NEW INTEGRABLE MODELS:

from short- to long-range deformations

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RAQIS 2024

PLAN

* Introduction.

* Construction of (exact) R-matrices.

* Construction of (perturbative) R-matrices.

Introduction.

IS A SPECIFIC HAMILTONIAN

INTEGRABLE ?

HOW TO SYSTEMATICALLY CONSTRUCT SOLUTIONS OF THE YANG-BAXTER EQUATION ?

(Algebraic) SYMMETRIES

Yangian symmetry

Quantum group symmetry

Temperley-Lieb

⋮

$\Rightarrow$ R-matrix

Yang, Baxter, Zamolodchikov, Zamolodchikov, Kulish, Drinfeld, Faddeev, Sklyanin, Kuniba, Jimbo, Bazhanov, Crampe, Frappat, Avan, Reshetikhin, Ragoucy, Vanicat, Shastry, Izergin, Korepin, Pakuliak, Belavin, Doikou, Fateev, Batchelor, Dorey, Beisert, Hoare, Torrieli, Sfondrini, Sax, Stefánski, Serban, ...

CBA

Models that can be solved via Coordinate Bethe ansatz

Crampé, Frappat, Ragoucy, Vanicat, Fonseca

Expansion

$$R(u) = P\left(\mathbb{1} + \sum_{n\geq 1} u^n A(u)\right)$$

Solve perturbatively and re-sum

Idzumi

Set-theoretic YBE

$\Rightarrow$ Generalisations of Yangian

Doikou, Vlaar, Rybolowicz

$R(u,v) = R(u-v)$ versus $R(u,v) \neq R(u-v)$

Difference form

Non-difference form

* MANY MODELS
* MANY FAMILIES OF MODELS

* FREE-FERMIONIC MODELS
* HUBBARD MODEL
* AdS_2, AdS_3, AdS_4, AdS_5

$\vdots$

$$R_{12}(u_1,u_2)\,R_{13}(u_1,u_3)\,R_{23}(u_2,u_3) = R_{23}(u_2,u_3)\,R_{13}(u_1,u_3)\,R_{12}(u_1,u_2)$$

For $\quad R_{ij}(u,v) = R_{ij}(u-v)$

$$R_{12}(u_1-u_2)\,R_{13}(u_1-u_3)\,R_{23}(u_2-u_3) = R_{23}(u_2-u_3)\,R_{13}(u_1-u_3)\,R_{12}(u_1-u_2)$$

$$\{u_1 \to u_1+u_3\;,\; u_2 \to u_2+u_3\}$$

$$R_{12}(u_1-u_2)\,R_{13}(u_1)\,R_{23}(u_2) = R_{23}(u_2)\,R_{13}(u_1)\,R_{12}(u_1-u_2)$$

$$\frac{d}{du_1} \, YBE(u_1,u_2)\Big|_{u_2=u_1}$$

Vieira '19

Classified all 4x4 rational R-matrices

Q1. HOW TO SOLVE THE YANG-BAXTER EQUATION WHEN NO ALGEBRAIC SYMMETRY IS ASSUMED AND WITHOUT KWOWING IF $R(u_1,u_2) = R(u_1,-u_2)$?

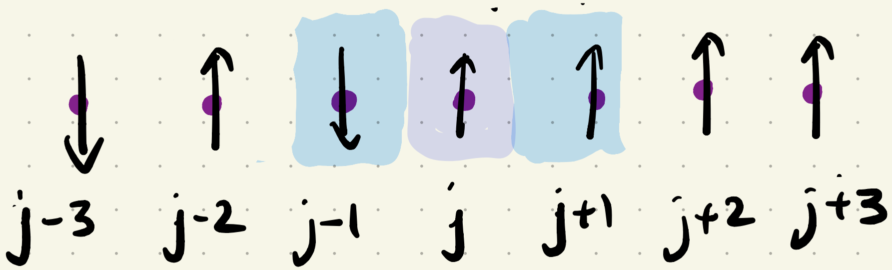
Work with M. de Leeuw, C. Paletta, A. Pribytok and P. Ryan, 2020

