F-basis, Bethe Ansatz, and Quantum Circuits

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in collaboration with

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Outline

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   - The Bethe Ansatz in the Era of Quantum Computing
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2. F-basis for Quantum Circuits
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   - F-basis and CBA
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The Bethe Ansatz in the Era of Quantum Computing

The Bethe Ansatz is an analytical method to tackle exactly solvable models in statistical and quantum mechanics.

The original coordinate Bethe Ansatz (CBA) is based on trial functions composed of linear superpositions of plane waves (‘magnons’) [Bethe, ’31].

The algebraic Bethe Ansatz (ABA) systematises the method by means of the R-matrix [Korepin, Bogoliubov, Izergin, ’93; Faddev, ’96].

The ABA can be realised via matrix-product states (MPS) [Alcaraz, Lazo, ’03–’06; Katsura, Maruyama, ’10; Murg, Korepin, Verstraete, ’12].

The Bethe Ansatz in the Era of Quantum Computing

Quantum computing calls for new testing grounds to push the boundaries of quantum supremacy further [Arute et al., '19].

- One-dimensional quantum spin-1/2 chains are suited to this task

$$\text{sites} \;:=\; \text{qubits}$$
$$|\uparrow\rangle := |0\rangle \ , \quad |\downarrow\rangle := |1\rangle$$

Bethe Ansatz:

- Systematic construction of eigenstates [KBI, '93; Faddev, '96].
- Computation of correlation functions [Kitanine, Maillet,Terras, '98].

The Bethe Ansatz in the Era of Quantum Computing

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$$\text{sites} := \text{qubits}$$
$$|\uparrow\rangle := |0\rangle$$
$$|\downarrow\rangle := |1\rangle$$

Challenges to quantum computing:

- Initialisation of algorithms $\begin{cases} \text{Real-time evolution of quenches} \\ \text{Adabatic preparation of ground states} \\ \text{etc.} \end{cases}$
- Benchmarking of quantum computers

The Bethe Ansatz in the Era of Quantum Computing

Can the Bethe Ansatz be adapted to quantum computers?

- Probabilistic algorithms [Van Dyke, Barron, Mayhall, Barnes, Economou, 21'; Van Dyke, Barnes, Economou, Nepomechie, '22; Li, Okyay, Nepomechie, '22].
- Deterministic algorithms [Sopena, Gordon, García-Martín, Sierra, López, '22; R. , Sopena, Gordon, Sierra, López, '23; Raveh, Nepomechie, '24].

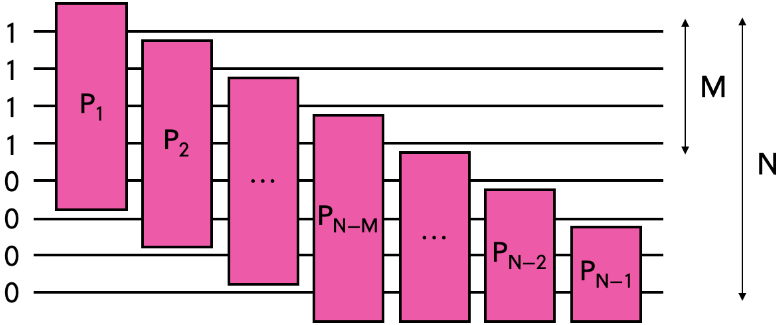
Outline

Algebraic Bethe Circuits

Algebraic Bethe circuits (ABC) are deterministic quantum circuits:

- a sequence of multi-qubit unitaries,
- an input state in the computational basis,
- and no ancillae.

