol{\xi}) |\, \mathbf{h} \,\rangle$$

  where P and Q are arbitrary functions

- The scalar products of separate states can be expressed (by construction) as determinants:

$$\langle \mathbf{h} \,|\, \mathbf{k} \rangle \propto \frac{\delta_{\mathbf{h},\mathbf{k}}}{V_{\mathbf{h}}(\boldsymbol{\xi})} \quad \text{with}$$

$$V_{\mathbf{h}}(\boldsymbol{\xi}) = \prod_{1 \le i,j \le L} \phi(\xi_i^{(h_i)} - \xi_j^{(h_j)}) \phi(\xi_i^{(h_i)} + \xi_j^{(h_j)}) = \det_L \left[\tilde{\phi}^{(j)}(\xi_i^{(h_i)}) \right]$$

$$\rightsquigarrow \;\langle\, P \,|\, Q \,\rangle \propto \det_{1 \le i,j \le L} \left[\sum_{h \in \{0,1\}} f(\xi_i^{(h_i)}) P(\xi_i^{(h_i)}) Q(\xi_i^{(h_i)})\, \sinh^{2(j-1)}(\xi_i^{(1-h_i)}) \right]$$

  However non directly usable for the consideration of the homogeneous/thermodynamic limit...

- For P and Q of the form $\prod_{j=1}^{M} \phi(\lambda - \lambda_j)\phi(\lambda + \lambda_j)$, and under the constraint, these determinants can be transformed into more usual Slavnov-type determinants both in the open XXZ [Kitanine, Maillet, Niccoli, VT 18] or open XYZ case [Niccoli, VT 24]

Eigenstates as generalised Bethe states

- In the range of Sklyanin's approach, separate states can be reformulated as generalised Bethe states:

$$| \, Q \, \rangle_{\mathrm{Sk}} \propto \prod_{j=1 \to M} \mathcal{B}(\lambda_j | \alpha, \beta - 2j + 1) \, | \, \Omega_{\alpha, \beta+1-2M} \, \rangle_{\mathrm{Sk}}$$

$$_{\mathrm{Sk}}\langle \, Q \, | \propto {}_{\mathrm{Sk}}\langle \, \Omega_{\alpha, \beta-1+2M} \, | \prod_{j=1 \to M} \mathcal{B}(\lambda_j | \alpha, \beta + 2M - 2j + 1)$$

for any $Q(\lambda) = \prod_{j=1}^{M} \phi(\lambda - \lambda_j)\phi(\lambda + \lambda_j)$

with $| \, \Omega_{\alpha, \beta+1-2M} \, \rangle_{\mathrm{Sk}}$ and $_{\mathrm{Sk}}\langle \, \Omega_{\alpha, \beta-1+2M} \, |$ special separate states

- With the special choice of α, β diagonalising K^+, and under the constraint, the reference state $| \, \Omega_{\alpha, \beta+1-2M} \, \rangle$ can be identified with the reference state of the generalized ABA construction of [Fan et al 96; Cao et al 03] :

$$| \, \eta, \alpha + \beta + L - 1 - 2M \, \rangle \equiv \prod_{n=1}^{L} S_n(-\xi_n | \alpha, \beta + n - 1 - 2M) \, | \, 0 \, \rangle$$

up to a proportionality coefficient which only depends on M

Computation of correlation functions: general strategy

Compute $\langle O_{1\to m}\rangle \equiv \dfrac{\langle\, Q\,|\, O_{1\to m}|\, Q\,\rangle}{\langle Q\,|\, Q\rangle}$ for $|\, Q\,\rangle =$ eigenstate described by homogeneous TQ-equation and $O_{1\to m} \in \mathrm{End}(\otimes_{n=1}^{m}\mathcal{H}_n)$ acts on sites 1 to m?

1. rewrite $|\, Q\,\rangle$ as a generalized Bethe state
$$\prod_{j=1\to M} \mathcal{B}(\lambda_j|\alpha,\beta-2j+1)\,|\,\eta,\alpha+\beta+L-1-2M\,\rangle$$

2. use a similar strategy as in the diagonal case [Kitanine et al. 07] to act with $O_{1\to m}$ on this Bethe state, i.e.
   - decompose the boundary Bethe state as a sum of bulk Bethe states
   - use the solution of the bulk inverse problem to act with local operators on bulk Bethe states
   - reconstruct the result of this action as sums over boundary Bethe states, and hence as a sum over separate states

3. compute the resulting scalar products using the determinant representation for the scalar products of separate states issued from SOV

but difficulties due to the use in all the steps of 2 of a gauged transformed boundary/bulk YB algebra !

