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Hybrid integrable systems

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Poisson manifold

Let $(\mathcal{M}, \omega)$ be a symplectic manifold. Think of it as the phase space of a Hamiltonian system.

We assume that for each point $x \in \mathcal{M}$ we have a $*$-algebra A_x and this construction varies smoothly over $\mathcal{M}$. It is a vector bundle over a symplectic manifold $(\mathcal{M}, \omega)$

Multiplication and center

The space of smooth sections $A = \Gamma(\mathcal{M}, E)$ has a natural pointwise multiplication

$$(s_1 s_2)_x = (s_1)_x (s_2)_x.$$

The center of A is $Z(A) = C(\mathcal{M}) \cdot \mathbf{1}$, the subalgebra of sections of the form

$$s(x) = f(x) \cdot 1_x, \quad f(x) \in C(\mathcal{M}).$$

We will identify $Z(A)$ with $C(\mathcal{M})$, the space of smooth functions on $\mathcal{M}$.

In the case of Calogero-Moser system, zero order Hamiltonians belong to the center $H_k^{(0)} = H_k^{(CM)} \in Z(A)$.

There is a natural Poisson structure on this center

$$\{z_1, z_2\} = \omega^{-1}(dz_1 \wedge dz_2), \quad z_1, z_2 \in Z(A).$$

We also can extend this brackets to the action on whole algebra A

$$\{z, s\} = \omega^{-1}(dz \wedge d_\alpha s), \quad z \in Z(A),\ s \in A.$$

where d_α is the de Rham differential twisted by α with local trivialization

$$d_\alpha s = ds + [\alpha, s].$$

Here $d_\alpha^2 = 0$ because the connection is flat. In local coordinates

$$\{z, s\} = (\omega^{-1})^{ij}\partial_i z \partial_j s + (\omega^{-1})^{ij}\partial_i z[\alpha_j, s].$$

Connection α is flat

Flatness of connection α ensures that the algebra A has a natural Poisson module structure over $Z(A)$, which means

$$\{z, s_1 s_2\} = s_1\{z, s_2\} + \{z, s_1\}s_2,$$

$$\{z_1, \{z_2, s\}\} = \{\{z_1, z_2\}, s\} + \{z_2, \{z_1, s\}\}.$$

Azumaya Poisson Algebra

The center of this algebra is the ring of algebraic functions on $\mathcal{M}$. The algebra A is finite-dimensional over its center. We add a Poisson structure on $Z(A) \subset A$, which acts on A by derivation, so it is natural to use the term **Poisson Azumaya algebras** for such structures.

Derivation

Definition

A **derivation of a Poisson Azumaya algebra** is a derivation $D\colon A \to A$ of the associative algebra A, i.e. a linear map $A \to A$ such that $D(ab) = D(a)b + aD(b)$ for $a, b \in A$ which is also a derivation of the Poisson structure, i.e.

$$D(\{z, a\}) = \{D(z), a\} + \{z, D(a)\}$$

for any $z \in Z(A)$ and $a \in A$.

Hybrid derivation

D is a **hybrid derivation** if

$$D(a) = \{H_D^{(0)}, a\} + i[H_D^{(1)}, a]$$

for some $H_D^{(0)} \in Z(A)$ and $H_D^{(1)} \in A$.

The time evolution

Given a derivation D of a Poisson Azumaya algebra A, we define the time evolution

$$\frac{\partial a_t}{\partial t} = D(a_t), \quad \text{with} \quad a_0 = a \in A.$$

This is a hybrid version of the Heisenberg evolution.

A hybrid integrable multi-time evolution

of $s \in A$ with the classical integrable background dynamics generated by $I_1, \ldots, I_n$ (such that $\{I_k, I_k\} = 0$) and with quantum Hamiltonians $M_1, \ldots, M_n \in A$ is the solution to the system of differential equations

$$\frac{\partial s_{\mathbf{t}}}{\partial t_k} = \{I_k, s_{\mathbf{t}}\} + i[M_k, s_{\mathbf{t}}],$$

with the initial condition $s_0 = s$. The quantum Hamiltonians should satisfy the compatibility condition

$$\{I_k, M_l\} + \{M_k, I_l\} + i[M_k, M_l] = 0.$$

Matrix quantum mechanics

Problem

The goal of this section is to describe semiclassical solutions to the non-stationary matrix valued Schödinger equation when quantum Hamiltonian is semiclassically proportional to the identity matrix.

Hamiltonian

Assume that as $\hbar \to 0$ the Hamiltonian of the system has the following structure

$$\hat{H} = H_0(-i\hbar\frac{\partial}{\partial q}, q)I + \hbar M(-i\hbar\frac{\partial}{\partial q}, q) + O(\hbar^2).$$

where H_0 is a symmetric scalar valued differential operator, I is the identity matrix in $\mathbb{C}^N$ and M is a matrix valued symmetric differential operator.