$$|\Phi_N^{[M]}\rangle =$$

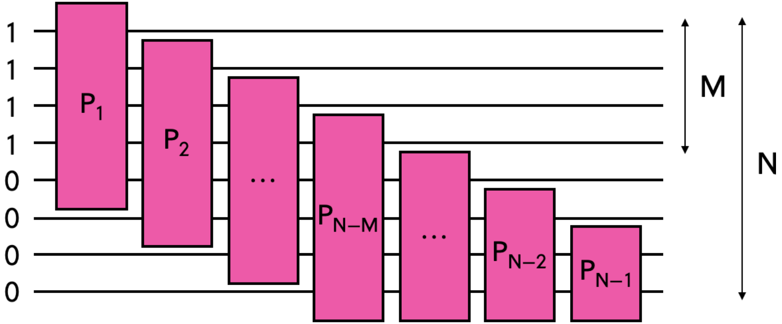


Algebraic Bethe Circuits

ABC apply to the XXZ model that

- has spin-1/2,
- is periodic,
- and is homogeneous.

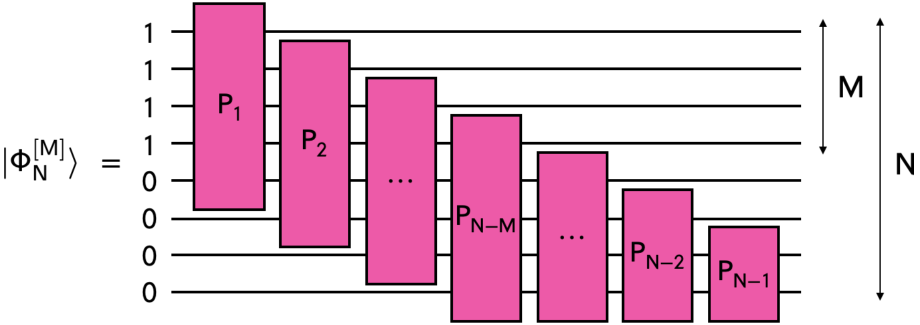
$|\Phi_N^{[M]}\rangle =$



Algebraic Bethe Circuits

ABC the prepare Bethe states of the Hamiltonian

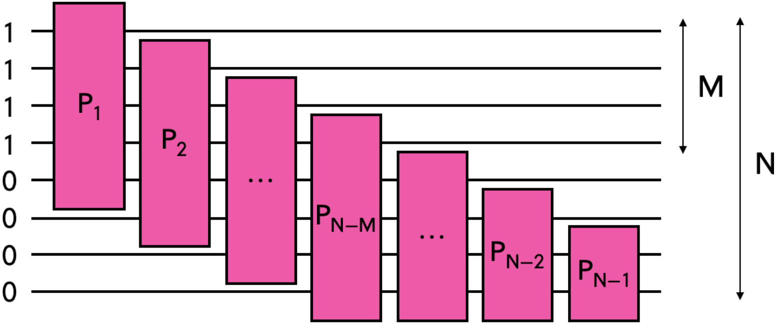
$$H = \sum_{j=1}^{N} \left(X_j X_{j+1} + Y_j Y_{j+1} + \Delta\, Z_j Z_{j+1} \right) .$$



Algebraic Bethe Circuits

- ABC is efficient in N: # of unitaries $\sim$ N.
- ABC is efficient in M at $\Delta = 0$: each unitary $\sim$ M two-qubit gates.
- If $\Delta \neq 0$, unclear: brute force on $P_j \sim \exp(M)$ one-/two-qubit gates.
- Additional cost of solving Bethe equations, imposed by hand.

$|\Phi_N^{[M]}\rangle =$



Algebraic Bethe Circuits

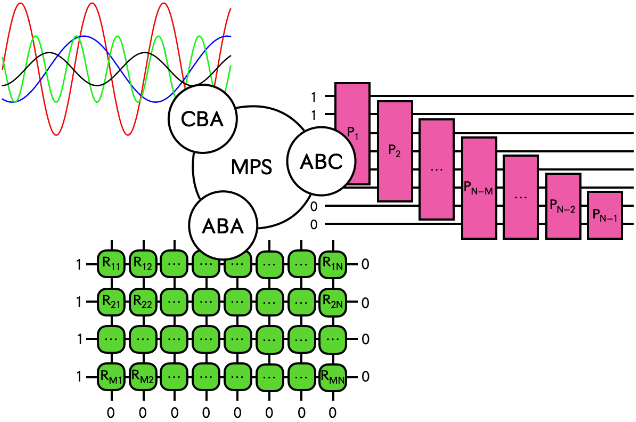
Problem

Search of models where ABC efficiently apply: systematisation of ABC.

