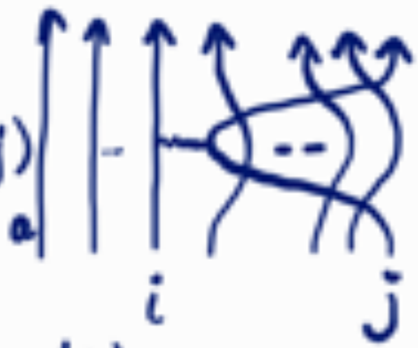
Landscapes of integrable spin chains

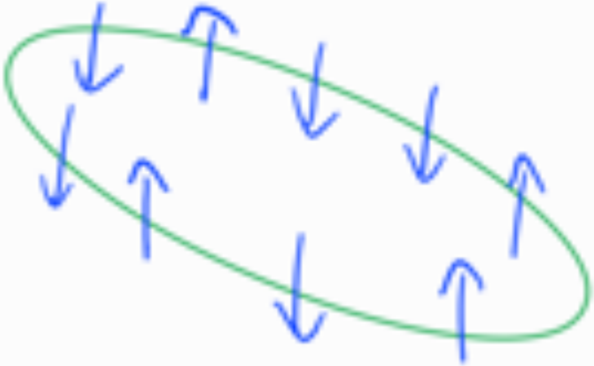
Face

$$\sum_{i<j}^{N} V(i-j)$$



Ψ

H_{XXX}



Vertex

$$\sum_{i<j}^{N} \sum_{\kappa=0}^{2} V(i-j+\omega_\kappa)$$

Ψ

H_{XX}

Rob Klabbers

Humboldt-Universität zu Berlin

Raqis '24

Annecy

based on:

arXiv:2306.13066

arXiv:2405.09718

with Jules Lamers

Nearest-neighbour integrability

Recipe

1) Take R-matrix

quantum Yang-Baxter eq'n (difference)

$$R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23}$$

2) Construct the transfer matrix

$$T(u) =$$

$$= tr_a(R_{aN} \ldots R_{a1})$$

3) Derive Hamiltonian

$$H = \frac{d}{du} \log T(u) \Big|_{u=u^*} = \sum_{i=1}^{N} R_{i,i+1}(u)^{-1} R'_{i,i+1}(u) \Big|_{u=u^*} =: H_{nn}[R]$$

$H_{nn}[R]$ is quantum-integrable, i.e. has commuting charges

Deforming Heisenberg

spin-1/2

$\mathcal{H} = \left(\mathbb{C}|\uparrow\rangle \oplus \mathbb{C}|\downarrow\rangle\right)^{\otimes N}$

rational
$su(2)$

$$H_{xxx} = H_{nn}\left[R^{rat}\right] \propto \sum_{i=1}^{N} \sigma_i^{(x)}\sigma_{i+1}^{(x)} + \sigma_i^{(y)}\sigma_{i+1}^{(y)} + \sigma_i^{(z)}\sigma_{i+1}^{(z)}$$

$$= \sum_{i=1}^{N} \left(1 - P_{i,i+1}\right)$$

Deforming Heisenberg

spin-1/2

$\mathcal{H} = \left(\mathbb{C}|\uparrow\rangle \oplus \mathbb{C}|\downarrow\rangle\right)^{\otimes N}$

Spin interaction

elliptic
anisotropic

trigonometric
$u(1)$

rational
$su(2)$

$$H_{XXX} = H_{nn}\left[R^{rat}\right] \propto \sum_{i=1}^{N} \sigma_i^{(x)}\sigma_{i+1}^{(x)} + \sigma_i^{(y)}\sigma_{i+1}^{(y)} + \sigma_i^{(z)}\sigma_{i+1}^{(z)}$$

nearest neighbour

$$= \sum_{i=1}^{N} \left(1 - P_{i,i+1}\right)$$

Deforming Heisenberg

spin-1/2

$\mathcal{H} = \left(\mathbb{C}|\uparrow\rangle \oplus \mathbb{C}|\downarrow\rangle\right)^{\otimes N}$

Spin interaction

elliptic
anisotropic

$$H_{xyz} = H_{nn}\left[R^{8V}\right] \propto \sum_{i=1}^{N} \sigma_i^{(x)}\sigma_{i+1}^{(x)} + \Gamma\,\sigma_i^{(y)}\sigma_{i+1}^{(y)} + \Delta\,\sigma_i^{(z)}\sigma_{i+1}^{(z)}$$

trigonometric
$u(1)$

$$H_{xxz} = H_{nn}\left[R^{6V}\right] \propto \sum_{i=1}^{N} \sigma_i^{(x)}\sigma_{i+1}^{(x)} + \sigma_i^{(y)}\sigma_{i+1}^{(y)} + \Delta\,\sigma_i^{(z)}\sigma_{i+1}^{(z)}$$

rational
$su(2)$

$$H_{xxx} = H_{nn}\left[R^{rat}\right] \propto \sum_{i=1}^{N} \sigma_i^{(x)}\sigma_{i+1}^{(x)} + \sigma_i^{(y)}\sigma_{i+1}^{(y)} + \sigma_i^{(z)}\sigma_{i+1}^{(z)}$$

$$= \sum_{i=1}^{N}\left(1 - P_{i,i+1}\right)$$

nearest
neighbour

spin-1/2

$\mathcal{H} = \left(\mathbb{C}|\uparrow\rangle \oplus \mathbb{C}|\downarrow\rangle\right)^{\otimes N}$

Deforming Heisenberg

Perturbative

* $[Q_r, Q_s] = \mathcal{O}(\lambda^{\text{length}})$

[Bargheer, Beisert, Loebbert, '08]

[Pozsgay, '21]

[Gombor, Pozsgay, '21]