For almost all the models we mentioned so far, if we compute the homogeneous spin chain

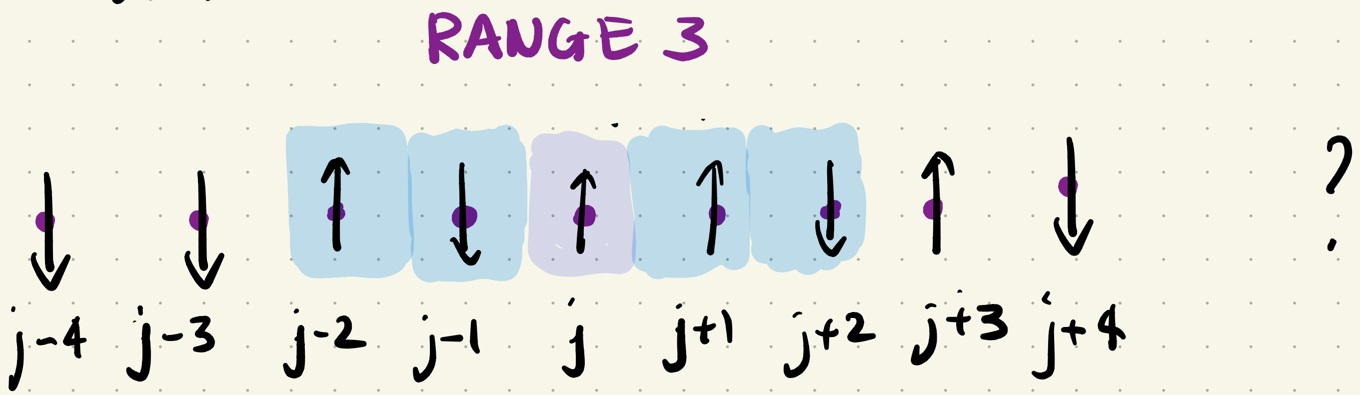
$$\mathbb{H} = \sum_{j=1} h_{j\,j+1},$$

NEAREST NEIGHBOUR (NN) HAMILTONIAN

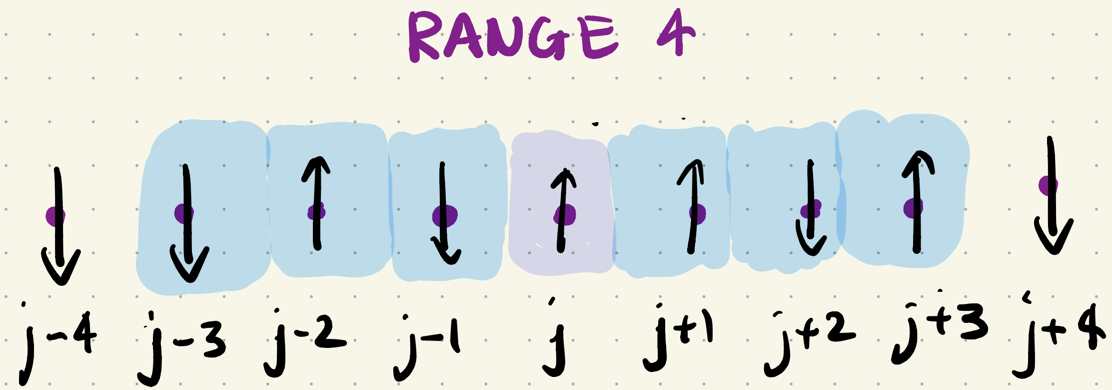
RANGE 2



But what about



And range 4 ?



And long-range chains ?

The understanding of such spin chains is very limited when compared with NN cases.

* Examples included:
  - Inhomogeneous spin chains
  - Inozemtsev, Haldane-Shastry, Calogero-Sutherland, and models by Fukui, Kawakami, Uglov, Polychronakos, Ruijsenaars, ...

* Recent progress due to Sechin, Zotov, Matushko, Serban, Chalykh, Klabbers, Lamers, ...

* Free-fermions and Free-parafermions in disguise

  (Fendley, Pimenta, Pozsgay, Vona, ...