Difficulties due to use of the gauged algebra

- the action of the usual basis of local operators given by $E_n^{i,j} \in \mathrm{End}(\mathcal{H}_n)$ (such that $(E^{i,j})_{k,\ell} = \delta_{i,k}\,\delta_{j,\ell}$) is very intricate on the gauged bulk Bethe states

  $\rightsquigarrow$ identification of a basis of $\mathrm{End}(\otimes_{n=1}^m \mathcal{H}_n)$ whose action is simpler to compute:

$$\mathbb{E}_m(\alpha,\beta) = \left\{ \prod_{n=1}^{m} E_n^{\epsilon'_n,\epsilon_n}(\xi_n|(a_n,b_n),(\bar{a}_n,\bar{b}_n)) \;\middle|\; \boldsymbol{\epsilon},\boldsymbol{\epsilon}' \in \{1,2\}^m \right\},$$

  where $E_n^{\epsilon'_n,\epsilon_n}(\lambda|(a_n,b_n),(\bar{a}_n,\bar{b}_n))) = S_n(-\lambda|\bar{a}_n,\bar{b}_n)\,E_n^{\epsilon'_n,\epsilon_n}\,S_n^{-1}(-\lambda|a_n,b_n)$ and the gauge parameters $a_n,\bar{a}_n,b_n,\bar{b}_n$, $1 \le n \le m$, are fixed in terms of α,β and of the m-tuples $\boldsymbol{\epsilon} \equiv (\epsilon_1,\ldots,\epsilon_m)$ and $\boldsymbol{\epsilon}' \equiv (\epsilon'_1,\ldots,\epsilon'_m)$ as

$$a_n = \alpha + 1, \qquad b_n = \beta - \sum_{r=1}^{n} (-1)^{\epsilon_r},$$

$$\bar{a}_n = \alpha - 1, \qquad \bar{b}_n = \beta + \sum_{r=n+1}^{m} (-1)^{\epsilon'_r} - \sum_{r=1}^{m} (-1)^{\epsilon_r} = b_n + 2\tilde{m}_{n+1},$$

  with $\tilde{m}_n = \sum_{r=n}^{m} (\epsilon'_r - \epsilon_r)$.

$\rightsquigarrow$ compute "elementary building blocks" $\langle \prod_{n=1}^m E_n^{\epsilon'_n,\epsilon_n}(\xi_n|(a_n,b_n),(\bar{a}_n,\bar{b}_n)) \rangle$

- the action of $\prod_{n=1}^{m} E_n^{\epsilon'_n,\epsilon_n}(\xi_n|(a_n,b_n),(\bar{a}_n,\bar{b}_n))$ for

$$\sum_{r=1}^{m}(\epsilon'_r-\epsilon_r)\neq 0$$

on the Bethe state

$$\prod_{j=1\to M}\mathcal{B}(\lambda_j|\alpha,\beta-2j+1)\,|\,\eta,\alpha+\beta+N-1-2M\,\rangle$$

produces a Bethe state with different number of B-operators and shifted gauge parameter β

$\rightsquigarrow$ we don't know how to express it simply in terms of separate states

$\rightsquigarrow$ the expression of the resulting scalar product is not known in that case

$\rightsquigarrow$ we had to restrict our study to the computation of "elementary blocks" $\langle\prod_{n=1}^{m} E_n^{\epsilon'_n,\epsilon_n}(\xi_n|(a_n,b_n),(\bar{a}_n,\bar{b}_n))\rangle$ for which

$$\sum_{r=1}^{m}(\epsilon'_r-\epsilon_r)=0$$

Hypothesis on the ground state

Based on Nepomechie-Ravanini's conjecture, we suppose that we are in a configuration of boundary fields such that the homogeneous TQ-equation yields the ground state close to half-filling

- $\rightsquigarrow$ the constraint can be maintained when taking the limit $L \to \infty$
- $\rightsquigarrow$ the Bethe equations are very similar to the diagonal case:

$$\frac{a(-\lambda_j)\, d(\lambda_j)}{a(\lambda_j)\, d(-\lambda_j)} \prod_{\substack{\sigma=\pm \\ i\in\{1,2\}}} \frac{\sinh(\lambda_j + \check{\lambda}^{(0)}_{\sigma,i})}{\sinh(\lambda_j - \check{\lambda}^{(0)}_{\sigma,i})} \prod_{\substack{k=1 \\ k\neq j}}^{M} \prod_{\sigma=\pm} \frac{\sinh(\lambda_j - \sigma\lambda_k + \eta)}{\sinh(\lambda_j - \sigma\lambda_k - \eta)} = 1, \quad j = 1, \ldots, M$$

except for the boundary factor $\to$ 4 boundary parameters instead of 2:

$$\check{\lambda}^{(0)}_{\sigma,1} = \eta/2 - \epsilon_{\varphi_\sigma}\varphi_\sigma, \quad \check{\lambda}^{(0)}_{\sigma,2} = \eta/2 - \sigma\epsilon_{\varphi_\sigma}\psi_\sigma + i\tfrac{\pi}{2}, \quad \sigma = \pm$$