Outline

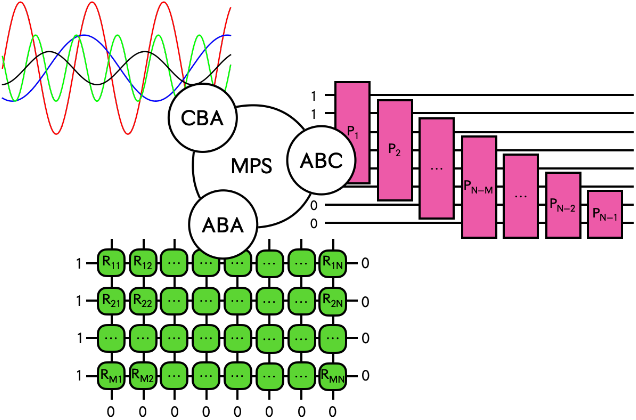
ABA = CBA = ABC

- ABC by direct unitarisation of the MPS of the ABA: P_j numerical
- ABC from proposed MPS of the CBA: P_j analytical
- ABC of the ABA and the CBA: unitarily equivalent



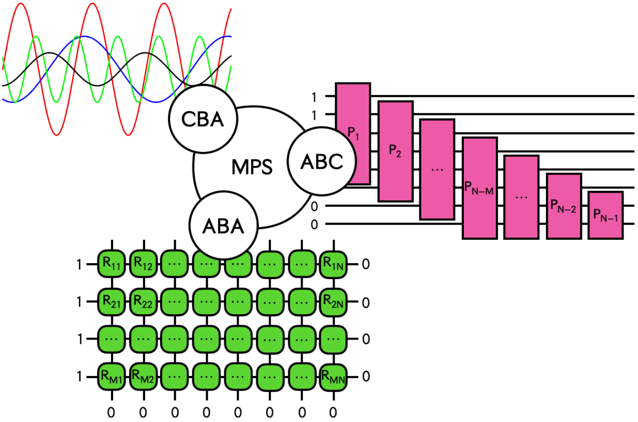
ABA = CBA = ABC

- ABA ↦ ABC: QR-factorisation.
- CBA ↦ ABC: Gram-Schmidt orthonormalisation.
- ABA ↦ CBA: ?



ABA = CBA = ABC

- ABA $\mapsto$ ABC: QR-factorisation.
- CBA $\mapsto$ ABC: Gram-Schmidt orthonormalisation.
- ABA $\mapsto$ CBA: F-basis and rescaling

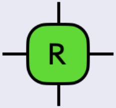


Outline

Inhomogeneous Periodic Spin-1/2 XXZ Model

R-matrix

$$R(u) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & f(u) & g(u) & 0 \\ 0 & g(u) & f(u) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$



,

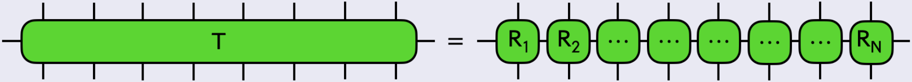
$$f(u) = \frac{\sinh u}{\sinh(u + i\gamma)} \, , \quad g(u) = \frac{\sinh(i\gamma)}{\sinh(u + i\gamma)} \, , \quad \Delta = \cos\gamma$$

- R acts on two qubits
- R solves the Yang-Baxter equation
- u denotes the spectral parameter

Inhomogeneous Periodic Spin-1/2 XXZ Model

Monodromy matrix

$$T(u) = \overleftarrow{\prod_{j=1}^{N}} R_{0j}(u - v_j)$$



- T acts on one ancilla and N physical qubits.
- T defines the exchange algebra of the ABA by the RTT-relation:

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v) \ .$$

- v_j denotes the inhomogeneity parameter of the j-th physical qubit.
- No local Hamiltonian for general v_j.