* Medium-range spin chains

  (Gombor, Pozsgay, '21)

* Gauge theories

  (Nieto-García's talk, ...)

PERTURBATIVE INTEGRABILITY

If someone gives you the following density Hamiltonian

$$h = 1 - P_{12} + g\,P_{13} + g^2\,\mathcal{H}^{(r=4)} + \mathcal{O}(g^3)$$

↳ What are you allowed to put here to have integrability up to order g^2 ?

Bargheer, Beisert & Loebbert, '09

PERTURBATIVE INTEGRABILITY

If someone gives you the following density Hamiltonian

$$h = 1 - P_{12} + g\, P_{13} + g^2\, \mathcal{H}^{(r=4)} + \mathcal{O}(g^3)$$

How to compute the Lax and the R-matrix

for this model such that

- RLL relations
- YBE
- $[t(u), t(v)] = 0$

are satisfied up to g^2?

Q_2: FOR

$$h = h^{NN} + g\, h^{NNN} + g^2\, h^{NNNN} + \cdots$$

HOW TO SYSTEMATICALLY CONSTRUCT

$\mathcal{L}$ & R ?

work with M. de Leeuw

Q1. HOW TO SOLVE THE YANG-BAXTER EQUATION WHEN NO ALGEBRAIC SYMMETRY IS ASSUMED AND WITHOUT KWOWING IF $R(u_1,u_2) = R(u_1-u_2)$?

Work with M. de Leeuw, C. Paletta, A. Pribytok and P. Ryan, 2020

BOOST METHOD: (WITHOUT ASSUMING $R(u,v) = R(u-v)$)

$$R(u,u) \sim P$$

$$\left.\frac{dR(u,v)}{du}\right|_{v=u} \sim P\, h(u)$$

BOUNDARY CONDITIONS

The key ingredient is the boost operator (Links, Zhou, McKenzie, Gold, 2001)

$$B[Q_2] = -\sum_n n\, h_{n,n+1}(u) + \frac{\partial}{\partial u}$$

It is very useful because

$$Q_{r+1} = [\mathbb{B}[Q_2], Q_r] \qquad r>1$$

$\mathbb{B}$: Boost operator

$$Q_3 = [\mathbb{B}[Q_2], Q_2]$$

$$= \sum_{i=1}^{L} [h_{i-1,i}(u), h_{i,i+1}(u)] + \frac{d}{du} \mathbb{H}(u)$$

It is very useful because

$$Q_{r+1} = [\underbrace{\mathbb{B}[Q_2]}_{\text{Boost operator}}, Q_r]$$

For $R(u,v) = R(u-v)$

$R'(0) = Ph$

$$Q_3 = [\mathbb{B}[Q_2], Q_2]$$

$$= \sum_{i=1}^{L} [h_{i-1,i}(u), h_{i,i+1}(u)] + \cancel{\frac{d}{du}\mathbb{H}(u)}$$

(Tetel'man, 1982)

STEP 1:

$$\text{YBE} = R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) - R_{23}(v,w)R_{13}(u,w)R_{12}(u,v) = 0$$

$$\left.\frac{d(\text{YBE})}{du}\right|_{v=u} = 0 \Rightarrow [R_{13}R_{23}, h_{12}] = \dot{R}_{13}R_{23} - R_{13}\dot{R}_{23}$$

SUTHERLAND EQUATION

Can we obtain extra information about $\underline{h}$?

YES, using the boost operator!

STEP 2: Ansatz

$$h(u) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_1(u) & a_3(u) & 0 \\ 0 & a_4(u) & a_2(u) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and compute periodic Hamiltonian.