[De Leeuw, Retore, '23]

elliptic
anisotropic $\quad H_{xyz}$

trigonometric
$U(1)$ $\quad H_{xxz}$

rational
$SU(2)$ $\quad H_{xxx}$

Spin interaction

nearest
neighbour

Interaction strength

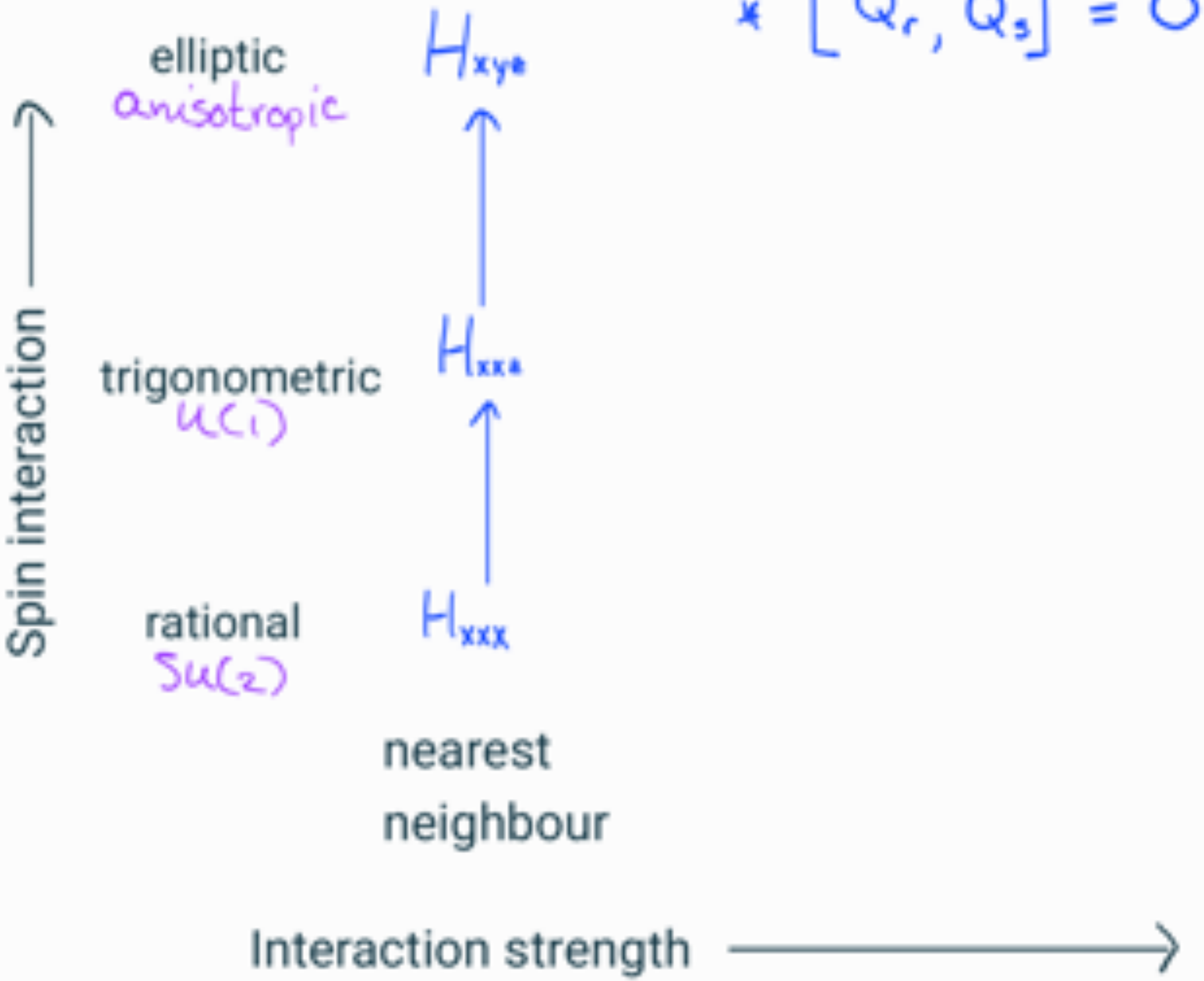
spin-1/2

$\mathcal{H} = \left(\mathbb{C}\,|\uparrow\rangle \oplus \mathbb{C}\,|\downarrow\rangle\right)^{\otimes N}$

Deforming Heisenberg

Non-perturbative

$* \; [Q_r, Q_s] = 0$



spin-1/2

$\mathcal{H} = \left(\mathbb{C}\,|\uparrow\rangle \oplus \mathbb{C}\,|\downarrow\rangle\right)^{\otimes N}$

Deforming Heisenberg

Non-perturbative

$* \; [Q_r, Q_s] = 0$

Spin interaction

elliptic
anisotropic

H_{xyz}

trigonometric
$u(1)$

H_{xxz}

q-deformed Inozemtsev
[RK, Lamers, '23]

q-deformed HS [Uglov, '95]
[Lamers, '18]

rational
$su(2)$

H_{xxx}

$$\sum_{i<j}^{N} \wp_2(i-j)\,(1-P_{ij})$$

[Inozemtsev, '90-'97]

$$\sum_{i<j}^{N} \frac{1-P_{ij}}{\sin^2\frac{\pi}{N}(i-j)}$$

[Haldane '88, Shastry '88]

nearest neighbour

elliptic

trigonometric

Interaction strength

$\wp_2(x) = \wp(x) + \mathrm{cst}$
Weierstraß P-function

Motivation

Inozemtsev

* Integrability far from understood (but see Oleg's talk)

$$H = \sum_{i<j}^{N} \wp_2(i-j)(1-P_{ij})$$

Theoretical

* SUSY gauge theory
  - Bethe-gauge correspondence [Nekrasov, Shatashvili, '09]
  - integrability of AdS/CFT [Minahan, Zarembo, '02] [Beisert, Staudacher, '05]

Mathematics

* quantum algebras
* representation theory
  - Hecke algebras
  - Macdonald theory

New:

* quantum-integrable models
* connections between models

Long-range recipe

SU(2)-symmetric

nearest-neighbour

elliptic

trigonometric

$$\sum_{i<j}^{N} \delta_{|i-j|,1}(1-P_{ij}) \longrightarrow \sum_{i<j}^{N} \wp_2(i-j)(1-P_{ij}) \longrightarrow \sum_{i<j}^{N} \frac{1}{\sin^2 \frac{\pi}{N}(i-j)}(1-P_{ij})$$

spin interaction

$$\sum_{i<j} V(i-j)\, E_{ij}$$

Potential

Long-range recipe

SU(2)-symmetric

nearest-neighbour

elliptic

trigonometric

$$\sum_{i<j}^{N} \delta_{|i-j|,1}(1-P_{ij}) \longrightarrow \sum_{i<j}^{N} \wp_2(i-j)(1-P_{ij}) \longrightarrow \sum_{i<j}^{N} \frac{1}{\sin^2\frac{\pi}{N}(i-j)}(1-P_{ij})$$

spin interaction

$$\sum_{i<j} V(i-j)\, E_{ij}$$

Potential

Idea 1

$$P_{j-1,j}\cdots P_{i+1,i+2}\; E_{i,i+1}\; P_{i+1,i+2}\cdots P_{j-1,j}$$

$$=$$

Long-range recipe

SU(2)-symmetric

nearest-neighbour

elliptic

trigonometric

$$\sum_{i<j}^{N} \delta_{|i-j|,1}(1-P_{ij}) \longrightarrow \sum_{i<j}^{N} \wp_2(i-j)(1-P_{ij}) \longrightarrow \sum_{i<j}^{N} \frac{1}{\sin^2\frac{\pi}{N}(i-j)}(1-P_{ij})$$

spin interaction

$$\sum_{i<j} V(i-j)\, E_{ij}$$

Potential

Idea 1

$$P_{j-1,j} \cdots P_{i+1,i+2} \, E_{i,i+1} \, P_{i+1,i+2} \cdots P_{j-1,j}$$

$$=$$

Idea 2

$$E = \check{R}(u)^{-1}\check{R}'(u)\Big|_{u=u^*}$$

R : Yang's R-matrix

$\check{R} = PR$

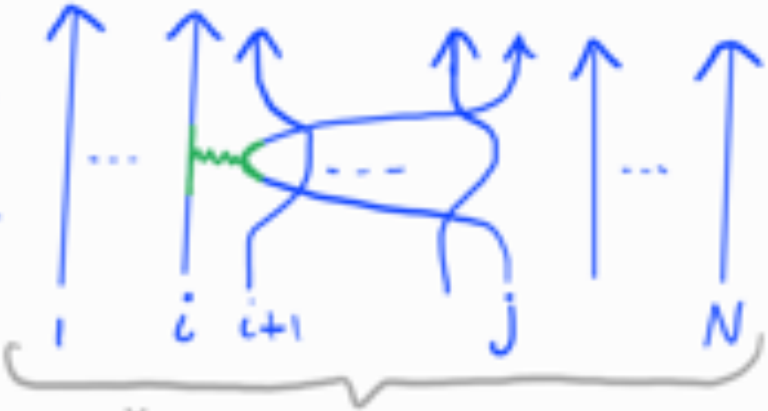
U(1)-symmetric

Long-range recipe

[Martin, Saleur]
[Hakobyan, Sedrakyan]
[Lamers]
[Matushko, Zotov]
[RK, Lamers]

Take $\check{R} = PR$

$$H^{\ell}_{\ell r}[R] = \sum_{i<j}^{N}$$



$$\check{R}_{j-1,j}(1)\dots\check{R}_{i+1,i+2}(j-i-1)$$
$$E_{i,i+1}(i-j)$$
$$\check{R}_{i+1,i+2}(i-j+1)\dots\check{R}_{j-1,j}(-1)$$

Transport

$$= \check{R}(u-v) \xrightarrow{\eta \to 0} P$$

Spin interaction

E :

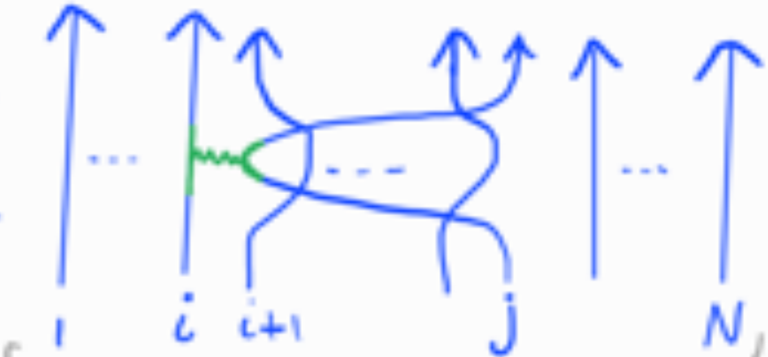

$$= \check{R}(z)^{-1}\,\check{R}'(z)\Big|_{z=u-v}$$

U(1)-symmetric

Long-range recipe

[Martin, Saleur]
[Hakobyan, Sedrakyan]
[Lamers]
[Matushko, Zotov]
[RK, Lamers]

$$H^{\ell}_{lr}[R] = \sum_{i<j}^{N}$$



$$\underbrace{\qquad\qquad}_{\check{R}_{j-1,j}(1)\dots\check{R}_{i+1,i+2}(j-i-1)\; E_{i,i+1}(i-j)\; \check{R}_{i+1,i+2}(i-j+1)\dots\check{R}_{j-1,j}(-1)}$$

