ENTANGLEMENT HAMILTONIAN FOR INHOMOGENEOUS FREE FERMIONS

RAQIS 2024
2-6 September, Annecy
Riccarda Bonsignori

R. B. and Viktor Eisler, J. Phys. A: Math. Theor. 57 275001 (2024).

TU Graz

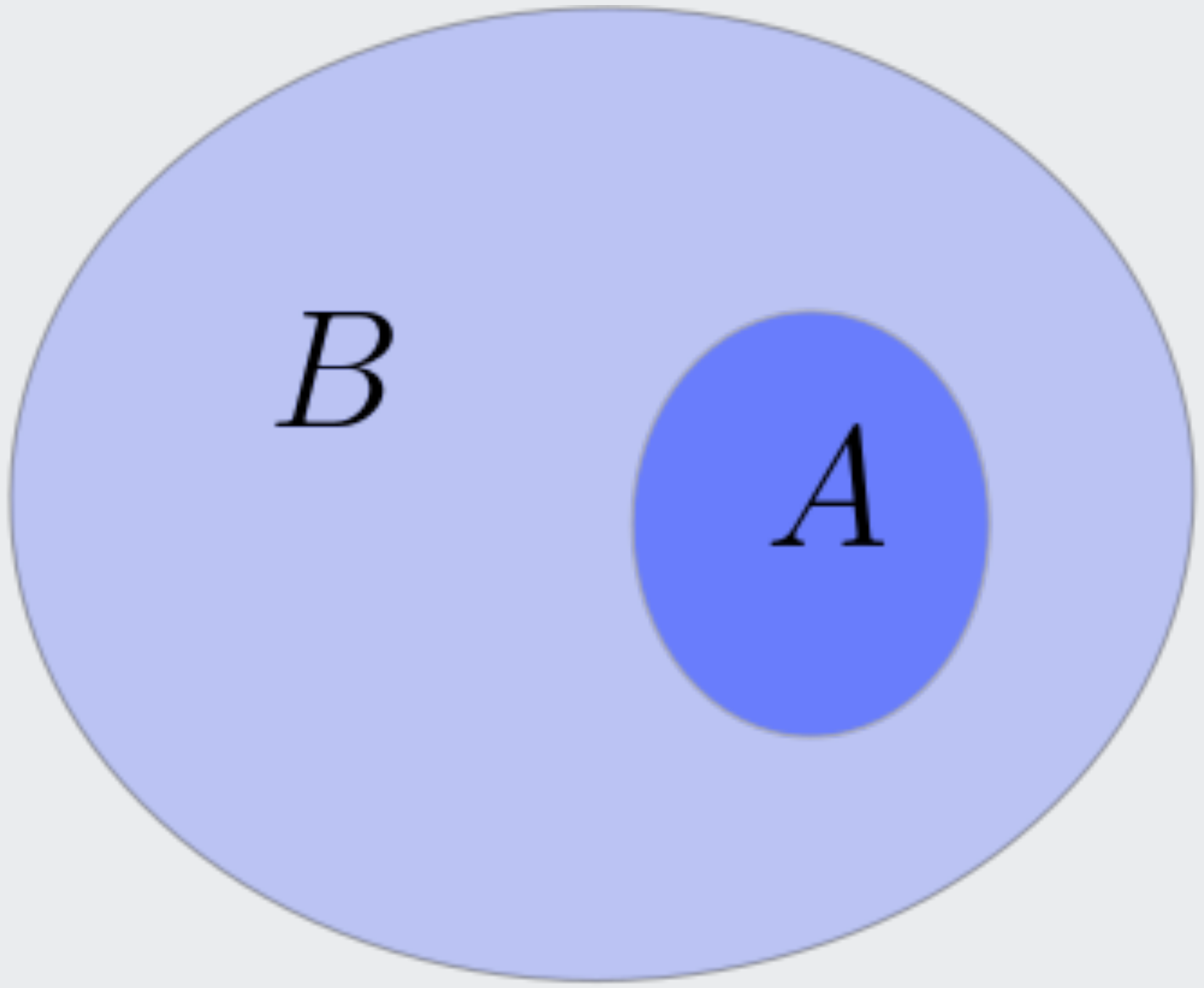
OUTLINE

- Entanglement Hamiltonian
  - Definitions and motivations
  - EH in (1+1)D CFTs
  - EH for free fermions on a lattice
- EH for the gradient chain
- EH for the free Fermi gas in a harmonic potential
- Conclusions

DEFINITIONS

ENTANGLEMENT HAMILTONIAN: DEFINITION

- Let us consider a bipartite quantum system $S = A \cup B$ in a pure state $\rho = |\psi\rangle\langle\psi|$
- Reduced density matrix of A: $\rho_A = \mathrm{Tr}_B \rho$



$$\rho_A = \frac{e^{-\mathcal{H}}}{Z}$$

- $\mathcal{H}$ is the *entanglement hamiltonian*.
- The spectrum of $\mathcal{H}$: *entanglement spectrum*
  - Entanglement Entropy (EE): $S = -\mathrm{Tr}\rho_A \log \rho_A$

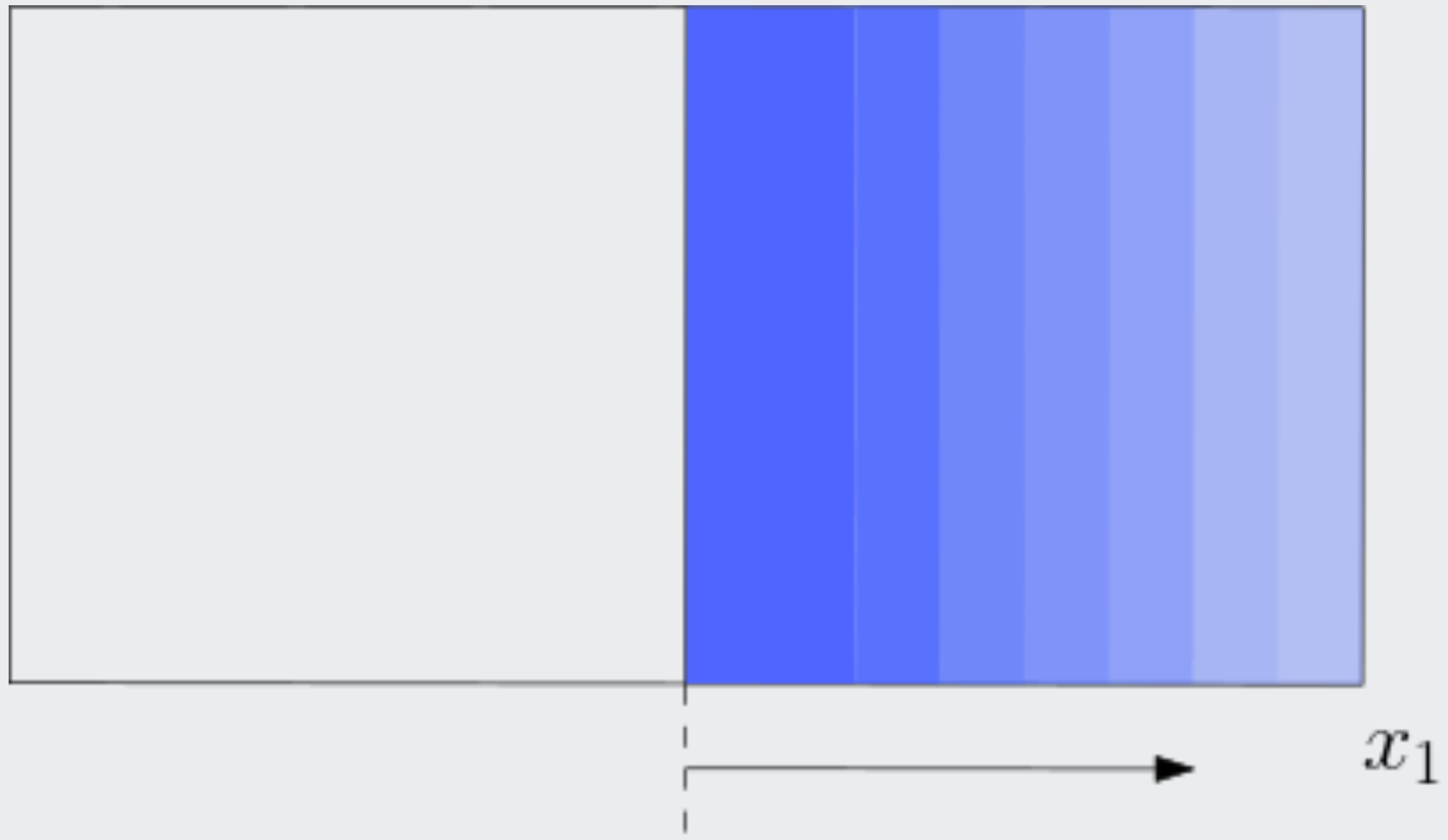
BISOGNANO-WICHMANN THEOREM

- Relativistic QFT in (D+1)dimensions, spatial coordinates $x = \{x_1, x_2, \cdots, x_D\}$, hamiltonian density $H(x)$.
- The partition A is the (right) half plane $x_1 > 0$.