STEP 3: Compute Q_3 using the Boost operator

$$Q_3 = [\mathbb{B}[Q_2], Q_2]$$

$$= -\sum_j [h_{j-1\,j}(u), h_{j\,j+1}(u)] + \frac{d\overset{Q_2}{\mathbb{H}}(u)}{du}$$

$$
q_3(u) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -a_1 a_3 & 0 & -a_3^2 & 0 & 0 & 0 \\
0 & a_1 a_4 & \dot{a}_1 & 0 & \dot{a}_3 - a_2 a_3 & 0 & 0 & 0 \\
0 & 0 & 0 & \dot{a}_1 & 0 & \dot{a}_3 + a_1 a_3 & a_3^2 & 0 \\
0 & a_4^2 & \dot{a}_4 + a_2 a_4 & 0 & \dot{a}_2 & 0 & 0 & 0 \\
0 & 0 & 0 & \dot{a}_4 - a_1 a_4 & 0 & \dot{a}_2 & a_2 a_4 & 0 \\
0 & 0 & 0 & -a_4^2 & 0 & -a_2 a_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$a_i \equiv a_i(u)$$

STEP 4 : Require $[Q_2, Q_3] = 0$

and carefully solve all ODEs

$$\dot{a}_3 (a_1 + a_2) = (\dot{a}_1 + \dot{a}_2) a_3 \qquad \dot{a}_4 (a_1 + a_2) = (\dot{a}_1 + \dot{a}_2) a_4$$

$$\boxed{a_3 = \frac{\alpha}{2}(a_1 + a_2)} \qquad \boxed{a_4 = \frac{\beta}{2}(a_1 + a_2)}$$

$$\Rightarrow \quad h(u) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_1 & \frac{\alpha}{2}(a_1+a_2) & 0 \\ 0 & \frac{\beta}{2}(a_1+a_2) & a_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\rightarrow$ Potentially integrable

STEP 5: Plug it in Sutherland equation:

$$[R_{13}R_{23}, h_{12}] = \dot{R}_{13}R_{23} - R_{13}\dot{R}_{23}$$

and solve the PDEs:

$$\Rightarrow R = e^{A_+}\begin{bmatrix} \cos wA_+ & -\sin w A_+ & 0 & 0 & 0 \\ 0 & \beta\dfrac{\sin w A_+}{w} & e^{-A_-} & 0 \\ 0 & e^{A_-} & \alpha\dfrac{\sin w A_+}{w} & 0 \\ 0 & 0 & 0 & \cos wA_+ & -\sin w A_+ \end{bmatrix}$$

where $w^2 = \alpha\beta - 1$, $A_\pm = \dfrac{A_1 \pm A_2}{2}$ where $A_i \equiv A_i(u,v) = \int_u^v a_i(x)\,dx$

Ansatz for Hamiltonian

$h(a_{ij}(u))$

Ansatz for Hamiltonian $h(a_{ij}(u))$ $\Rightarrow$ Compute Q_3 using the Boost op.

$$= -\sum_j \left[h_{j-1\,j}(u), h_{j\,j+1}(u) \right] + \frac{d\mathbb{H}}{du}$$

Ansatz for Hamiltonian $h(a_{ij}(u))$

$\Longrightarrow$

Compute Q_3 using the Boost op.

$$= -\sum_j \left[h_{j-1\,j}(u), h_{j\,j+1}(u) \right] + \frac{d\mathbb{H}}{du}$$

$\Downarrow$

Compute $[Q_2, Q_3] = 0$ and solve ODEs:

$$\Longleftarrow \quad \underbrace{h_1(u) \qquad h_2(u) \qquad \dots \qquad h_n(u)}_{\text{Potentially integrable}}$$

Ansatz for Hamiltonian $h(a_{ij}(u))$ $\Longrightarrow$ Compute Q_3 using the Boost op.

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$\Longleftarrow$ Plug each h_i in the Sutherland equation and find R-matrix

Ansatz for Hamiltonian $h(a_{ij}(u))$ $\Rightarrow$ Compute Q_3 using the Boost op.

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$\Downarrow$

Compute $[Q_2, Q_3] = 0$ and solve ODEs:

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$\Leftarrow$ Plug each h_i in the Sutherland equation and find R-matrix

$\Uparrow$

Check YBE

$[Q_2, Q_3] = 0$ was enough to fix $\mathbb{H}$ and obtain R

If you would like to find difference-form models just make h constant.