Transport

$$= \check{R}(u-v) \xrightarrow{\eta \to 0} P$$

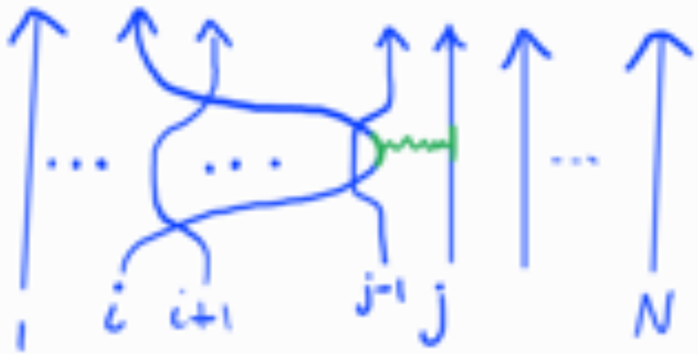
Spin interaction

$$E :$$

$$= \check{R}(z)^{-1}\,\check{R}'(z)\Big|_{z=u-v}$$

Chiral partner

$$H^{r}_{lr}[R] = \sum_{i<j}^{N}$$



Two options

[RK, Lamers, '23-'24]

Face

[Felder, '94]

$$\check{R}^F(u,a;\eta) = \frac{1}{\phi(u,\eta)} \times \begin{pmatrix} \phi(u,\eta) & 0 & 0 & 0 \\ 0 & \phi(u,\eta_a) & \phi(\eta_a,\eta) & 0 \\ 0 & -\phi(\eta_a,-\eta) & \phi(u,-\eta_a) & 0 \\ 0 & 0 & 0 & \phi(u,\eta) \end{pmatrix}$$

Vertex

[Baxter, '72]

$$\check{R}^{(8v)}(u;\eta) = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix}$$

$$= \frac{1}{\phi(u,\eta)} \sum_{\alpha=0}^{z} e^{-\kappa u \, \delta^{\alpha=x,y}} \phi\left(u, \frac{\eta+\omega_\alpha}{2}\right) \sigma^{(\alpha)} \otimes \sigma^{(\alpha)}$$

ϕ : Kronecker elliptic function : $\theta(u+v)/\theta(u)\theta(v)$

$\mathbb{L} = N\mathbb{Z} \oplus \frac{i\pi}{\kappa}\mathbb{Z}$, $(\omega_0, \omega_x, \omega_y, \omega_z) = \left(0, N, N-\frac{i\pi}{\kappa}, -\frac{i\pi}{\kappa}\right)$

$\tau = \omega_x/\omega_z$

Two options

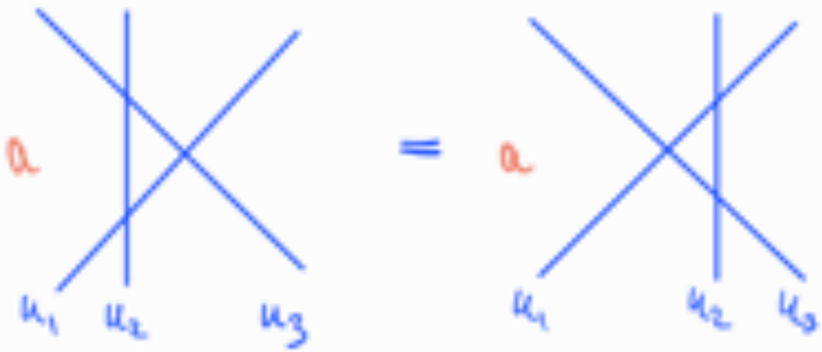
[RK, Lamers, '23-'24]

Face

[Felder, '94]

$$\check{R}^F(u,a;\eta) = \frac{1}{\phi(u,\eta)} \times \begin{pmatrix} \phi(u,\eta) & 0 & 0 & 0 \\ 0 & \phi(u,\eta a) & \phi(\eta a,\eta) & 0 \\ 0 & -\phi(\eta a,-\eta) & \phi(u,-\eta a) & 0 \\ 0 & 0 & 0 & \phi(u,\eta) \end{pmatrix}$$

dynamical YBe



$$\check{R}_{12}(u_1-u_2,a)\,\check{R}_{23}(u_1-u_3,a-\sigma_1^{(z)})\,\check{R}_{12}(u_2-u_3,a) =$$
$$= \check{R}_{23}(u_2-u_3,a-\sigma_1^{(z)})\,\check{R}_{12}(u_1-u_3,a)\,\check{R}_{23}(u_1-u_2,a-\sigma_1^{(z)})$$

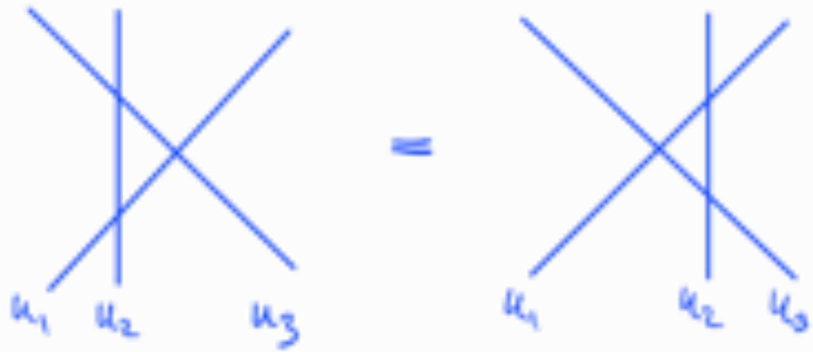
Vertex

[Baxter, '72]

$$\check{R}^{(8v)}(u;\eta) = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix}$$

$$= \frac{1}{\phi(u,\eta)} \sum_{\alpha=0}^{z} e^{-\kappa u\, \delta^{\alpha=x,y}}\, \phi\!\left(u, \frac{\eta+\omega_\alpha}{2}\right) \sigma^{(\alpha)} \otimes \sigma^{(\alpha)}$$

Braided YBe



$$\check{R}_{12}(u_1-u_2)\,\check{R}_{23}(u_1-u_3)\,\check{R}_{12}(u_2-u_3) = \check{R}_{23}(u_2-u_3)\,\check{R}_{12}(u_1-u_3)\,\check{R}_{23}(u_1-u_2)$$