- *Bisognano-Wichmann theorem:* the EH of the vacuum state is

$$\mathcal{H} = \frac{2\pi}{c} \int_{x \in A} \mathrm{d}x x_1 H(x)$$

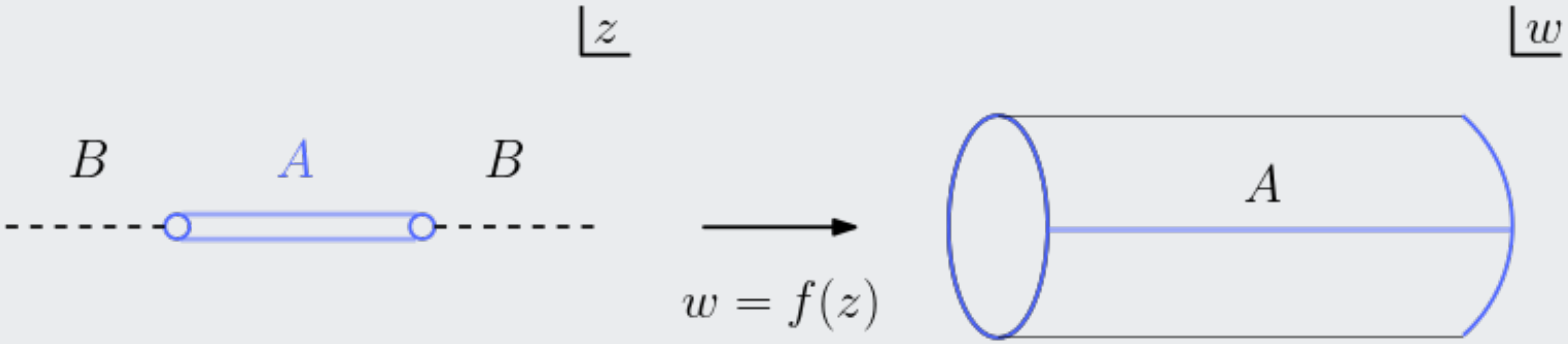
- The RDM has the form of a thermal state wrt to the original Hamiltonian, with a position-dependent inverse temperature $\frac{2\pi}{c} x_1$.



J. J. Bisognano and E. H. Wichmann, *J. Math. Phys.* **16**, *985–1007 (1975)*
J. J. Bisognano and E. H. Wichmann, *J. Math. Phys.* **17**, *303–321 (1976)*

MAPPING TO THE ANNULUS

- (1+1)D CFTs

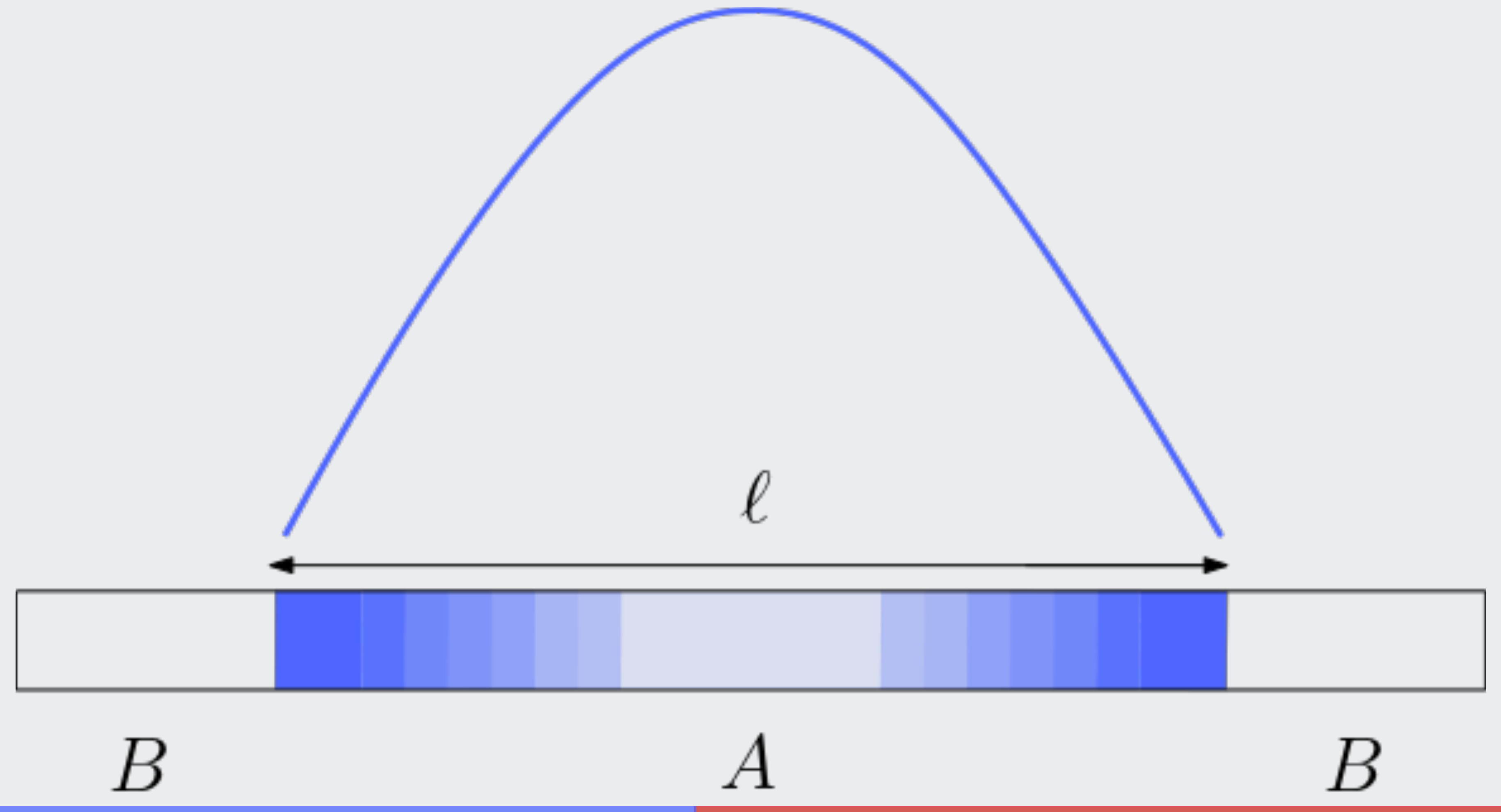


- The EH can be written as a local integral over the energy-momentum tensor

$$\mathcal{H} = \frac{2\pi}{c} \int_A \beta(x) T_{00}(x) \mathrm{d}x$$

- *Example:* finite interval $A = [0, \ell]$

$$\beta(x) \propto \frac{x}{\ell}\left(1 - \frac{x}{\ell}\right)$$

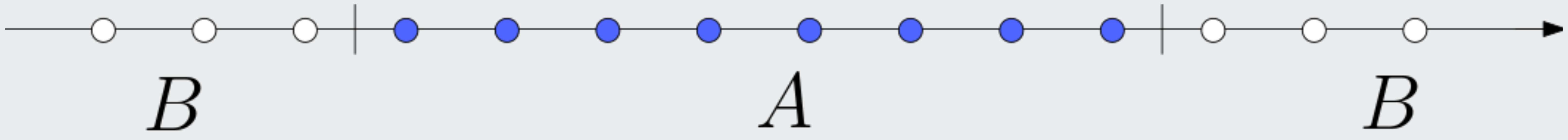


J. Cardy and E. Tonni, *J. Stat. Mech. 123103 (2016)*

EH ON THE LATTICE: FREE FERMIONS

- Free fermions hopping on a lattice: $\hat{H} = -\frac{1}{2}\sum_n t_n(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n)$
- Subsystem $A = [1,\cdots,L_A]$

$$\rho_A = \frac{e^{-\mathscr{H}}}{Z}, \qquad \mathscr{H} = \sum_{i,j\in A}^{L_A} H_{ij} c_i^\dagger c_j = \sum_k^{L_A} \varepsilon_k \tilde{c}_k^\dagger \tilde{c}_k$$



I. Peschel and V. Eisler, J. Phys. A **42** 504003 (2009)

EH ON THE LATTICE: FREE FERMIONS

- Free fermions hopping on a lattice: $\hat{H} = -\frac{1}{2}\sum_n t_n(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n)$
- Subsystem $A = [1,\cdots,L_A]$

$$\rho_A = \frac{e^{-\mathscr{H}}}{Z}, \qquad \mathscr{H} = \sum_{i,j\in A}^{L_A} H_{ij} c_i^\dagger c_j = \sum_k^{L_A} \varepsilon_k \tilde{c}_k^\dagger \tilde{c}_k$$

- Correlation matrix restricted to $A = [1,\cdots,L_A]$: $(C_A)_{i,j} = \mathrm{Tr}[\rho_A c_i^\dagger c_j] = \langle c_i^\dagger c_j \rangle, \qquad i,j \in A$

$$H = \ln[(\mathbb{I} - C)/C]$$

$$\varepsilon_k = \ln[(1-\zeta_k)/\zeta_k]$$ eigenvalues

I. Peschel and V. Eisler, *J. Phys. A* **42** 504003 (2009)

BW ON THE LATTICE?

- The lattice EH cannot be obtained as a simple discretization of the continuum result.