Transformations like

- $R(u,v) \to R(f(u), f(v))$ — reparametrisation
- $R(u,v) \to f(u,v)\, R(u,v)$ — normalisation
- $R(u,v) \to (V \otimes V)\, R(u,v)\, (V \otimes V)^{-1}$ — basis transformations
- $R(u,v) \to P\, R(u,v)\, P$
- $R(u,v) \to R(u,v)^T$ — transposition

and

- $R(u,v) \to (U(u) \otimes 1)\, R(u,v)\, (1 \otimes U(v)^{-1})$ — twist

  if $[R(u,v),\, U(u) \otimes U(v)] = 0$

preserve YBE!

"Disadvantages:"

Normalisation

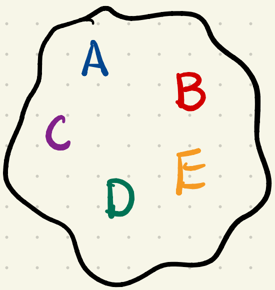
Reparametrisation

Basis transformation

Discrete transformations

Twists

Expectation:



Some interesting solutions

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Normalisation

Reparametrisation

Basis transformations

Discrete transformations

Twists

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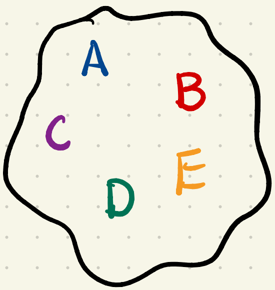


Some interesting solutions

Reality:

Lots of dependent solutions

"Disadvantages:"

Normalisation

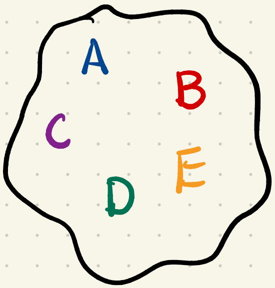
Reparametrisation

Basis transformations

Discrete transformations

Twists

Expectation:

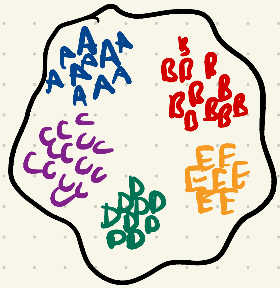


Some interesting solutions

Reality:

Lots of dependent solutions

reasonable amount of work $\Longrightarrow$



This disadvantage can be turned into an advantage

We can use this freedom to

- Put one diagonal element of the Hamiltonian to zero,
- Put two elements the same

This makes the system much easier to solve!

By doing this, our example becomes the XXZ

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We applied the method to

- $su(2) \oplus su(2)$ diff. and non-diff
- 15-vertex model non-diff.

de Leeuw, Paletta
Pribytok, ALR, Ryan, 2020, 2021

- Full 4x4 non-difference form — Corcoran, de Leeuw, 2023
- Flag models — de Leeuw, Nepomechie, ALR, 2022
- Systematic construction of integrable Lindblad systems — de Leeuw, Paletta, Pozsgay, 2021
- AdS_3 and AdS_2 integrable deformations — de Leeuw, Paletta, Pribytok, ALR, Ryan, 2020

FLAG MODELS (work with de Leeuw and Nepomechie, 2022)

Usually the building blocks of $SO(n)$ are (for a local Hilbert space of dimension $\underline{n}$)

$$\mathbb{P} = \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji}$$

$$\mathbb{I} = \sum_{i,j=1}^{n} e_{ii} \otimes e_{jj} \qquad (e_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$$

$$\mathbb{K} = \sum_{i,j=1}^{n} e_{ij} \otimes e_{ij}$$

$$\mathbb{P}^{(m,n)} = \sum_{i,j=1}^{m} e_{ij} \otimes e_{ji} \,, \qquad \mathbb{I}^{(m,n)} = \sum_{i,j=1}^{m} e_{ii} \otimes e_{jj}$$

$$\mathbb{K}^{(m,n)} = \sum_{i,j=1}^{m} e_{ij} \otimes e_{ij} \,, \qquad 1 \leq m \leq n$$

$$n = k_0 > k_1 > \cdots > k_{d-1} \geq 1 \qquad \vec{k} = \{ k_0, k_1, \ldots, k_{d-1} \}$$

$$h^{\vec{K}} = \sum_{i=0}^{d-1} \left(a_i \mathbb{I}^{(k_i,n)} + b_i \mathbb{P}^{(k_i,n)} + c_i \mathbb{K}^{(k_i,n)} \right)$$