Non-stationary Schödinger equation

We want to describe semiclassical solutions to the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(t,q) = \hat{H}\psi(t,q),$$

with initial conditions

$$\psi(0,q) = e^{\frac{i}{\hbar}f(q)}\varphi(q).$$

Theorem

As $\hbar \to 0$, the solution has the following asymptotic

$$\psi(q,t) = e^{\frac{iS(q,t)}{\hbar}} D(q,t)\Psi(q_0(q,t),t),$$

where $S(q,t)$ is the Hamilton-Jacobi action

$$S[\sigma] = \int_0^t (p(\tau)\dot{q}(\tau) - H(p(\tau),q(\tau)))d\tau + f(q(0)).$$

Function $q_0(q,t)$ is the solution of equation $q = q(t,q_0)$, and $D(q,t) = \left|\frac{\partial q_0(q,t)}{\partial q}\right|^{\frac{1}{2}}$.

Linear ODE

And $\Psi(q,t)$ would be the solution of the vector valued ODE

$$i\frac{d}{dt}\Psi(q,t) = M\left(\frac{\partial S(q,t)}{\partial q}, q\right)\Psi(q,t).$$

with the initial condition

$$\Psi(q,0) = \varphi(q).$$

Multitime matrix quantum mechanics

Now assume that we have n commuting

$$[\widehat{H}_k, \widehat{H}_l] = 0, , \qquad k, l = 1, \dots, n,$$

matrix valued differential operators on n-dimesional manifold Q of the form

$$\widehat{H}_k = H_k(-i\hbar\frac{\partial}{\partial q}, q)I + \hbar M_k(-i\hbar\frac{\partial}{\partial q}, q) + O(\hbar^2), \quad k = 1, \dots, n.$$

Multi-time evolution

The multi-time evolution $\psi \mapsto \psi_{\mathbf{t}}$ is a solution to the system of equations

$$-i\hbar \frac{\partial \psi(\mathbf{t})}{\partial t_k} = \widehat{H}_k \psi(\mathbf{t}), \quad \psi(\mathbf{0}) = \psi,$$

where $\mathbf{t} = (t_1, \ldots, t_n)$.

Linear ODEs

Similarly to the one time evolution we have the

$$i\frac{\partial\Psi(\mathbf{t},\mathbf{q_0})}{\partial t_k} = M_k\Big(\frac{\partial S}{\partial q}(q(\mathbf{t}),\mathbf{t})\Big)\Psi(\mathbf{t},\mathbf{q_0})$$

Zero curvative equation

as consictency condition

$$[i\partial_k - M_k, i\partial_l - M_l] = 0.$$

Fixed point

Integrability

If there is a classical configuration and trajectory such that all the coordinates stay umovement in all the times

$$[\partial_k, M_l] = 0,$$

then the zero curvative equation implies existence of family of commuting operators

$$[M_k, M_l] = 0.$$

Qunatum spin trigonometrical Calogero-Moser-Sutherland model

The Hamiltonian of this model is

$$\hat{H}_2 = -\frac{1}{2}\sum_{i=1}^{n}\hbar^2\frac{\partial^2}{\partial q_i^2} + \frac{\pi^2}{2L^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{1+\hbar P_{ij}}{\sin^2\frac{\pi(q_i-q_j)}{L}}.$$

Here we use coordinates $q_i \in \mathbb{R}/\mathbb{Z}L \simeq S^1, L \in \mathbb{R}_{>0}$.
And P_{ij} are permutation operators for two m-dimentional vectors spaces each.

Higher Hamiltonians

Introduce new variables $z_j = \exp\left(\frac{2\pi i q_j}{L}\right)$. Then a few fisrt Hamiltonians have the form

$$\widehat{H}_1 = \sum_{i=1}^{n} \hat{p}_i,$$

$$\widehat{H}_2 = \frac{1}{2}\sum_{i} \hat{p}_i^2 - \frac{1}{2}\sum_{j\neq i} \frac{z_i z_j}{(z_i - z_j)^2}\left(1 + \hbar P_{ij}\right),$$

$$\widehat{H}_3 = \frac{1}{3}\sum_{i=1}^{n} \hat{p}_i^3 - \sum_{i\neq j} \frac{z_i z_j\ \left(1 + \hbar P_{ij}\right)}{(z_i - z_j)^2}\hat{p}_i - \frac{\hbar}{3}\sum_{k\neq j\neq i} \frac{z_i z_j z_k\, P_{jk} P_{ij}}{(z_i - z_j)(z_j - z_k)(z_k - z_i)}$$

where $\hat{p}_i = \hbar\, z_i \frac{\partial}{\partial z_i}$.

Quantum integrable system

is formed by the Hamiltonians $\widehat{H}_k$ such that

$$\left[\widehat{H}_i, \widehat{H}_j\right] = 0, \qquad i, j = 1, \ldots, n.$$

Semiclassical expantion

All the Hamiltonians in the classical limit $\hbar \to 0$ have the form

$$\widehat{H}_k = \widehat{H}_k^{(0)} + \hbar\,\widehat{H}_k^{(1)} + O(\hbar^2).$$

Here $\widehat{H}_k^{(0)} = H_k^{CM}(p,z)$ are Hamiltonians of "spinless" CM system.