Inhomogeneous Periodic Spin-1/2 XXZ Model

R-matrix for general permutations

$$R^{\sigma}_{12\ldots M} T_1 T_2 \ldots T_M = T_{\sigma_1} T_{\sigma_2} \ldots T_{\sigma_M} R^{\sigma}_{12\ldots M} \ .$$

F-matrix

$$R^{\sigma}_{12\ldots M}(u_1, \ldots, u_M) = F^{-1}_{\sigma_1\sigma_2\ldots\sigma_M}(u_{\sigma_1}, \ldots, u_{\sigma_M}) F_{12\ldots M}(u_1, \ldots, u_M) \ .$$

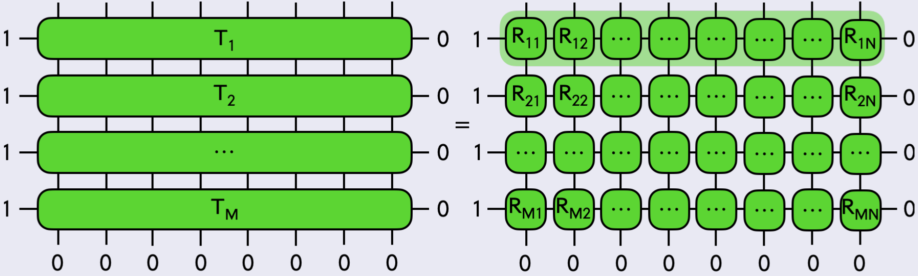
- The existence of F_{12} follows from

$$R_{12}(u) R_{21}(-u) = 1_2 \ .$$

- $F_{12\ldots M}$ admits a closed formula [Maillet, Sánchez de Santos, '96]
- $F_{12\ldots M}$ act on ancillae [Fehér, Pozsgay,'18], not physical qubits [MSS,'96; KMT,'98].

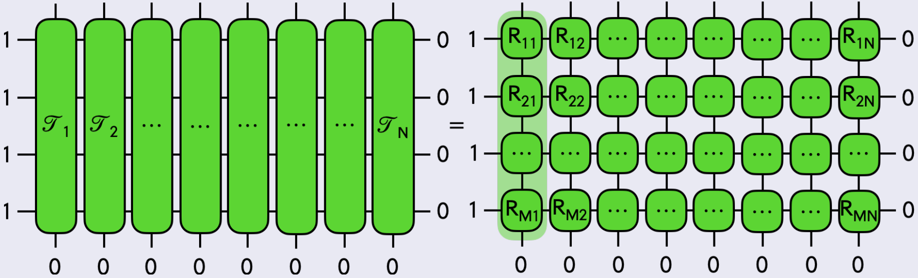
F-basis and ABA

$$B(u_1)B(u_2)\dots B(u_M)\,|0\rangle^{\otimes M}$$



F-basis and ABA

$$B(u_1)B(u_2)\dots B(u_M)\,|0\rangle^{\otimes M}$$

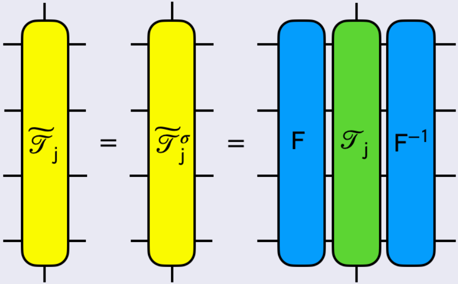


$\mathscr{T}_j$ is a monodromy matrix whose physical qubits and ancillae swap roles:

- carries the spectral parameter $-v_j$ of the j-th physical qubit
- and inhomogeneity $-u_a$ of the a-th ancilla.