For which values of a_i, b_i and c_i is the model integrable?

$$n = k_0 > k_1 > \cdots > k_{d-1} \geq 1 \qquad \vec{k} = \{k_0, k_1, \ldots, k_{d-1}\}$$

$$h^{\vec{k}} = \sum_{i=0}^{d-1} \left(a_i \mathbb{I}^{(k_i, n)} + b_i \mathbb{P}^{(k_i, n)} + c_i \mathbb{K}^{(k_i, n)} \right)$$

- $\vec{k} = \{4, 3\}$ (*)
- $\vec{k} = \{4, 2\}$
- $\vec{k} = \{4, 1\}$

$n = 4 \qquad \vec{k} = \{k_0, k_1, k_2\}$

$d = 2$

$$\left. \begin{aligned} h^{\vec{k}} &= a_1 \mathbb{I}^{(4,4)} + b_1 \mathbb{P}^{(4,4)} + c_1 \mathbb{K}^{(4,4)} \\ &+ a_2 \mathbb{I}^{(3,4)} + b_2 \mathbb{P}^{(3,4)} + c_2 \mathbb{K}^{(3,4)} \end{aligned} \right\}$$ Apply the Boost method

We found four classes of models:

Model 2:

$$R^{\pm,\vec{k}}(u) = (u+1)\Big(u\,\mathbb{I}^{(n,n)} + \alpha\,\mathbb{P}^{(n,n)} + 2u\sum_{j=1}^{d-2} (-1)^j\,\mathbb{I}^{(k_j,n)} - u(-1)^d\,\mathbb{I}^{(k_{d-1},n)} \pm u\,\mathbb{P}^{(k_{d-1},n)} \Big)$$

For d=2 and +, these models correspond to a basis transformation of the rational limit of Maassarani's models

(There is a generalisation of Maassarani's models to GL(M|N) by Drummond, Feverati, Frappat and Ragoucy, 07)

BETHE ANSATZ

It has a generalized graded symmetry + discrete symmetries + ... ?

Model II$^+$ (n = 5)

	d = 2			d = 3			d = 4
l = 0	$\{5,4\}^+$	$\{5,3\}^+$	$\{5,2\}^+$	$\{5,4,3\}^+$	$\{5,4,2\}^+$	$\{5,3,2\}^+$	$\{5,4,3,2\}^+$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
l = 1	$\mathbb{P}^{(4)}$	$\{4,3\}^+$	$\{4,2\}^+$	$\{4,3\}^-$	$\{4,2\}^-$	$\{4,3,2\}^+$	$\{4,3,2\}^-$
		$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
l = 2		$\mathbb{P}^{(3)}$	$\{3,2\}^+$	$\mathbb{P}^{(3)}$	$\{3,2\}^-$	$\{3,2\}^-$	$\{3,2\}^+$
			$\downarrow$		$\downarrow$	$\downarrow$	$\downarrow$
l = 3			$\mathbb{P}^{(2)}$		$\mathbb{P}^{(2)}$	$\mathbb{P}^{(2)}$	$\mathbb{P}^{(2)}$

$$\Lambda_{aux}^{(k_0-k_{d-1})} = \begin{cases} \exp\left(\frac{2\pi i p}{m_{k_0-k_{d-1}}}\right), & p=0,1,\dots,m_{k_0-k_{d-1}}-1 \quad \text{if } m_{k_0-k_{d-1}} \neq 0 \\ k_{d-1}, & \text{if } m_{k_0-k_{d-1}} = 0 \end{cases}$$

Q_2: FOR

$$h = h^{NN} + g\, h^{NNN} + g^2\, h^{NNNN} + \cdots$$

HOW TO SYSTEMATICALLY CONSTRUCT

$\mathcal{L}$ & R ?

work with M. de Leeuw, 2022

"WARM-UP CASE" SU(2)

$$h = 1 - P_{12} + g\, P_{13} + \mathcal{O}(g^2)$$

How to obtain $\mathcal{L}$ and R ?

STEP 1: Increase the auxiliar space.