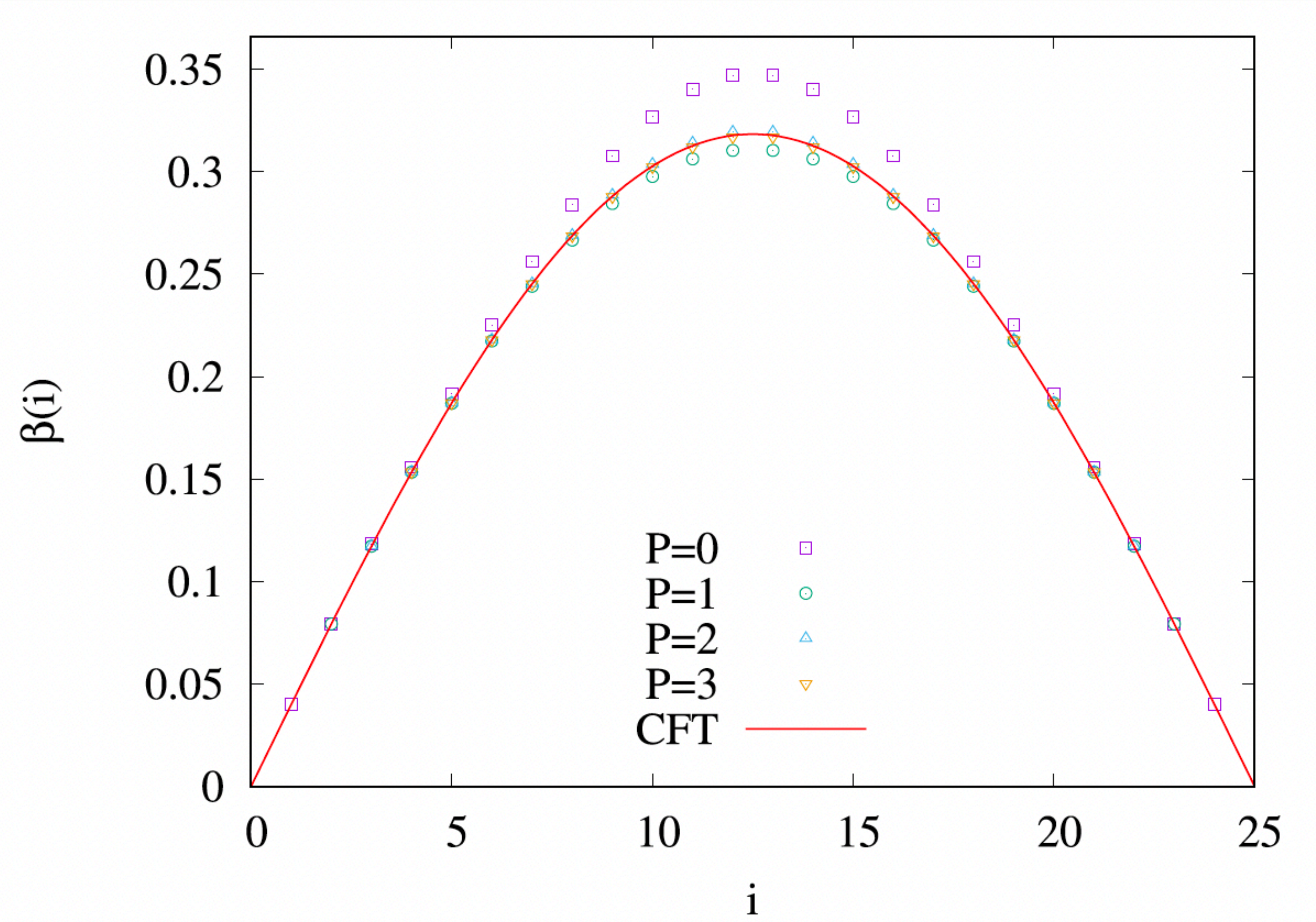
- Results:

  - The CFT result is recovered from the lattice EH including all the long range hopping terms in the continuum limit

V. Eisler, E. Tonni and I. Peschel, *J. Stat. Mech. 073101 (2019)*

  - *Example:* finite interval $A = [1,\cdots,L]$ in a finite chain

$$\beta(x) = \frac{1}{\pi L}\sum_{p=0}^{P}(-1)^p(2p+1)H_{i-p,i+p+1}$$



BW ON THE LATTICE?

- The latticeEH cannot be in general obtained as a simple discretization of the continuum result.
- Results:
  - The CFT result is recovered from the lattice EH including all the long range hopping terms in the continuum limit

V. Eisler, E. Tonni and I. Peschel, *J. Stat. Mech. 073101 (2019)*

  - A tridiagonal matrix exists that commutes with the lattice EH and has the form of the discretized CFT ansatz.

I. Peschel, *J. Stat. Mech. P06004 (2004)*

EH FOR INHOMOGENEOUS FREE FERMIONS: RESULTS

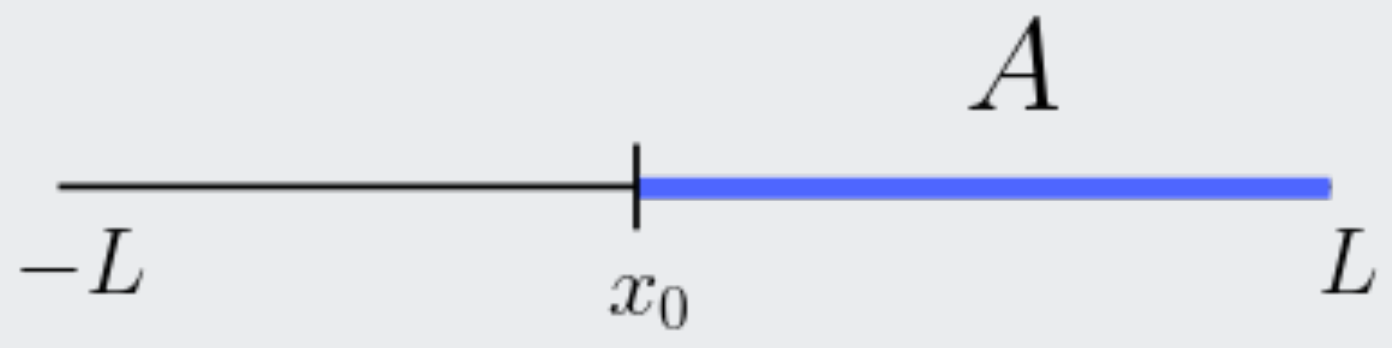
EH IN CURVED SPACE CFT

J. Dubail, J.-M. Stéphan, J. Viti, and P. Calabrese, *SciPost Phys.* **2** *002 (2017)*

- Free Dirac fermion in curved space: $\mathcal{S} = \frac{1}{2\pi}\int \mathrm{d}z\mathrm{d}\bar{z}\mathrm{e}^{\sigma(x)}\left[\psi_R^\dagger \overleftrightarrow{\partial}_{\bar{z}}\psi_R + \psi_L^\dagger \overleftrightarrow{\partial}_z \psi_L\right]$

$$\mathrm{d}s^2 = \mathrm{e}^{2\sigma(x)}\mathrm{d}z\mathrm{d}\bar{z}, \qquad \sigma(x) = v_F(x) \quad \text{Weil factor}$$

- Chain of size $2L$, bipartition $A \in [x_0, L]$



  - Homogeneous system

$$\beta(x) = \frac{2L}{\pi}\frac{\sin\left(\frac{\pi x}{2L}\right) - \sin\left(\frac{\pi x_0}{2L}\right)}{\cos\left(\frac{\pi x_0}{2L}\right)}$$

E. Tonni, J. Rodriguez-Laguna, and G. Sierra, *J. Stat. Mech. (2018) 043105*

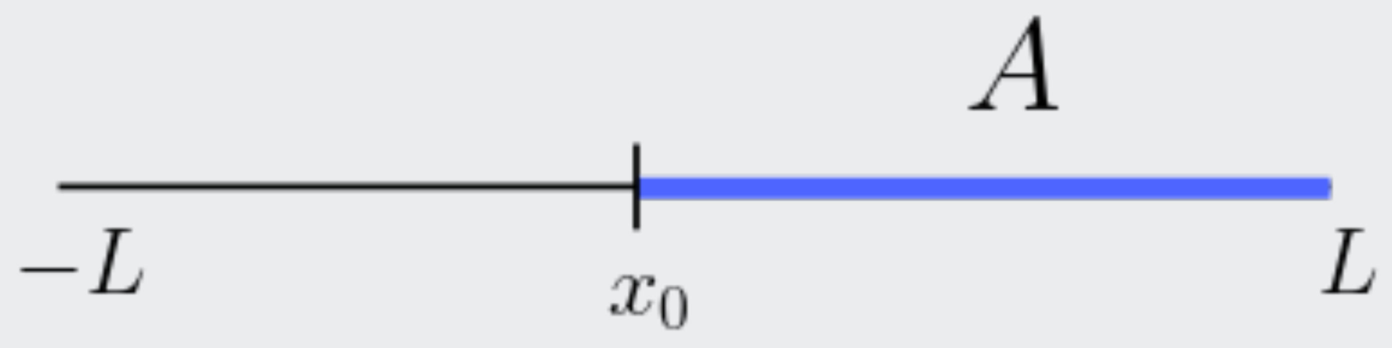
EH IN CURVED SPACE CFT

J. Dubail, J.-M. Stéphan, J. Viti, and P. Calabrese, *SciPost Phys.* **2** 002 (2017)

- Free Dirac fermion in curved space: $\mathcal{S} = \frac{1}{2\pi}\int \mathrm{d}z\mathrm{d}\bar{z}\, e^{\sigma(x)}\left[\psi_R^{\dagger}\overleftrightarrow{\partial}_{\bar{z}}\psi_R + \psi_L^{\dagger}\overleftrightarrow{\partial}_{z}\psi_L\right]$

$$\mathrm{d}s^2 = e^{2\sigma(x)}\mathrm{d}z\mathrm{d}\bar{z}, \qquad \sigma(x) = v_F(x) \quad \text{Weil factor}$$