F-basis and ABA

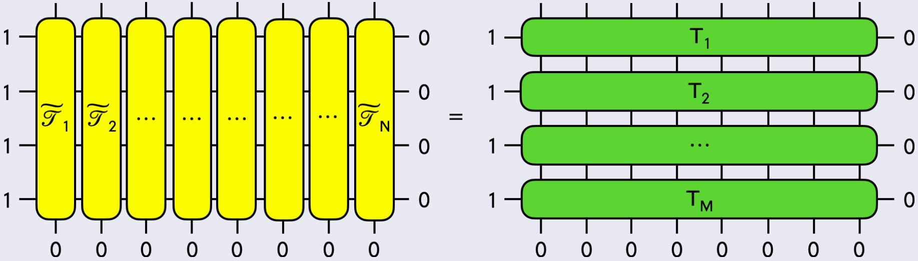
$$\widetilde{\mathscr{T}}_k = F_{12\ldots M}\mathscr{T}_k F^{-1}_{12\ldots M} \ ,$$



$$\langle i_1 \ldots i_M | \widetilde{\mathscr{T}}_k(-v_k; -u_1, \ldots, -u_M) \, | j_1 \ldots j_M \rangle$$
$$= \langle i_{\sigma_1} \ldots i_{\sigma_M} | \, \widetilde{\mathscr{T}}_k(-v_k; -u_{\sigma_1}, \ldots, -u_{\sigma_M}) \, | j_{\sigma_1} \ldots j_{\sigma_M} \rangle$$

F-basis and ABA

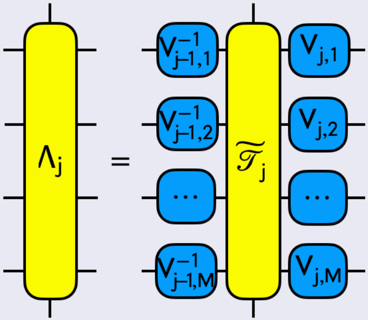
$$B(u_1)B(u_2)\dots B(u_M)\,|0\rangle^{\otimes M}$$



Outline

F-basis and CBA

$$\Lambda_j = \left[\bigotimes_{a=1}^{M} V_{j,a}\right] \widetilde{\mathscr{T}}_j \left[\bigotimes_{a=1}^{M} V_{j-1,a}^{-1}\right]$$



F-basis and CBA

$$\Lambda_j^i := \langle i|_j \Lambda_j |0\rangle_j$$

$$\Lambda_j^0 = \bigotimes_{a=1}^{M} \begin{bmatrix} 1 & 0 \\ 0 & x_{aj} \end{bmatrix}$$

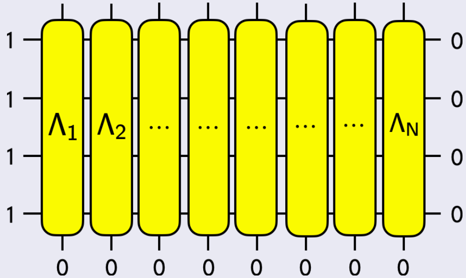
$$\Lambda_j^1 = \sum_{a=1}^{M} \bigotimes_{b=1}^{a-1} \begin{bmatrix} 1 & 0 \\ 0 & s_{ab}x_{bj} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \bigotimes_{c=a+1}^{M} \begin{bmatrix} 1 & 0 \\ 0 & s_{ac}x_{cj} \end{bmatrix}$$

$$x_{aj} = \frac{\sinh(u_a - v_j)}{\sinh(u_a - v_j + i\gamma)}, \quad s_{ab} = \frac{\sinh(u_a - u_b)}{\sinh(u_a - u_b + i\gamma)}$$