Gombor and Pozsgay 21'

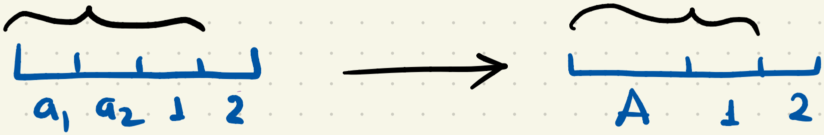
For range 3 : Lax: $\mathcal{L}_{A_j}$ $\uparrow$ $\mathcal{L}_{a_1 a_2 j}$

double the auxiliar space

$$\mathcal{L}_{A_j} : V \otimes V \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H} \mapsto V \otimes V \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}$$

$\vartheta_{a_1 1}$

↳ range 3



ϑ_{A1}

↓

NN

As long as $\mathcal{L}_{a_1 a_2 j}(0) = P_{a_1 j} P_{a_2 j}$ a local Hamiltonian can be constructed with density

$$P_{13} P_{23} h_{123} = \mathcal{L}'_{123}(0)$$

$$\mathbb{H} = \sum_j^L P_{j+1\, j+2} P_{j\, j+2} \mathcal{L}'_{j\, j+1\, j+2}(0)$$

Similarly for range 4   triplicate the auxiliary space

$$\mathcal{L}_{a_1 a_2 a_3 j}(0) = P_{a_1 j} P_{a_2 j} P_{a_3 j}$$

$\vdots$

range r need $r-1$ copies of the auxiliary space

...

Let us use the same trick but with perturbative spin chain?

$$h = 1 - P_{12} - g P_{13} + O(g^2), \qquad \mathcal{L}^{(2)}_{a j}(u) = \frac{u 1 - P_{a i}}{u-1}$$

c.f. Gombor 22'

Assume

$$\mathcal{L}_{a_1 a_2 j}(u) = P_{a_1 j} P_{a_2 j} \left(P_{a_1 a_2} \mathcal{L}^{(2)}_{a_1 a_2}(u) + f(u)\, g\, P_{13} \right)$$

Compute

$$t(u) = \mathrm{tr}_A \left(\mathcal{L}_{A_L}(u)\, \mathcal{L}_{A_{L-1}}(u) \dots \mathcal{L}_{A_1}(u) \right)$$

↳ doubled auxiliary space

and then

$$[t(u), \mathbb{H}] = \mathcal{O}(g^2)$$

$$\Rightarrow \mathcal{L}_{a_1 a_2 j}(u) = P_{a_1 j} P_{a_2 j} \left(P_{a_1 a_2} \mathcal{L}^{(2)}_{a_1 a_2}(u) \frac{-2u}{(u-2)(u^2-1)} g\, P_{13} \right) +$$

$$+ \mathcal{O}(g^2)$$

Plug the Lax in $R\mathcal{L}\mathcal{L}$:

$$R_{AB}(u,v)\,\mathcal{L}_{A_j}(u)\,\mathcal{L}_{B_j}(v) - \mathcal{L}_{B_j}(v)\,\mathcal{L}_{A_j}(u)\,R_{AB}(u,v) = \mathcal{O}(g^2)$$

$\mathbb{C}^2 \otimes \mathbb{C}^2$ $\mathbb{C}^2 \otimes \mathbb{C}^2$ $\mathbb{C}^2$

Compute the R-matrix

$$R(u,v) = R^{(2)}(u,v) + g\,R^{(3)}(u,v) + \mathcal{O}(g^2)$$

$$[R] = 16 \times 16$$

Range 4: Auxiliary space: $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

$$\mathcal{H}_{1234} = \mathcal{H}_{123} + g^2 \sum_i \rho_i$$

with

$$\rho_1 = A_1 \sum_{i=1}^{3} \sigma_i \otimes 1 \otimes 1 \otimes \sigma_i , \qquad \rho_2 = A_2 \sum_{i,j=1}^{3} \sigma_i \otimes \sigma_j \otimes \sigma_j \otimes \sigma_i$$