- Chain of size $2L$, bipartition $A \in [x_0, L]$



  - **In**-Homogeneous system

$$\beta(x) = \frac{2\tilde{L}}{\pi}\,\frac{\sin\left(\frac{\pi\tilde{x}(x)}{2\tilde{L}}\right) - \sin\left(\frac{\pi\tilde{x}_0}{2\tilde{L}}\right)}{\cos\left(\frac{\pi\tilde{x}_0}{2\tilde{L}}\right)}\,v_F(x)$$

$$\begin{cases} z = \tilde{x} + it \\ \tilde{x} = \int_0^x \frac{\mathrm{d}y}{v_F(y)} \\ \tilde{L} = \int_0^L \frac{\mathrm{d}y}{v_F(y)} \end{cases}$$

E. Tonni, J. Rodriguez-Laguna, and G. Sierra, *J. Stat. Mech. (2018) 043105*

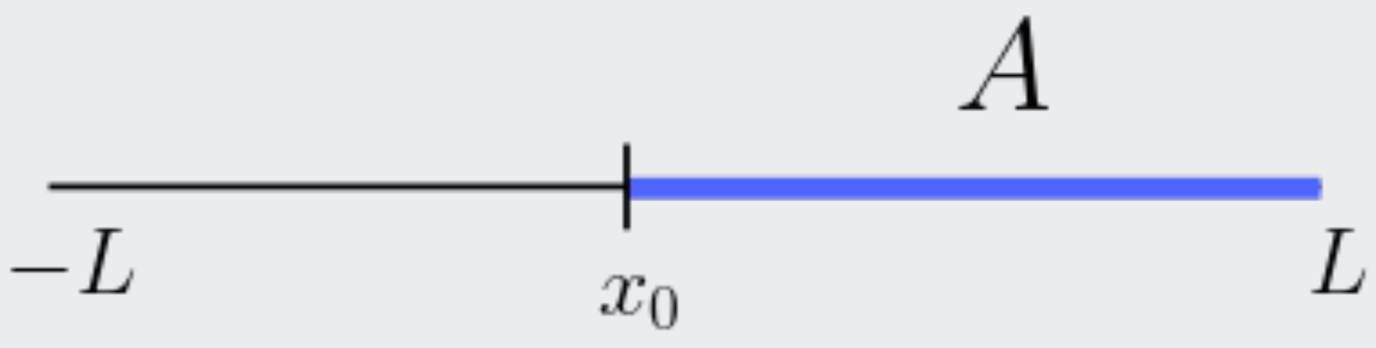
EH IN CURVED SPACE CFT

J. Dubail, J.-M. Stéphan, J. Viti, and P. Calabrese, *SciPost Phys.* **2** *002 (2017)*

- Free Dirac fermion in curved space: $\mathcal{S} = \frac{1}{2\pi}\int \mathrm{d}z\mathrm{d}\bar{z}\mathrm{e}^{\sigma(x)}\left[\psi_R^\dagger \overleftrightarrow{\partial}_{\bar{z}}\psi_R + \psi_L^\dagger \overleftrightarrow{\partial}_z \psi_L\right]$

$$\mathrm{d}s^2 = \mathrm{e}^{2\sigma(x)}\mathrm{d}z\mathrm{d}\bar{z}, \qquad \sigma(x) = v_F(x) \quad \text{Weil factor}$$

- Chain of size $2L$, bipartition $A \in [x_0, L]$



  - **In**-Homogeneous system

$$\beta(x) = \frac{2\tilde{L}}{\pi}\frac{\sin\left(\frac{\pi\tilde{x}(x)}{2\tilde{L}}\right) - \sin\left(\frac{\pi\tilde{x}_0}{2\tilde{L}}\right)}{\cos\left(\frac{\pi\tilde{x}_0}{2\tilde{L}}\right)} v_F(x) \longrightarrow \beta(x) = \tilde{\beta}(x)v_F(x)$$

spatially varying inverse temperature $\tilde{\beta}(x)$

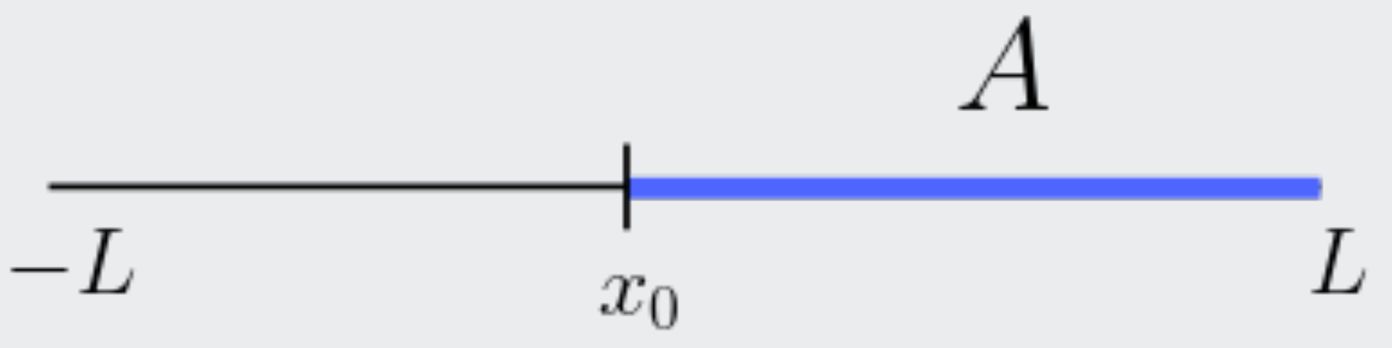
EH IN CURVED SPACE CFT

J. Dubail, J.-M. Stéphan, J. Viti, and P. Calabrese, *SciPost Phys.* **2** 002 (2017)

- Free Dirac fermion in curved space: $\mathcal{S} = \frac{1}{2\pi}\int \mathrm{d}z\mathrm{d}\bar{z}\mathrm{e}^{\sigma(x)}\left[\psi_R^\dagger \overleftrightarrow{\partial}_{\bar{z}}\psi_R + \psi_L^\dagger \overleftrightarrow{\partial}_z \psi_L\right]$

$\mathrm{d}s^2 = \mathrm{e}^{2\sigma(x)}\mathrm{d}z\mathrm{d}\bar{z}, \qquad \sigma(x) = v_F(x)$ Weil factor

- Chain of size $2L$, bipartition $A \in [x_0, L]$



  - **In**-Homogeneous system

$$\mathcal{H} = \frac{2\pi}{c}\int_A \tilde{\beta}(x) v_F(x) T_{00}(x)\mathrm{d}x$$

spatially varying inverse temperature $\tilde{\beta}(x)$

Hamiltonian density

$H(x) = v_F(x)T_{00}(x)$

EH FOR THE GRADIENT CHAIN

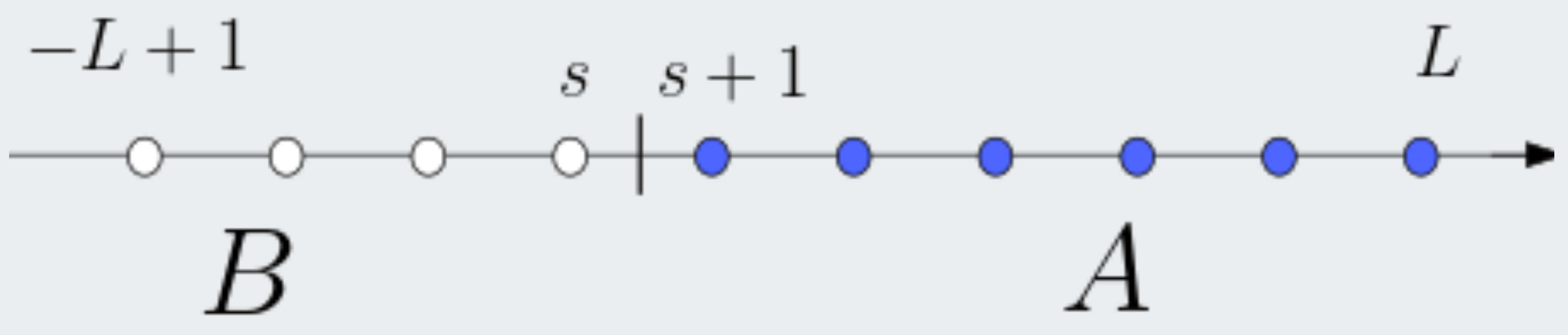
- Hamiltonian: $$\hat{H} = -\frac{1}{2}\sum_{n=-L+1}^{L-1}(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) + \frac{1}{\xi}\sum_{n=-L+1}^{L}\left(n-\frac{1}{2}\right)c_n^\dagger c_n,$$

- Correlation matrix: $$C_{m,n} = \langle c_m^\dagger c_n\rangle = \sum_{\kappa\in F}\Phi_\kappa(m)\Phi_\kappa(n)$$