F-basis and CBA

Inhomogeneous CBA

$$|\Psi_N^{[M]}\rangle = \sum_{i_j=0,1} \langle 0|^{\otimes M} \Lambda_N^{i_N} \dots \Lambda_1^{i_1} |1\rangle^{\otimes M} |i_1 \dots i_N\rangle$$
$$= \sum_{1\leq n_1 < \dots < n_M \leq N} \psi_N^{[M]}(n_1, \dots, n_M) |n_1 \dots n_M\rangle$$



F-basis and CBA

Inhomogeneous CBA

Bethe states of the inhomogeneous CBA [also Ovchinnikov, '10] ($M = 2$):

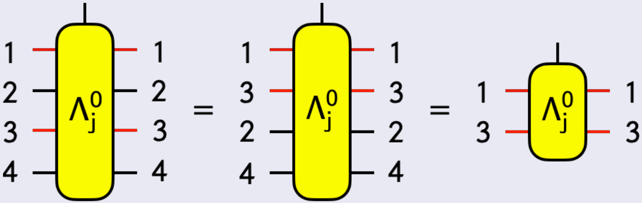
$$\Psi_N^{[2]}(n_1, n_2) = \left(s_{12} \left[\prod_{j=1}^{n_1 - 1} x_{1j} \right] \left[\prod_{k=1}^{n_2 - 1} x_{2k} \right] + s_{21} \left[\prod_{j=1}^{n_1 - 1} x_{2j} \right] \left[\prod_{k=1}^{n_2 - 1} x_{1k} \right] \right)$$

- Site-dependent quasi-momenta: $x_{aj} = \exp(ip_{aj})$
- Scattering amplitudes: s_{ab}
- Scattering matrix: $S_{ab} = \frac{s_{ba}}{s_{ab}}$
- $v_j = 0$: $\Psi_N^{[2]}(n_1, n_2) = s_{12} x_1^{n_1 - 1} x_2^{n_2 - 1} + s_{21} x_2^{n_1 - 1} x_1^{n_2 - 1}$

F-basis and CBA

$$\langle 0|^{\otimes M}\left[\overleftarrow{\prod_{j=1}^{k}}\Lambda^{i_j}(u_1,\ldots,u_M)\right]|m_1\ldots m_r\rangle_M$$

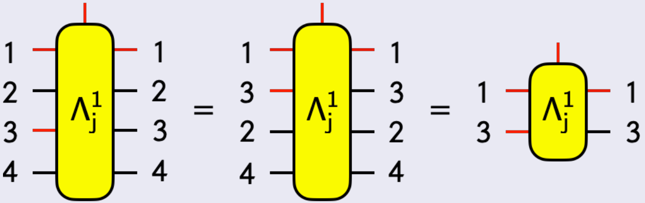
$$=\langle 0|^{\otimes r}\left[\overleftarrow{\prod_{j=1}^{k}}\Lambda^{i_j}(u_{m_1},\ldots,u_{m_r})\right]|1\rangle^{\otimes r}$$



F-basis and CBA

$$\langle 0|^{\otimes M}\left[\overleftarrow{\prod_{j=1}^{k}}\Lambda^{i_j}(u_1,\dots,u_M)\right]|m_1\dots m_r\rangle_M$$

$$=\langle 0|^{\otimes r}\left[\overleftarrow{\prod_{j=1}^{k}}\Lambda^{i_j}(u_{m_1},\dots,u_{m_r})\right]|1\rangle^{\otimes r}$$



F-basis and CBA

Partial Bethe states

$$\left|\Psi_k^{[r]}\right\rangle = \sum_{i_\ell=0,1} \langle 0|^{\otimes M} \Lambda_N^{i_N} \dots \Lambda_{j_k}^{i_{j_k}} \left|m_1 \dots m_r\right\rangle_M \left|i_{j_k} \dots i_N\right\rangle_k$$