ϕ : Kronecker elliptic function $\theta(u+v)/\theta(u)\theta(v)$

$\mathbb{L} = N\mathbb{Z} \oplus \frac{i\pi}{\kappa}\mathbb{Z}$, $(\omega_0,\omega_x,\omega_y,\omega_z) = (0, N, N-\frac{i\pi}{\kappa}, -\frac{i\pi}{\kappa})$

$\tau = \omega_x/\omega_0$

Two options

[RK, Lamers, '23-'24]

Face

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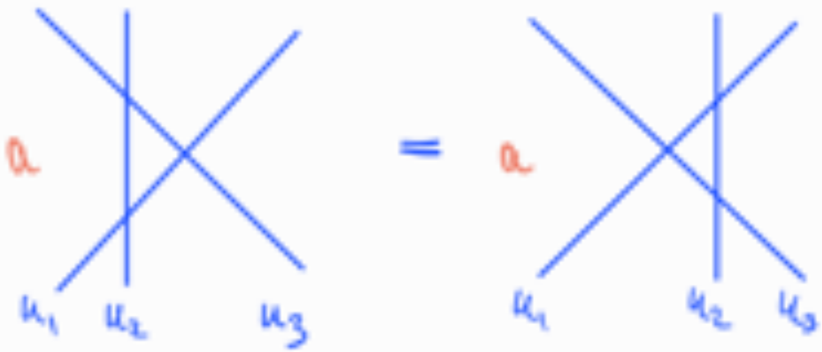
$$\check R^{F}(u,a;\eta) = \frac{1}{\phi(u,\eta)}\times\begin{pmatrix}\phi(u,\eta) & 0 & 0 & 0\\ 0 & \phi(u,\eta a) & \phi(\eta a,\eta) & 0\\ 0 & -\phi(\eta a,-\eta) & \phi(u,-\eta a) & 0\\ 0 & 0 & 0 & \phi(u,\eta)\end{pmatrix}$$

dynamical YBe



$$\check R_{12}(u_1-u_2,a)\,\check R_{23}(u_1-u_3,a-\sigma_1^{(z)})\,\check R_{12}(u_2-u_3,a) =$$
$$= \check R_{23}(u_2-u_3,a-\sigma_1^{(z)})\,\check R_{12}(u_1-u_3,a)\,\check R_{23}(u_1-u_2,a-\sigma_1^{(z)})$$

$$\check R^{(F)}(-u)\,\check R^{(F)}(u) = \Theta(\eta)\,V(u;\eta)\,e(u,a;\eta)$$

η-deformed Potential

spin interaction

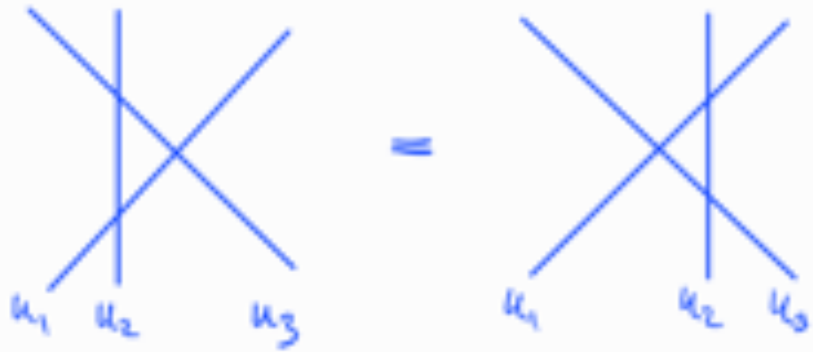
Vertex

[Baxter, '72]

$$\check R^{(8v)}(u;\eta) = \begin{pmatrix} a & 0 & 0 & d\\ 0 & b & c & 0\\ 0 & c & b & 0\\ d & 0 & 0 & a\end{pmatrix}$$

$$= \frac{1}{\phi(u,\eta)}\sum_{\alpha=0}^{z} e^{-\kappa u\,\delta^{\alpha=x,y}}\,\phi\!\left(u,\frac{\eta+\omega_\alpha}{2}\right)\sigma^{(\alpha)}\otimes\sigma^{(\alpha)}$$

Braided YBe



$$\check R_{12}(u_1-u_2)\,\check R_{23}(u_1-u_3)\,\check R_{12}(u_2-u_3) = \check R_{23}(u_2-u_3)\,\check R_{12}(u_1-u_3)\,\check R_{23}(u_1-u_2)$$

$$\check R^{(8v)}(-u)\,\check R^{(8v)'}(u) = \frac{1}{4}\sum_{\alpha=0}^{z} V\!\left(\frac{u+\omega_\alpha}{2};\frac{\eta}{2}\right)\left(1-P\,\sigma^{(\alpha)}\otimes\sigma^{(\alpha)}\right)$$

ϕ : Kronecker elliptic function $\quad \Theta(u+v)/\Theta(u)\Theta(v)$

$$\mathbb{L} = N\mathbb{Z}\oplus\frac{i\pi}{\kappa}\mathbb{Z},\qquad (\omega_0,\omega_x,\omega_y,\omega_z) = \left(0,N,N-\frac{i\pi}{\kappa},-\frac{i\pi}{\kappa}\right)$$

$$\tau = \omega_x/\omega_0$$

Two options

[RK, Lamers, '23-'24]

Face

$$\check{R}^{(F)}(-u)\,\check{R}^{(F)}(u) = \Theta(\eta)\,V(u;\eta)\,e(u,a;\eta)$$

$$H_{\ell r}\left[R^{(F)}\right] = \sum_{i<j}^{N} V(i-j)$$

- $V(u;\eta) \sim \dfrac{1}{\operatorname{sn}(u+\eta)\,\operatorname{sn}(u-\eta)}$

-  $= \check{R}^{(F)}_{i,i+1}\left(u-v,\ a-\sum_{j=1}^{i-1}\sigma_j^{(z)};\eta\right)$