$\rightsquigarrow$ G.S. described when $L \to \infty$ by the same density $\rho(\lambda)$ of "real" Bethe roots over the same Fermi zone $[-\Lambda, \Lambda]$ as in the diagonal case
+ possibly isolated "complex" roots (boundary roots) of the form

$$\check{\lambda}_{\sigma,i} = \check{\lambda}^{(0)}_{\sigma,i} + \varepsilon_{\sigma,i}, \quad \sigma = \pm, \quad i \in \{1,2\}, \quad \varepsilon_{\sigma,i} = O(L^{-\infty})$$

$\to$ 4 possible boundary roots instead of 2

"Elementary building blocks" in the ground state

As in the diagonal case, the result is given as a multiple sum over scalar products, which turn in the half-infinite chain limit into multiple integrals over the Fermi zone $[-\Lambda, \Lambda]$ on which the Bethe roots condensate with density $\rho(\lambda)$ + possible contribution of the two (instead of one in the diagonal case) boundary roots $\check{\lambda}_{-,i}, \ i = 1, 2$ corresponding to the 2 boundary parameters at site 1

$$\langle \prod_{n=1}^{m} E_n^{\epsilon'_n, \epsilon_n}(\xi_n | (a_n, b_n), (\bar{a}_n, \bar{b}_n)) \rangle = \prod_{n=1}^{m} \frac{e^{\eta}}{\sinh(\eta b_n)} \frac{(-1)^s}{\prod_{j<i} \sinh(\xi_i - \xi_j) \prod_{i \le j} \sinh(\xi_i + \xi_j)}$$

$$\times \int_{\mathcal{C}} \prod_{j=1}^{s} d\lambda_j \int_{\mathcal{C}_{\boldsymbol{\xi}}} \prod_{j=s+1}^{m} d\lambda_j \ \underbrace{H_m(\{\lambda_j\}_{j=1}^{M}; \{\xi_k\}_{k=1}^{m})}_{\substack{\text{similar to the diagonal case} \\ \text{except that it has poles} \\ \text{in both parameters } \check{\lambda}^{(0)}_{-,i}, \ i=1,2}} \ \underbrace{\det_{1 \le j,k \le m}\left[\Phi(\lambda_j, \xi_k)\right]}_{\text{determinant of densities}},$$

The contours $\mathcal{C}$ and $\mathcal{C}_{\boldsymbol{\xi}}$ are defined as

$$\mathcal{C} = [-\Lambda, \Lambda] \cup \Gamma_{BR}, \qquad \mathcal{C}_{\boldsymbol{\xi}} = \mathcal{C} \cup \Gamma(\{\xi_k^{(1)}\}_{k=1}^{m})$$

where Γ_{BR} surrounds with index 1 the point(s) $\check{\lambda}^{(0)}_{-,i}$ iff the set of Bethe roots for the GS contains the boundary root(s) $\check{\lambda}_{-,i}$,
and $\Gamma(\{\xi_k^{(1)}\}_{k=1}^{m})$ the points $\xi_1^{(1)}, \ldots, \xi_m^{(1)}$, all other poles being outside.

Conclusion, perspectives and open problems

1. thermal form factor expansion of finite-temperature correlation functions

   - to be done : the low temperature limit (difficulties : complicated representation of (part of) the boundary factor, can it be simplified ?)
     $\rightsquigarrow$ explicit dependence on m of the magnetization at distance m from the boundary at $T = 0$?

2. multiple integral representation for some matrix elements of the open XXZ chain with non-longitudinal boundary fields (case with a constraint)

   - compute more general matrix elements with $\sum_{r=1}^{m}(\epsilon'_r - \epsilon_r) \neq 0$?
     $\rightarrow$ the action modifies the number of B-operators in the Bethe states and shifts the dynamical parameter β
     $\rightarrow$ no simple known expression of the resulting state in terms of separate states
     $\rightarrow$ no known formula for the resulting scalar product
   - case without constraint ?