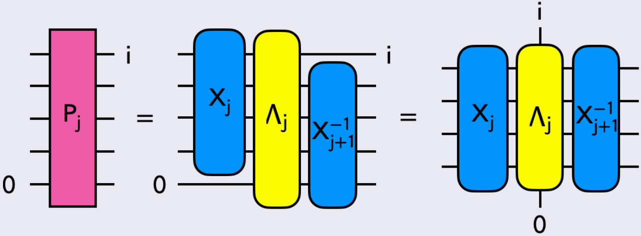
$$j_k = N + 1 - k$$

Outline

F-basis and ABC

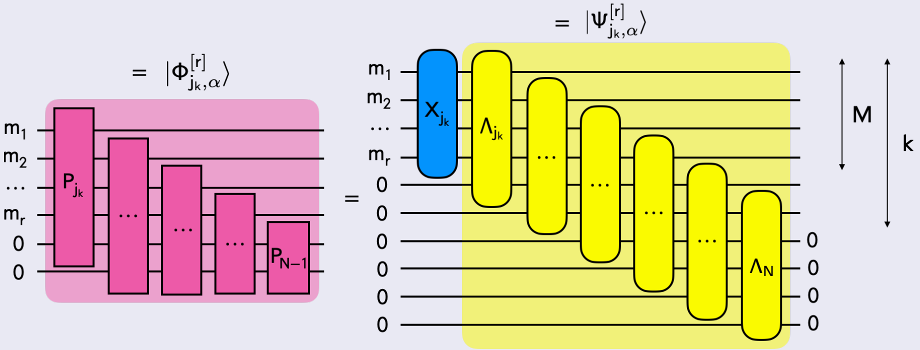
Unitaries of the ABC

$$P_j = X_{j+1}^{-1} \Lambda_j X_j \ ,$$



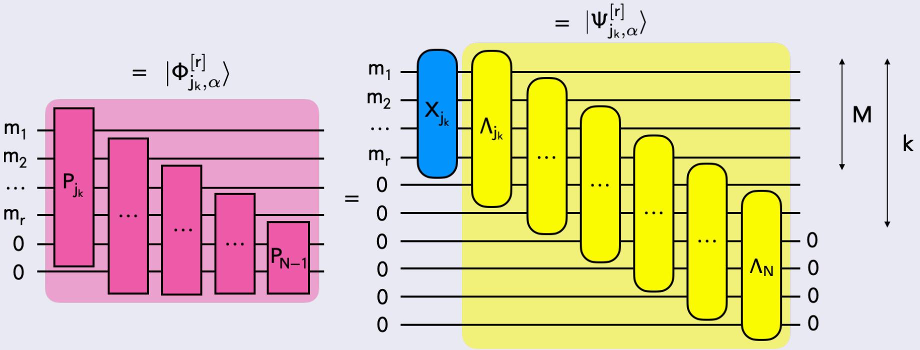
F-basis and ABC

$$\text{MPS}_{j_k} : |m_1 m_2 \dots m_r\rangle_M \mapsto |\Psi^{[r]}_{k,\alpha}\rangle$$



F-basis and ABC

$$\mathrm{ABC}_{j_k} : |m_1 m_2 \dots m_r\rangle_M \mapsto |\Phi^{[r]}_{k,\alpha}\rangle$$



F-basis and ABC

$$|\Phi^{[r]}_{k,\beta}\rangle = \sum_{\alpha} X^{[r]}_{j_k,\alpha\beta} |\Psi^{[r]}_{k,\alpha}\rangle \ .$$