V. Eisler, F. Igloi, I. Peschel, *J. Stat. Mech.* *(2009) P02011*

- Bipartition $A \in [s+1, L]$:

$$\sum_{j\in A} C_{i,j}\,\phi_k(j) = \zeta_k\,\phi_k(i)$$



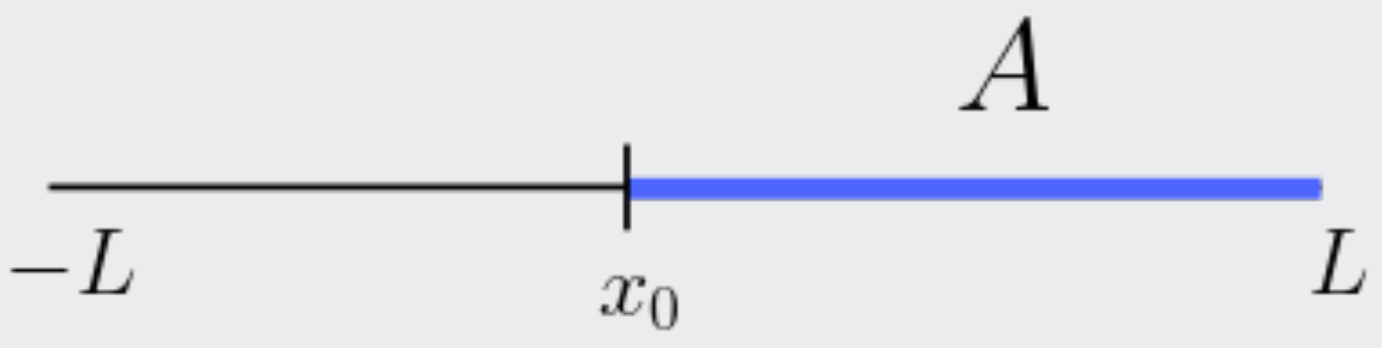
  - Entanglement hamiltonian: $$\mathscr{H} = \sum_{i,j\in A} H_{i,j}c_i^\dagger c_j$$

$$H_{i,j} = \sum_k \varepsilon_k\,\phi_k(i)\phi_k(j), \qquad \varepsilon_k = \ln\frac{1-\zeta_k}{\zeta_k}$$

EH FOR THE GRADIENT CHAIN

- Fermi velocity: $$v_F(x) = \left.\frac{\mathrm{d}\omega_q(x)}{\mathrm{d}q}\right|_{q_F} = \sqrt{1 - \left(\frac{x}{\xi}\right)^2}$$

Local Density Approx



- Weak gradient regime $L < \xi$:

$$\tilde{\beta}(x) = \frac{2}{\pi}\xi \arcsin\left(\frac{L}{\xi}\right) \frac{\sin\left(\frac{\pi}{2}\frac{\arcsin(x/\xi)}{\arcsin(L/\xi)}\right) - \sin\left(\frac{\pi}{2}\frac{\arcsin(x_0/\xi)}{\arcsin(L/\xi)}\right)}{\cos\left(\frac{\pi}{2}\frac{\arcsin(x_0/\xi)}{\arcsin(L/\xi)}\right)}$$

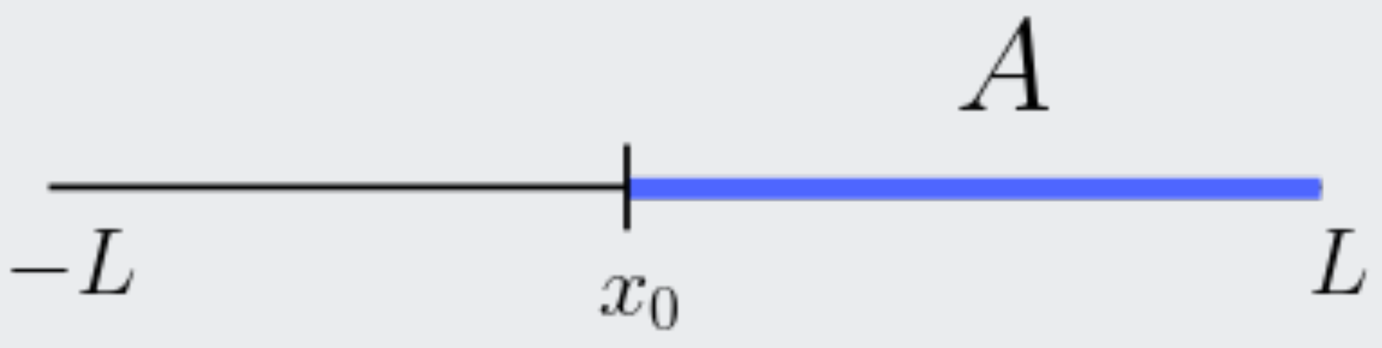
- Infinite system $L \to \infty$:

$$\tilde{\beta}(x) = \frac{x - x_0}{\sqrt{1 - \left(\frac{x_0}{\xi}\right)^2}} = \frac{x - x_0}{v_F(x_0)}$$

EH FOR THE GRADIENT CHAIN: $L \to \infty$

- Fermi velocity: $$v_F(x) = \left.\frac{\mathrm{d}\omega_q(x)}{\mathrm{d}q}\right|_{q_F} = \sqrt{1 - \left(\frac{x}{\xi}\right)^2}$$

Local Density Approx



- Infinite system $L \to \infty$:

$$\tilde{\beta}(x) = \frac{x - x_0}{\sqrt{1 - \left(\frac{x_0}{\xi}\right)^2}} = \frac{x - x_0}{v_F(x_0)}$$

- CFT prediction for the EH has the BW form:

$$\mathcal{H} = \frac{2\pi}{v_F(x_0)} \int_A (x - x_0) H(x)$$

EH FOR THE GRADIENT CHAIN: $L \to \infty$

$$\mathcal{H} = \frac{2\pi}{v_F(x_0)} \int_A (x - x_0) H(x)$$

- Tridiagonal T matrix $[C_A, T] = 0$:

A. Borodin, A. Okounkov, G. Olshanski, *J. Amer. Math. Soc. 13 (2000) 3, 481-515*

$$t_i = i - s$$

$$d_i = -\frac{2}{\xi}\left(i - \frac{1}{2} - s\right)\left(i - \frac{1}{2}\right)$$

$$T = \begin{pmatrix} d_1 & t_1 & & & \\ t_1 & d_2 & t_2 & & \\ & t_2 & d_3 & t_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

- *The BW structure is inherited by the T matrix, which has the form of the discretized CFT ansatz*

EH FOR THE GRADIENT CHAIN: $L \to \infty$

$$\mathscr{H} = \frac{2\pi}{v_F(x_0)} \int_A (x - x_0) H(x)$$

- Tridiagonal T matrix $[C_A, T] = 0$:

$$t_i = i - s$$

$$d_i = -\frac{2}{\xi}\left(i - \frac{1}{2} - s\right)\left(i - \frac{1}{2}\right)$$

Lattice hamiltonian:

$$\hat{H} = -\frac{1}{2}\sum_{n=-L+1}^{L-1}(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) + \frac{1}{\xi}\sum_{n=-L+1}^{L}\left(n - \frac{1}{2}\right)c_n^\dagger c_n,$$

- *The BW structure is inherited by the T matrix, which has the form of the discretized CFT ansatz*

EH FOR THE GRADIENT CHAIN: $L \to \infty$

$$\mathscr{H} = \frac{2\pi}{v_F(x_0)} \int_A (x - x_0) H(x)$$

- Tridiagonal T matrix $[C_A, T] = 0$:

Lattice hamiltonian:

$$t_i = i - s$$

$$d_i = -\frac{2}{\xi}\left(i - \frac{1}{2} - s\right)\left(i - \frac{1}{2}\right)$$

$$\hat{H} = -\frac{1}{2}\sum_{n=-L+1}^{L-1}(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) + \frac{1}{\xi}\sum_{n=-L+1}^{L}\left(n - \frac{1}{2}\right)c_n^\dagger c_n,$$