-  $= e(u-v,a;\eta)$

Vertex

$$\check{R}^{(8V)}(-u)\,\check{R}^{(8V)}(u) = \frac{1}{4}\sum_{\kappa=0}^{z} V\left(\frac{u+\omega_\kappa}{2};\frac{\eta}{2}\right)\left(1-P\,\sigma^{(\kappa)}\otimes\sigma^{(\kappa)}\right)$$

ϕ : Kronecker elliptic function $\Theta(u+v)/\Theta(u)\Theta(v)$

$\mathbb{L} = N\mathbb{Z}\oplus\frac{i\pi}{\kappa}\mathbb{Z}$, $(\omega_0,\omega_x,\omega_y,\omega_z) = (0,\ N,\ N-\frac{i\pi}{\kappa},\ -\frac{i\pi}{\kappa})$

$\tau = \omega_x/\omega_0$

Two options

[RK, Lamers, '23-'24]

Face

$\check{R}^{(F)}(-u)\,\check{R}^{(F)}(u) = \Theta(\eta)\,V(u;\eta)\,e(u,a;\eta)$

$$H_{\ell r}[R^{(F)}] = \sum_{i<j}^{N} V(i-j)$$

- $V(u;\eta) \sim \dfrac{1}{\mathrm{sn}(u+\eta)\,\mathrm{sn}(u-\eta)}$

- 

$= \check{R}^{(F)}_{i,i+1}\Big(u-v,\ a-\sum_{j=1}^{i-1}\sigma_j^{(z)};\eta\Big)$

- 

$= e(u-v,a;\eta)$

Vertex

$\check{R}^{(8v)}(-u)\,\check{R}^{(8v)}(u) = \frac{1}{4}\sum_{\alpha=0}^{z} V\Big(\frac{u+\omega_\alpha}{2};\frac{\eta}{2}\Big)\Big(1-P\,\sigma^{(\alpha)}\otimes\sigma^{(\alpha)}\Big)$

$$H_{\ell r}[R^{(8v)}] = \sum_{i<j}^{N}\sum_{\alpha=0}^{z}\frac{1}{4}V\Big(\frac{i-j+\omega_\alpha}{2};\frac{\eta}{2}\Big)\times$$

- 

$= \check{R}^{(8v)}(u-v;\eta)$

- 

$= \Big(1-P\,\sigma^{(\alpha)}\otimes\sigma^{(\alpha)}\Big)$

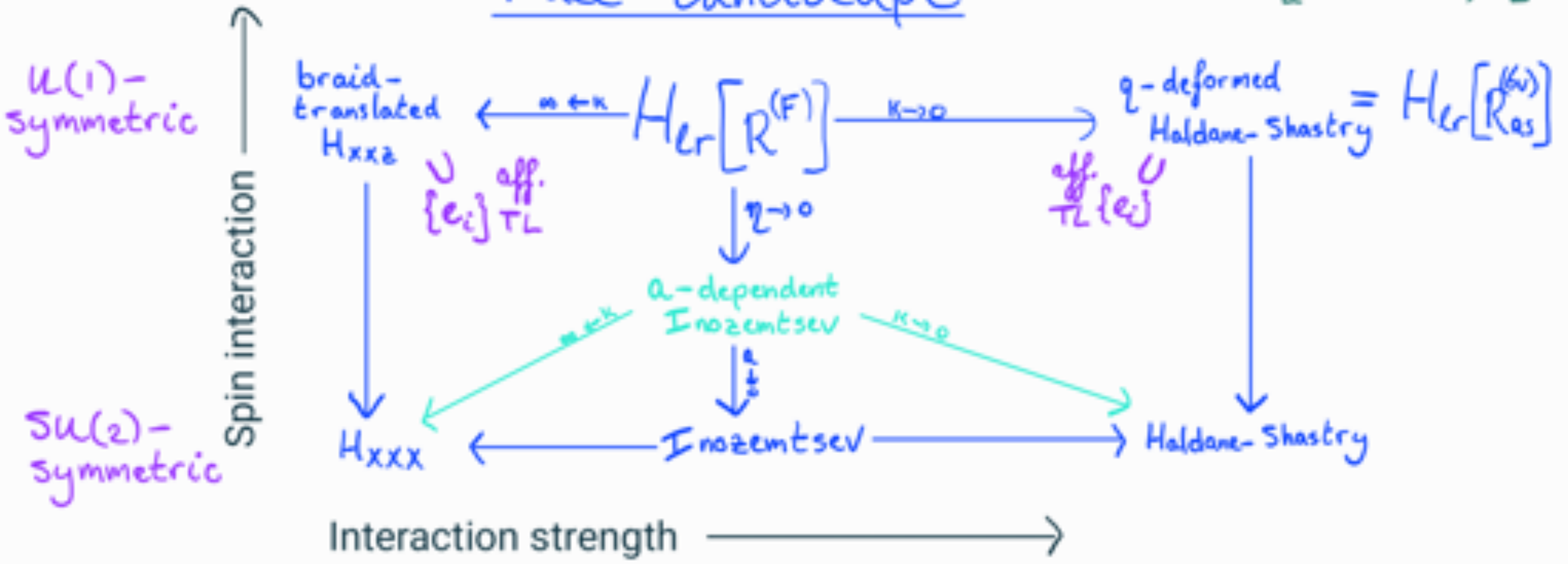
ϕ : Kronecker elliptic function $\Theta(u+v)/\Theta(u)\Theta(v)$

$\mathbb{L} = N\mathbb{Z}\oplus\frac{i\pi}{\kappa}\mathbb{Z}$, $\quad(\omega_0,\omega_x,\omega_y,\omega_z) = \Big(0,N,N-\frac{i\pi}{\kappa},-\frac{i\pi}{\kappa}\Big)$

$\tau = \omega_x/\omega_0$

Face landscape

[RK, Lamers,'23]