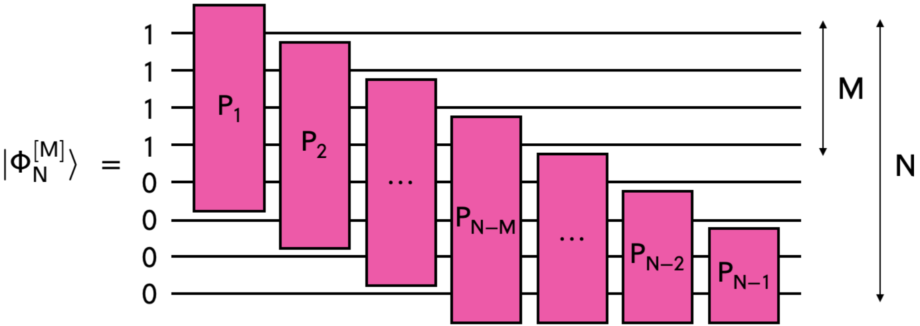
- Gram matrix: $C^{[r]}_{k,\alpha\beta} = \langle \Psi^{[r]}_{k,\alpha} | \Psi^{[r]}_{k,\beta} \rangle$
- $$\begin{cases} X^{[r]}_{j_k,\alpha\alpha} & = \sqrt{\dfrac{\det_{\alpha-1} C^{[r]}_k}{\det_{\alpha} C^{[r]}_k}} \ , \\ X^{[r]}_{j_k,\alpha\beta} & = 0 \ \text{ if } \ \alpha > \beta \ , \\ X^{[r]}_{j_k,\alpha\beta} & = -\dfrac{\det_{\beta-1} C^{[r]}_{k,\alpha\to\beta}}{\sqrt{\det_{\beta-1} C^{[r]}_k \det_{\beta} C^{[r]}_k}} \quad \text{if} \ \ \alpha < \beta \ , \end{cases}$$

F-basis and ABC

$$|\Phi^{[r]}_{k,\beta}\rangle = \sum_{\alpha} X^{[r]}_{j_k,\alpha\beta}|\Psi^{[r]}_{k,\alpha}\rangle \ .$$

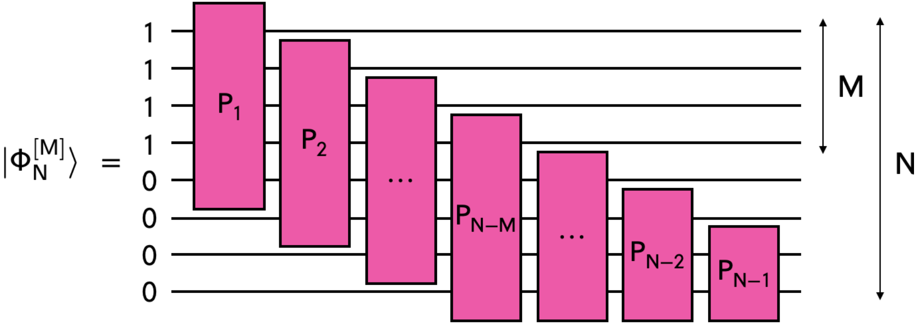
- Gram matrix: $C^{[r]}_{k,\alpha\beta} = \langle\Psi^{[r]}_{k,\alpha}|\Psi^{[r]}_{k,\beta}\rangle$
- $X^{-1[r]}_{j_k,\alpha\beta} = \dfrac{\det_{\alpha} C^{[r]}_{k,\alpha\to\beta}}{\sqrt{\det_{\alpha-1} C^{[r]}_{k} \det_{\alpha} C^{[r]}_{k}}} \ .$

F-basis and ABC



- Ancillae are eliminated.
- Short unitaries follow from a new tensor that prepares Bethe states.

F-basis an



- The output is normalised.
- The determination of unitaries is analytical.
- The recurrence relation $C_k^{[r]} = \sum_{i=0,1} \Lambda_{j_k}^{[i,r]\dagger} C_{k-1}^{[r-i]} \Lambda_{j_k}^{[i,r]}$ implies unitarity.

Summary

Results

- The F-basis and the CBA by MPS.
- Systematisation ABC.
- ABC for the inhomogeneous periodic spin-1/2 XXZ model.
- (Demostration that the ABC exactly prepares Bethe states).

Open Problems

Use v_j as variational parameters to find an efficient factorisation of P_j.

Open Problems

Test our approach by building the ABC of the models with F-basis, e.g. the spin-1/2 XXZ model with open boundary conditions [Kitanine, Kozlowski, Maillet, Niccoli, Slavnov, Terras, '07].

Thank you very much for your attention.