$$\rho_3 = A_3 \sum_{i=1}^{3} \sigma_i \otimes \sigma_j \otimes \sigma_i \otimes \sigma_j , \qquad \rho_4 = A_4 \sum_{i,j=1}^{3} \sigma_i \otimes \sigma_i \otimes \sigma_j \otimes \sigma_j$$

$$\rho_5 = A_5 \sum_{i,j,k=1}^{3} \varepsilon^{ijk} \sigma_i \otimes \sigma_j \otimes 1 \otimes \sigma_k , \qquad \rho_6 = A_6 \sum_{i,j,k=1}^{3} \varepsilon^{ijk} \sigma_i \otimes 1 \otimes \sigma_j \otimes \sigma_k$$

is SU(2) invariant.

we find

$$A_3 = \frac{1-2A_1-4A_2}{4}, \qquad A_4 = \frac{1-2A_1}{4} \qquad (*)$$

so anything <u>NOT</u> satisfying $(*)$ is not integrable

we cannot put all A_i to zero at the same time

the g^2 range 3 existence $\Longrightarrow$ g^4 range 4

WHAT IF NO SU(2) ?

Ansatz Hamiltonian: $\mathcal{H}_{NNN} = \mathcal{H}^{(2)}_{12} + g\, \mathcal{H}^{(3)}_{123} + \cdots$

$\overset{NN}{\nearrow}$

$$\mathcal{H}^{(2)}_{12} = h_1 \mathbb{I} + h_2 \left(\sigma^z \otimes \mathbb{1} - \mathbb{1} \otimes \sigma^z \right) + h_3\, \sigma^+ \otimes \sigma^- + h_4\, \sigma^- \otimes \sigma^+ + h_5 \left(\sigma^z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma^z \right) + h_6 \left(\sigma^z \otimes \sigma^z \right)$$

STEP 1: For range-r take (r-1) copies of your auxiliary space

STEP 2: $h_{NNN} = h^{(2)}_{12} + g\, h^{(3)}_{123} + \cdots$

Impose required symmetry

$$\left[h_{NNN} ,\ \sum_j^L \sigma_j^z \right] = 0$$

STEP 3: Write $\mathcal{L}$ as an expansion on the spectral parameter

$$\mathcal{L}^{(\text{range } 3)}_{ab_j}(u) = P_{aj} P_{bj} \left(P_{ab} \mathcal{L}^{(0)}_{ab}(u) + g \sum_{n \geq 1} \frac{\ell^{(n)} u^n}{n!} + \mathcal{O}(g^2) \right)$$

$$\mathcal{L}'_{ab_j}(u)\Big|_{u=0} = P_{aj} P_{bj}\, h_{NNN} \quad \text{and} \quad \mathcal{L}_{ab_j}(0) = P_{aj} P_{bj}$$

$$Q_2 = H = \frac{d}{du} \log t(u) \Big|_{u=0} = t^{-1}(0)\, t'(0)$$

$$Q_3 = \frac{d^2}{du^2} \log t(u) \Big|_{u=0} = t^{-1}(0)\, t''(0) \qquad Q_2^2$$

NOT RELEVANT FOR COMMUTATOR PURPOSES

$$[Q_2, Q_3] = 0$$

$\Rightarrow$ fixes two matrix elements of $\ell^{(2)}$ in terms of the rest

$$Q_4 = t^{-1}(0)\, t'''(0) + f(Q_2, Q_3)$$

$$[Q_2, Q_4] = 0$$

Again only two conditions and all other $\ell_{ij}^{(3)}$ remain free

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$$\vdots$$

$$[Q_2, Q_n] = 0$$

Trick: $$\boxed{[t(u), H] = 0}$$

All Hamiltonian deformations (range 3) computed in

Beisert, Fiévet, de Leeuw & Loebbert, 2013

are generated by a Lax operator.

CONCLUSIONS / FUTURE

* I presented a new method to systematically construct solutions of the Yang-Baxter equation

  NEW MODELS : SPECTRUM & SYMMETRIES REMAIN TO BE UNDERSTOOD

* ... And a strategy to construct perturbatively long-range deformations of spin chains

  OTHER SECTORS LIKE $su(1|1)$

THANK YOU!