Zero part

of the Hamiltonians have the form

$$\widehat{H}_1^{(0)} = \sum_{i=1}^{n} \hat{p}_i, \quad \widehat{H}_2 = \frac{1}{2}\sum_{i} \hat{p}_i^2 - \frac{1}{2}\sum_{j\neq i} \frac{z_i z_j}{(z_i - z_j)^2},$$

$$\widehat{H}_3 = \frac{1}{3}\sum_{i=1}^{n} \hat{p}_i^3 - \sum_{i\neq j} \frac{z_i z_j}{(z_i - z_j)^2}\hat{p}_i,$$

and they commute also

$$\left[\widehat{H}_k^{(0)}, \widehat{H}_k^{(0)}\right] = 0.$$

In the classical limit

commutator becomes Poisson brackets

$$[f,g] = -i\hbar\{f,g\} + O(\hbar^2),$$

which implies qunatum integrable system generates classical integrable system in the first order of expanssion in $\hbar$

$$\left[\widehat{H}_k^{(0)}, \widehat{H}_l^{(0)}\right] = 0 \qquad \Longrightarrow \qquad \left\{H_k^{CM}, H_l^{CM}\right\} = 0.$$

Multi-time evolution

is generated by CM Hamiltonians:

$$\frac{\partial p_i}{\partial t_k} = z_i \frac{\partial H_k^{CM}}{\partial z_i}, \quad \frac{\partial z_i}{\partial t_k} = -z_i \frac{\partial H_k^{CM}}{\partial p_i}.$$

Dynamical Haldane-Shastry model

In the second order of expantion $\hbar \to 0$ we obtain

$$\left\{H_k^{CM}, M_l\right\} + \left\{M_k, H_l^{CM}\right\} + i\,[M_k, M_l] = 0,$$

or in the form of zero curvation equation

$$\left[\frac{\partial}{\partial t_k} + M_k(z,p,P), \frac{\partial}{\partial t_l} + M_l(z,p,P)\right] = 0.$$

Very precise example

Time dynamics

$$\frac{\partial z_i}{\partial t_2} = p_i z_i, \qquad \frac{\partial p_i}{\partial t_2} = -\sum_{j\neq i} \frac{z_i z_j (z_i + z_j)}{(z_i - z_j)^3},$$

$$\frac{\partial z_i}{\partial t_3} = p_i^2 z_i - \sum_{j\neq i} \frac{z_i^2 z_j}{(z_i - z_j)^2}, \qquad \frac{\partial p_i}{\partial t_3} = -\sum_{j\neq i} \frac{z_i z_j (z_i + z_j)}{(z_i - z_j)^3}(p_i + p_j).$$

M-operators

$$M_2 = -\frac{1}{2} \sum_{i\neq j} \frac{z_i z_j}{(z_i - z_j)^2} P_{ij},$$

$$M_3 = -\sum_{i\neq j} \frac{z_i z_j p_i}{(z_i - z_j)^2} P_{ij} + \frac{1}{3} \sum_{i\neq j\neq k} \frac{z_i z_j z_k P_{jk} P_{ij}}{(z_i - z_j)(z_j - z_k)(z_k - z_i)}$$

Then one can directly calculate

$$[M_2(z,p,P), M_3(z,p,P)] = \frac{\partial M_3(z,p,P)}{\partial t_2} - \frac{\partial M_2(z,p,P)}{\partial t_3}.$$

Low dimentional tori

The statement

The generic orbits of the multi-time Calogero-Moser dynamics are n-dimentional tori, but it also admits k-dimetional tori for any $k = 1, \ldots, n-1$ for special initial conditions.

The fixed point

Theorem

The point $x^* = (p^*, z^*)$

$$p_i^* = 0, \qquad z_k^* = \exp\left(\frac{2\pi i}{N} k\right).$$

is the fixed point of the multitime CM evolution, i.e.

$$dH_k^{CM}(x^*) = 0$$

Characteristic equation

The Calogero-Moser coordinates on this trajectory are the roots of equation

$$z^N + 1 = 0$$

Haldane-Shastry model

As a corollary we have the commutativity of corresponding M-operators

$$[M_k(z^*,p^*),M_l(z^*,p^*)]=0.$$

Where the operator M_2 is the Hamiltonian of the of Haldane-Shastry model

$$M_2=-\frac{1}{8}\sum_{i\neq j}\frac{P_{ij}}{\sin^2(\pi\frac{i-j}{n})},$$

and M_3 is symmetry of it

$$M_3=\frac{1}{24}\sum_{i\neq j\neq k}\frac{P_{jk}P_{ij}}{\sin(\pi\frac{i-j}{n})\sin(\pi\frac{j-k}{n})\sin(\pi\frac{k-i}{n})}.$$

1-dimentional tori

Characteristic equation

There are $N-1$ trajectories of 1-dimensional torus type.
The Calogero-Moser coordinates on these trajectories are the roots of equations

$$z^N + u\left(\beta z^{N-k} + \beta^{-1} z^k\right) + 1 = 0,$$

where

$$k = 1, \ldots, N-1, \qquad 0 < u < 1, \qquad |\beta| = 1,$$

and

$$\beta(\bar{t}) = \exp\,(i \sum_{i=1}^{N} v_i t_i).$$

Thank you for your attention!