All chains

* are partially isotropic
* are integrable
* have (diagonally twisted) boundary conditions

Translation operator

$G =$  $= e^{-\kappa\eta\underline{a}\,\sigma_N^{(z)}} \prod_{N\geq i>1} \check{R}^{(F)}_{i-1,i}(1-i,\underline{a};\eta)$

Generalised Inozemtsev

$$H_{Ino}(a) = \frac{1}{2}\sum_{i<j} \phi'(i-j,a)\frac{\sigma_i^+\sigma_j^-}{2} + \phi'(i-j,-a)\frac{\sigma_i^-\sigma_j^+}{2} + \wp_2(i-j)(1-\sigma_i^{(z)}\sigma_j^{(z)})$$

Vertex landscape

[RK, Lamers, '24]

[Matushko, Zotov, '22]

Spin interaction ↑

$$
\begin{tikzcd}
\begin{array}{c}\text{braid-}\\\text{translated}\\ H_{xx}\end{array} \arrow[d] & H_{\ell r}\left[R^{(8v)}\right] \arrow[l, "\infty \leftarrow \kappa"'] \arrow[r, "\kappa \to 0", "+\text{conjugation}"'] \arrow[d, "\eta \to 0"] & H_{\ell r}\left[R^{(6v)}_{\text{sym}}\right] \arrow[d] \\
\begin{array}{c}\text{anti-periodic}\\ H_{xx}\end{array} & \begin{array}{c}\text{anti-periodic}\\ \text{Inozemtsev}\end{array} \arrow[l] \arrow[r] & \begin{array}{c}\text{anti-periodic}\\ \text{Haldane-Shastry}\end{array}
\end{tikzcd}
$$

[Sechin, Zotov, '18]

[Fukui, Kawakami, '96]

$$\sum_{i<j}^{N} \sum_{\alpha=0}^{3} \wp_2\left(\frac{u+\omega_\alpha}{2}\right)\left(1 - P_{ij}\, \sigma_i^{(\alpha)} \sigma_j^{(\alpha)}\right)$$

Interaction strength →

All chains

* are anisotropic
* are integrable
* have antiperiodic (twisted) boundary conditions

Translation operator

$G =$ 

$$G^N = \prod_{i=1}^{N} \sigma_i^{(x)}$$

$$= e^{-(N-1)i\kappa\eta/2}\, \sigma_N^{(x)} \prod_{N \geq i > 1} \check{R}^{(8v)}_{i-1,i}(1-i;\eta)$$

Wrapping

Periodic

$$: \sigma_{i+N}^{(\alpha)} = \sigma_i^{(\alpha)}$$

$$H_{HS}^{rat} = \sum_{i<j}^{\infty} \frac{1-P_{ij}}{(i-j)^2}$$

Wrapping

$$H_{HS} = \sum_{i<j}^{N} \frac{1-P_{ij}}{\sin^2\frac{\pi}{N}(i-j)}$$

$$H_{HS}^{rat} = \sum_{i<j}^{\infty} \frac{1-P_{ij}}{(i-j)^2}$$

Periodic

$$: \sigma_{i+N}^{(\alpha)} = \sigma_i^{(\alpha)}$$

Wrapping

$$H_{HS} = \sum_{i<j}^{N} \frac{1-P_{ij}}{\sin^2\frac{\pi}{N}(i-j)}$$

$$\frac{i\pi}{\kappa} \leftrightarrow N$$

$$H^{hyp}_{Ino} = \sum_{i<j}^{\infty} \frac{1-P_{ij}}{\sinh^2\kappa(i-j)}$$

$$H^{rat}_{HS} = \sum_{i<j}^{\infty} \frac{1-P_{ij}}{(i-j)^2}$$

Periodic

$$: \sigma^{(\alpha)}_{i+N} = \sigma^{(\alpha)}_{i}$$

Wrapping

Face

$$H_{\mathrm{Ino}}$$

$$H_{HS} = \sum_{i<j}^{N} \frac{1-P_{ij}}{\sin^2\frac{\pi}{N}(i-j)}$$

$$\frac{i\pi}{\kappa} \longleftrightarrow N$$

$$H^{hyp}_{\mathrm{Ino}} = \sum_{i<j}^{\infty} \frac{1-P_{ij}}{\sinh^2\kappa(i-j)}$$

$$H^{rat}_{HS} = \sum_{i<j}^{\infty} \frac{1-P_{ij}}{(i-j)^2}$$

Periodic

$$: \sigma^{(\alpha)}_{i+N} = \sigma^{(\alpha)}_{i}$$

Wrapping

Face

$$H_{I_{no}}$$

$$H_{HS} = \sum_{i<j}^{N} \frac{1-P_{ij}}{\sin^2\frac{\pi}{N}(i-j)}$$

$$\frac{i\pi}{\kappa} \longleftrightarrow N$$

$$H^{hyp}_{I_{no}} = \sum_{i<j}^{\infty} \frac{1-P_{ij}}{\sinh^2\kappa(i-j)}$$

$$H^{rat}_{HS} = \sum_{i<j}^{\infty} \frac{1-P_{ij}}{(i-j)^2}$$

Periodic

$$:\ \sigma^{(\alpha)}_{i+N} = \sigma^{(\alpha)}_{i}$$

Anti-periodic

α,β :

$\gamma = \alpha,\beta$:

$$\sigma^{(\gamma)}_{i+kN} = (-1)^k \sigma^{(\gamma)}_{i}$$

$\gamma \neq \alpha,\beta$

$$\sigma^{(\gamma)}_{i+kN} = \sigma^{(\gamma)}_{i}$$

x,y

Wrapping

Face

$$H_{\mathrm{Ino}}$$

$$H_{\mathrm{HS}} = \sum_{i<j}^{N} \frac{1-P_{ij}}{\sin^2\frac{\pi}{N}(i-j)}$$

$$\frac{i\pi}{\kappa} \longleftrightarrow N$$

$$H_{\mathrm{Ino}}^{\mathrm{hyp}} = \sum_{i<j}^{\infty} \frac{1-P_{ij}}{\sinh^2\kappa(i-j)}$$

$$H_{\mathrm{HS}}^{\mathrm{rat}} = \sum_{i<j}^{\infty} \frac{1-P_{ij}}{(i-j)^2}$$