λ_k eigenvalues of T

- $[H, T] = 0$, spectra:

$$\varepsilon_k = -\frac{\pi}{v_F(s)}\lambda_k$$

EH FOR THE GRADIENT CHAIN: $L \to \infty$

$$\mathscr{H} = \frac{2\pi}{v_F(x_0)} \int_A (x - x_0) H(x)$$

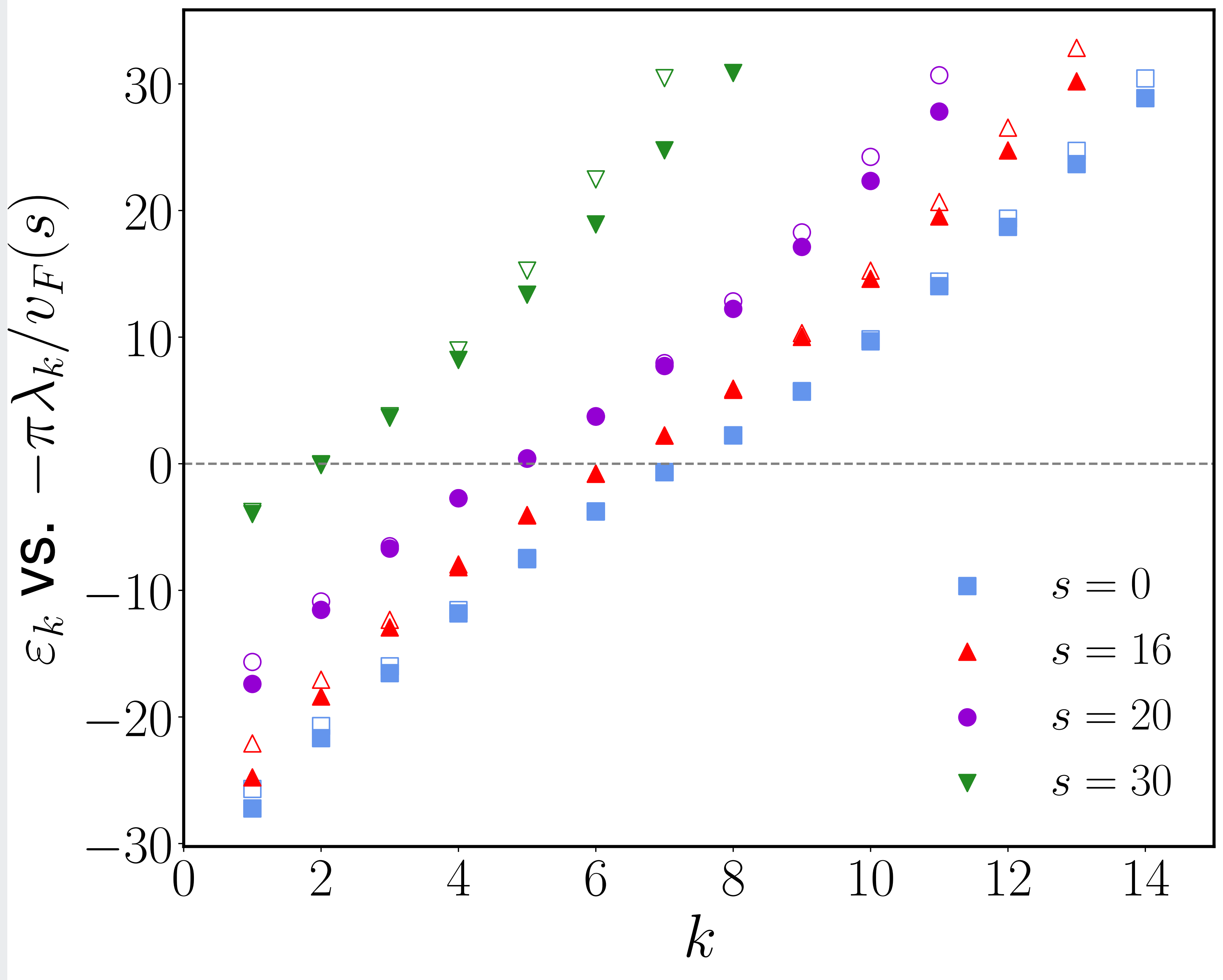
- Tridiagonal T matrix $[C_A, T] = 0$:

$$t_i = i - s$$

$$d_i = -\frac{2}{\xi}\left(i - \frac{1}{2} - s\right)\left(i - \frac{1}{2}\right)$$

- $[H, T] = 0$, spectra:

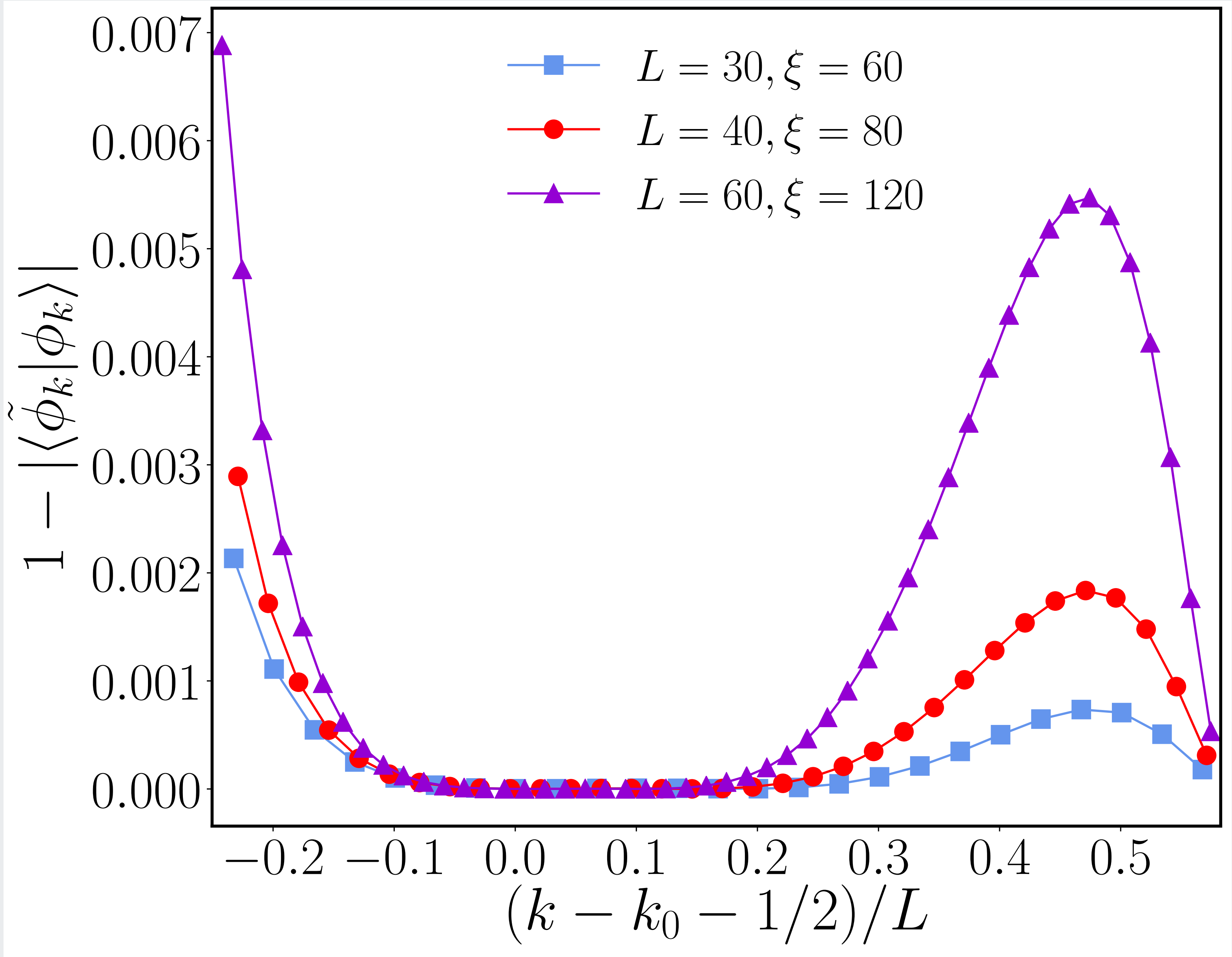
$$\varepsilon_k = -\frac{\pi}{v_F(s)} \lambda_k$$

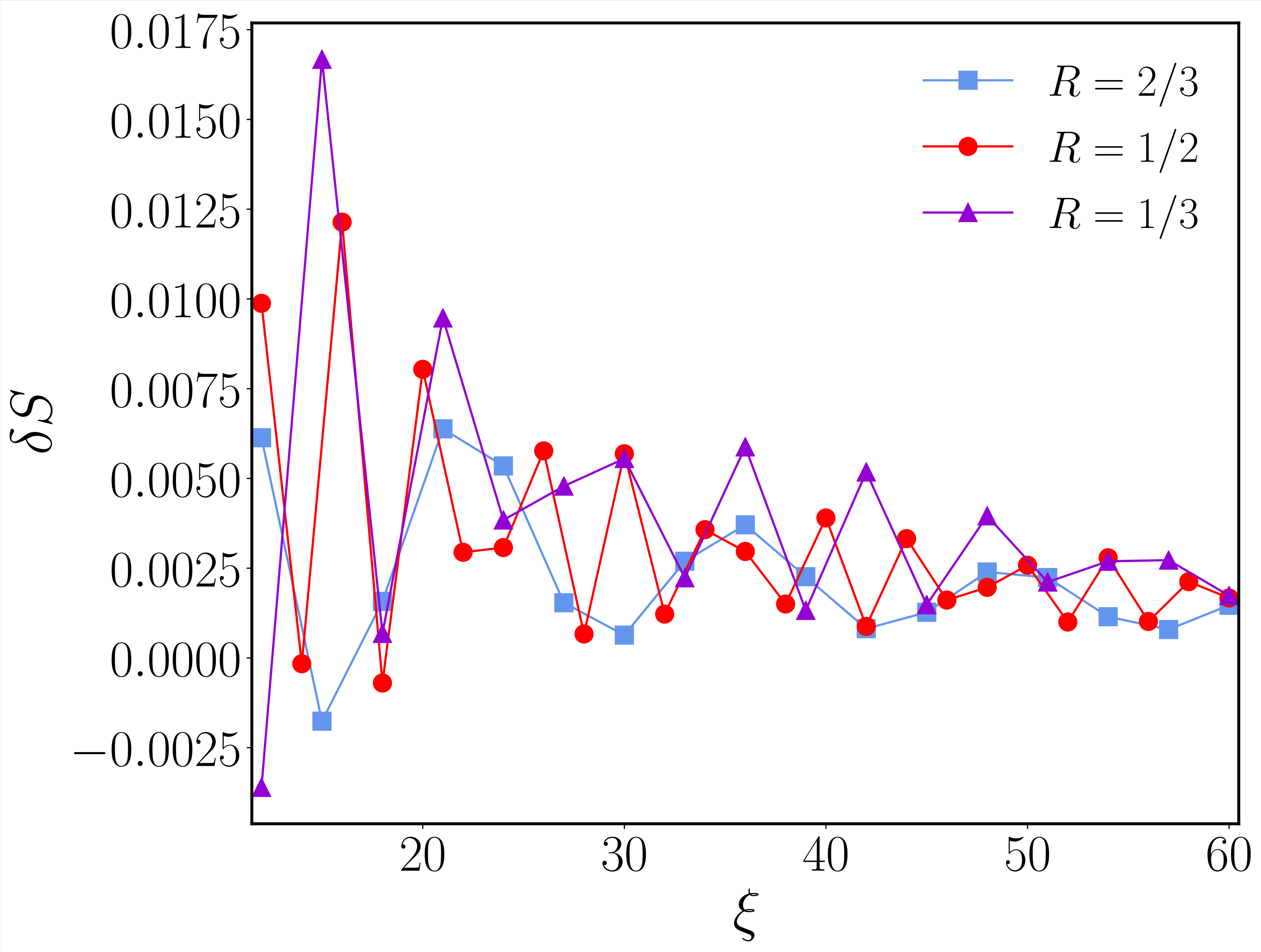


EH FOR THE GRADIENT CHAIN: $L < \xi$

- Almost commuting tridiagonal matrix obtained from discretisation of CFT ansatz.

$$t_i = \tilde{\beta}(i) \qquad d_i = -\frac{2}{\xi}\left(i - \frac{1}{2}\right)\tilde{\beta}(i - 1/2)\,.$$



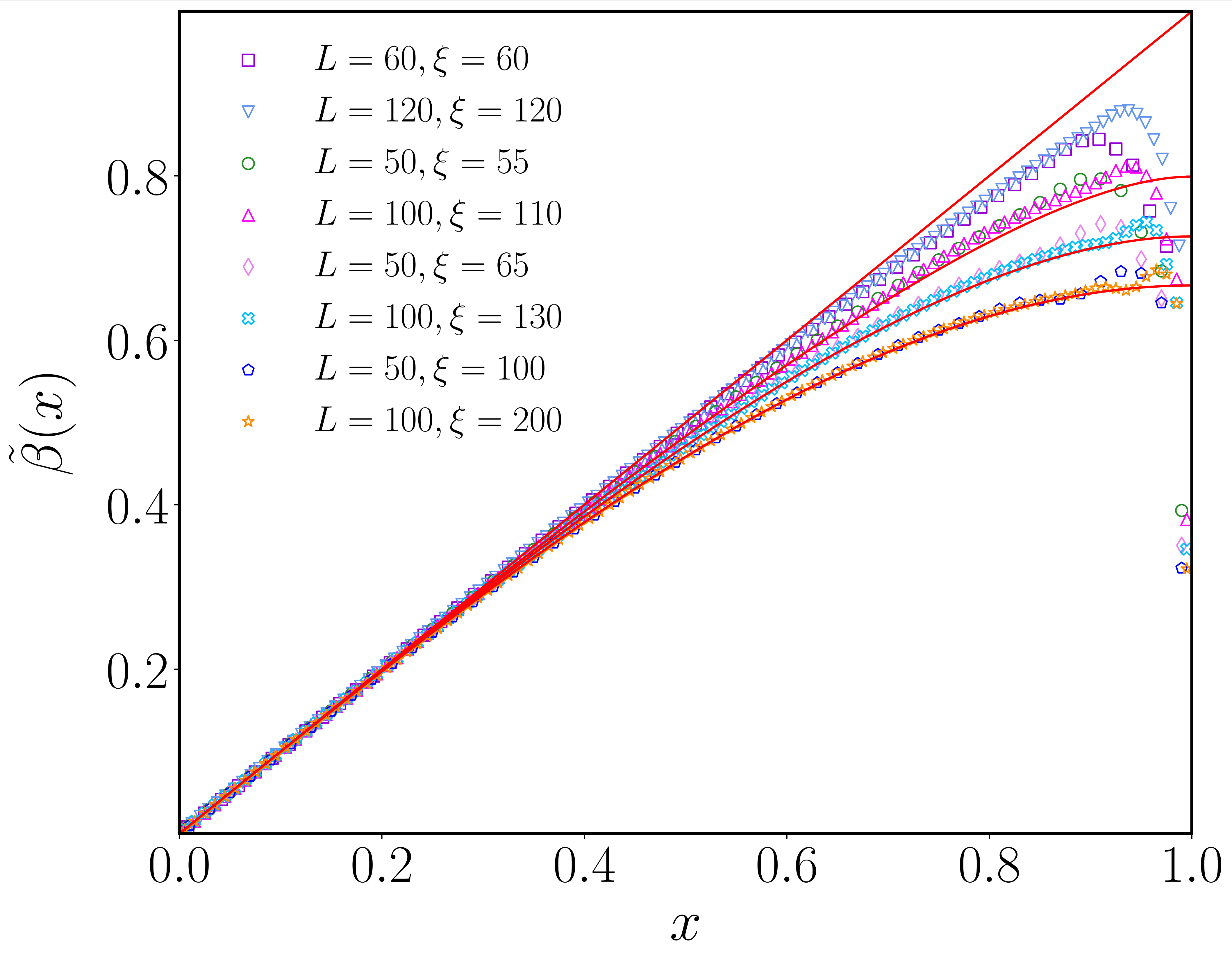


CONTINUUM LIMIT FOR THE GRADIENT CHAIN

- Both regimes $\xi > L$ and $L \to \infty$:

  - The CFT result is recovered from the lattice EH including all the long range hopping terms in the continuum limit

$$2\pi\tilde{\beta}(x) = -\frac{2a}{v_F(x)}\sum_{r>0} r\sin(q_F(x)ar)H_{i,i+r}$$



EH FOR THE HARMONIC TRAP

- Single particle hamiltonian: $\hat{H} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}m\,\Omega^2 x^2 - \mu$ $\qquad (\hbar = m = \Omega = 1)$

- Correlation matrix: $K(x,y) = \sum_{n=0}^{N-1} \Phi_n(x)\Phi_n(y), \qquad \Phi_n(x) = c_n^{-1/2} H_n(x) \mathrm{e}^{-\frac{x^2}{2}}$

  N-particle GS

- Bipartition $A \in [x_0, \infty)$:

$$\hat{\mathscr{K}}\phi_k(x) = \int_A \mathrm{d}y\, K(x,y)\phi_k(y) = \zeta_k \phi_k(x)$$

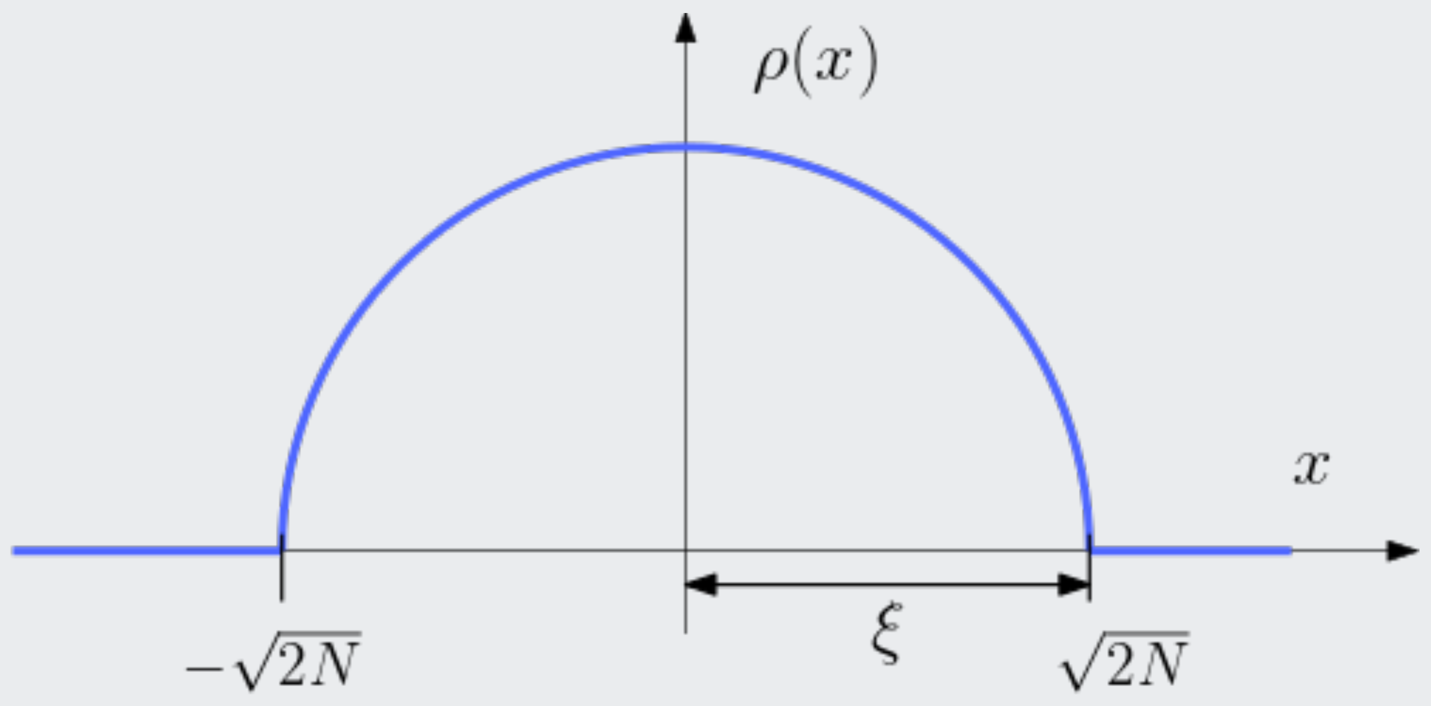
  - Overlap matrix: $\mathbb{A}_{m,n} = \int_A \mathrm{d}x\, \Phi_m(x)\Phi_n(x)$

    Same eigenvalues $\zeta_k \neq 0$

  - Entanglement hamiltonian: $\hat{\mathscr{H}} = \ln(\hat{\mathscr{K}}^{-1} - 1)$

EH FOR THE HARMONIC TRAP

- Fermi velocity: $v_F(x) = \xi\sqrt{1-\left(\frac{x}{\xi}\right)^2}$, $\xi = \sqrt{2N}$



- Curved CFT prediction:

$$\tilde{\beta}(x) = \frac{x - x_0}{\xi\sqrt{1-\left(\frac{x_0}{\xi}\right)^2}} = \frac{x - x_0}{v_F(x_0)}$$

- CFT prediction for the EH has the BW form analogous to that of the gradient chain:

$$\mathcal{H} = \frac{2\pi}{v_F(x_0)}\int_A (x - x_0)H(x)$$

EH FOR THE HARMONIC TRAP

$$\mathscr{H} = \frac{2\pi}{v_F(x_0)} \int_A (x - x_0) H(x)$$

- Differential operator $[\mathscr{K}, \hat{D}] = 0$:

$$\hat{D} = \frac{\mathrm{d}}{\mathrm{d}x}(x - x_0)\frac{\mathrm{d}}{\mathrm{d}x} - x^2(x - x_0) + 2N(x - x_0)$$

A. Grunbaum, *J. Math. Anal. Appl. 95 no. 2, 491–500 (1983)*

Single-particle hamiltonian:

$$\hat{H} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2 + \mu$$

- *The BW structure is inherited by the differential operator* $\hat{D}$.

EH FOR THE HARMONIC TRAP

$$\mathscr{H} = \frac{2\pi}{v_F(x_0)} \int_A (x - x_0) H(x)$$

- Differential operator $[\mathscr{K}, \hat{D}] = 0$:

$$\hat{D} = \frac{\mathrm{d}}{\mathrm{d}x}(x - x_0)\frac{\mathrm{d}}{\mathrm{d}x} - x^2(x - x_0) + 2N(x - x_0)$$

Single-particle hamiltonian:

$$\hat{H} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2 + \mu$$

- BW Ansatz for the EH:

$$\mathscr{H}_{BW} = -\frac{\pi}{v_F(x_0)}\hat{D}$$

$$\varepsilon_k = -\frac{\pi}{v_F(x_0)}\chi_k$$

χ_k eigenvalues of $\hat{D}$

EH FOR THE HARMONIC TRAP

$$\mathscr{H} = \frac{2\pi}{v_F(x_0)} \int_A (x - x_0) H(x)$$

- Differential operator $[\mathscr{K}, \hat{D}] = 0$:

$$\hat{D} = \frac{\mathrm{d}}{\mathrm{d}x}(x - x_0)\frac{\mathrm{d}}{\mathrm{d}x} - x^2(x - x_0) + 2N(x - x_0)$$

Single-particle hamiltonian:

$$\hat{H} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2 + \mu$$

- The operator $\hat{D}$ has the same spectrum $\{\chi_k\}$ of the tridiagonal matrix M

$$M_{m,n} = 2(N - n)\sqrt{\frac{n}{2}}\delta_{m,n-1} + 2(N - m)\sqrt{\frac{m}{2}}\delta_{n,m-1} - 2\left(N - n - \frac{1}{2}\right)x_0\delta_{m,n}$$

A. Grunbaum, *J. Math. Anal. Appl. 95 no. 2, 491–500 (1983)*

EH FOR THE HARMONIC TRAP

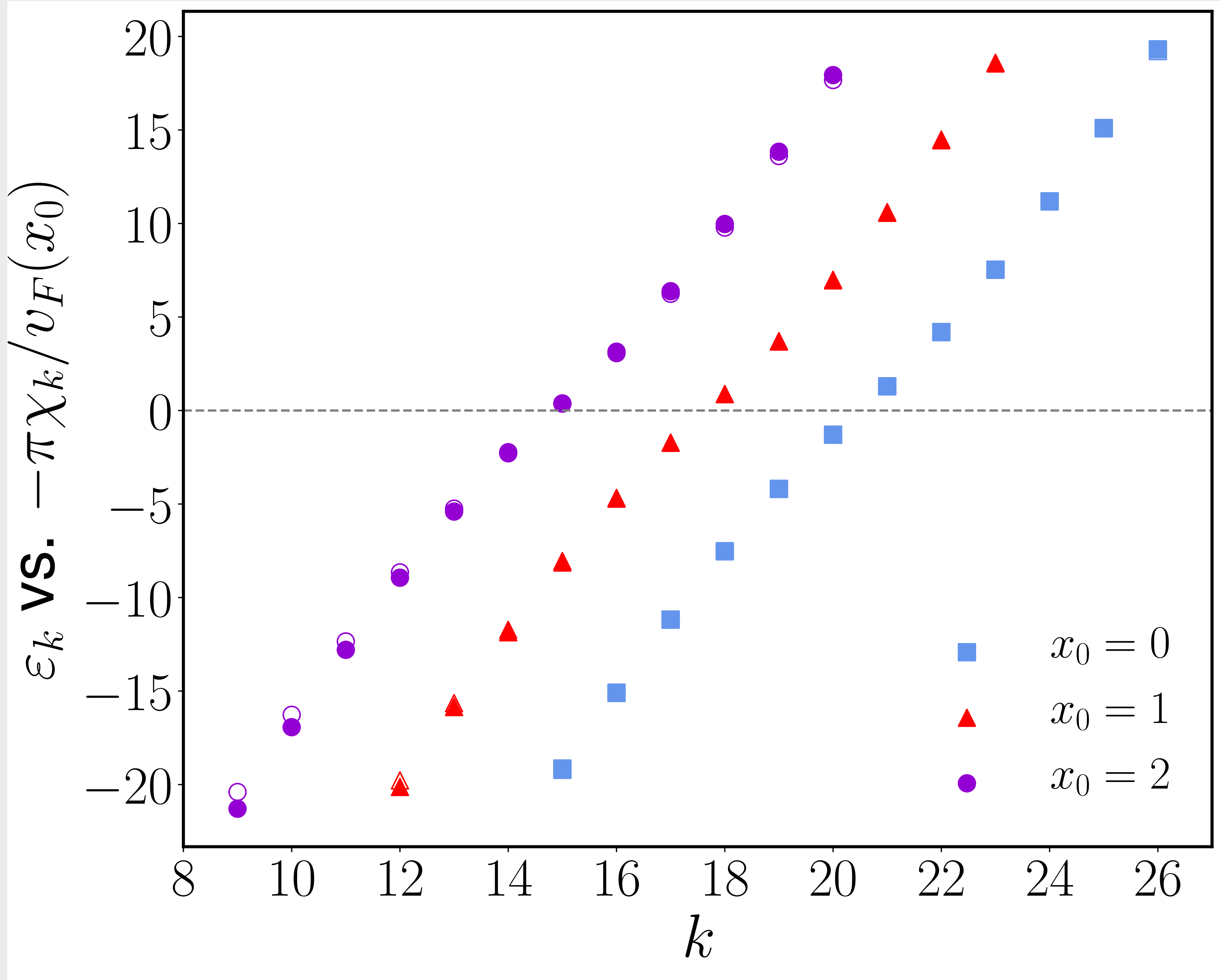
$$\mathscr{H} = \frac{2\pi}{v_F(x_0)} \int_A (x - x_0) H(x)$$

- Differential operator $[\mathscr{K}, \hat{D}] = 0$:

$$\hat{D} = \frac{\mathrm{d}}{\mathrm{d}x}(x - x_0)\frac{\mathrm{d}}{\mathrm{d}x} - x^2(x - x_0) + 2N(x - x_0)$$

$$\varepsilon_k = -\frac{\pi}{v_F(x_0)}\chi_k$$

χ_k eigenvalues of M



CONCLUSIONS

- EH of gradient chain and Fermi gas in an harmonic trap:
    - Effective curved-space CFT description with a space-dependent Fermi velocity, which yields the EH in a Bisognano-Wichmann form for both cases
    - The BW structure is inherited by operators that commute exactly with the EH
    - The low-lying entanglement spectra can be well approximated by rescaling the spectra of the commuting operators by a local v_F
- Possible future investigations:
    - Extend to more general potentials
    - Dilute edge regime

THANK YOU!