Periodic

$$:\ \sigma_{i+N}^{(\alpha)} = \sigma_i^{(\alpha)}$$

Vertex

$$H_{\mathrm{FK}}^{\mathrm{hyp}}$$

$$\frac{i\pi}{\kappa} \longleftrightarrow N$$

x,y

y,z

anti-periodic Inozemtsev

$$H_{\mathrm{FK}} = -\frac{1}{2}\sum_{i<j}^{N} \frac{\cos\frac{\pi}{N}(i-j)\left[\sigma_i^{(\alpha)}\sigma_j^{(\alpha)} + \sigma_i^{(\beta)}\sigma_j^{(\beta)}\right] + \sigma_i^{(\gamma)}\sigma_j^{(\gamma)} - 1}{\sin^2\frac{\pi}{N}(i-j)}$$

Anti-periodic

α,β :

$\gamma = \alpha, \beta$:

$$\sigma_{i+kN}^{(\gamma)} = (-1)^k \sigma_i^{(\gamma)}$$

$\gamma \neq \alpha, \beta$

$$\sigma_{i+kN}^{(\gamma)} = \sigma_i^{(\gamma)}$$

Quantum many body systems

When is $H_{\ell r}[\check{R}]$ integrable?

$$A_i(\vec{x}) = \prod_{j\neq i}^{N} \frac{\theta(x_i - x_j + \eta)}{\theta(x_i - x_j)}$$

Elliptic spin Ruijsenaars models

$$\tilde{D}_1 = \sum_{i=1}^{N} A_i(\vec{x})$$

$$= \Gamma_i = e^{-i\hbar\epsilon\partial_{x_i}}$$

$$x_i^- = x_i - i\hbar\epsilon$$

$$= \check{R}(u-v)$$

Quantum many body systems

When is $H_{lr}[R]$ integrable?

$$A_i(\vec{x}) = \prod_{j\neq i}^{N} \frac{\theta(x_i - x_j + \eta)}{\theta(x_i - x_j)}$$

Elliptic spin Ruijsenaars models

$$\tilde{D}_1 = \sum_{i=1}^{N} A_i(\vec{x})$$

$$= \Gamma_i = e^{-i\hbar\epsilon\partial_{x_i}}$$

$$x_i^- = x_i - i\hbar\epsilon$$

$$= \check{R}(u-v)$$

* natural equivalent of the (scalar)

$$\tilde{D}_1 = \sum_{i=1}^{N} A_i(\vec{x})\Gamma_i$$ on deformed bosons / fermions:

$$P^{tot}_{i,i+1}|\psi\rangle = \pm|\psi\rangle$$

$$P^{tot}_{i,i+1} = s^{(x)}_{i,i+1}\check{R}_{i,i+1}(x_i - x_{i+1})$$

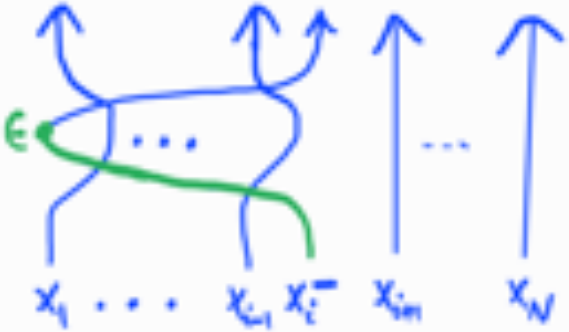
$$s^{(x)}_{12}\psi(x_1, x_2) = \psi(x_2, x_1)$$

Quantum many body systems

When is $H_{\ell r}[R]$ integrable?

$$A_i(\vec{x}) = \prod_{j\neq i}^{N} \frac{\theta(x_i - x_j + \eta)}{\theta(x_i - x_j)}$$

Elliptic spin Ruijsenaars models

$\tilde{D}_1 = \sum_{i=1}^{N} A_i(\vec{x})$ 

$= \Gamma_i = e^{-i\hbar\epsilon\partial_{x_i}}$

$x_i^- = x_i - i\hbar\epsilon$

$= R(u-v)$

* natural equivalent of the (scalar)

$\tilde{D}_1 = \sum_{i=1}^{N} A_i(\vec{x})\Gamma_i$ on deformed bosons/fermions: $P^{tot}_{i,i+1}|\psi\rangle = \pm|\psi\rangle$

$P^{tot}_{i,i+1} = S^{(x)}_{i,i+1}\check{R}_{i,i+1}(x_i - x_{i+1})$

$S^{(x)}_{12}\psi(x_1,x_2) = \psi(x_2,x_1)$

* Integrable for $R^{(8v)}$ [Matushko, Zotov, '23] and $R^{(\sigma)}$ [RK, Lamers, unpub]

* Freezing yields $H_{\ell r}[R]$ [Polychronakos, '93] [Lyashik, Sechin, Reshetikhin, '24] [RK, Lamers, unpub]

Conclusions

1) Both elliptic R-matrices generate a separate landscape of integrable spin chains

2) Connecting known and new models:

* braid-translated $\begin{matrix} H_{xx} \\ H_{xxz} \end{matrix} = \sum_{i=1}^{N-1} e_{i,i+1} + G e_{12} G^{-1}$

* Generalised Inozemtsev

$$H_{Ino}(a) = \frac{1}{2}\sum_{i<j} \phi'(i-j,a)\frac{\sigma_i^+\sigma_j^-}{2} + \phi'(i-j,-a)\frac{\sigma_i^-\sigma_j^+}{2} + \wp_1(i-j)(1-\sigma_i^{(0)}\sigma_j^{(0)})$$

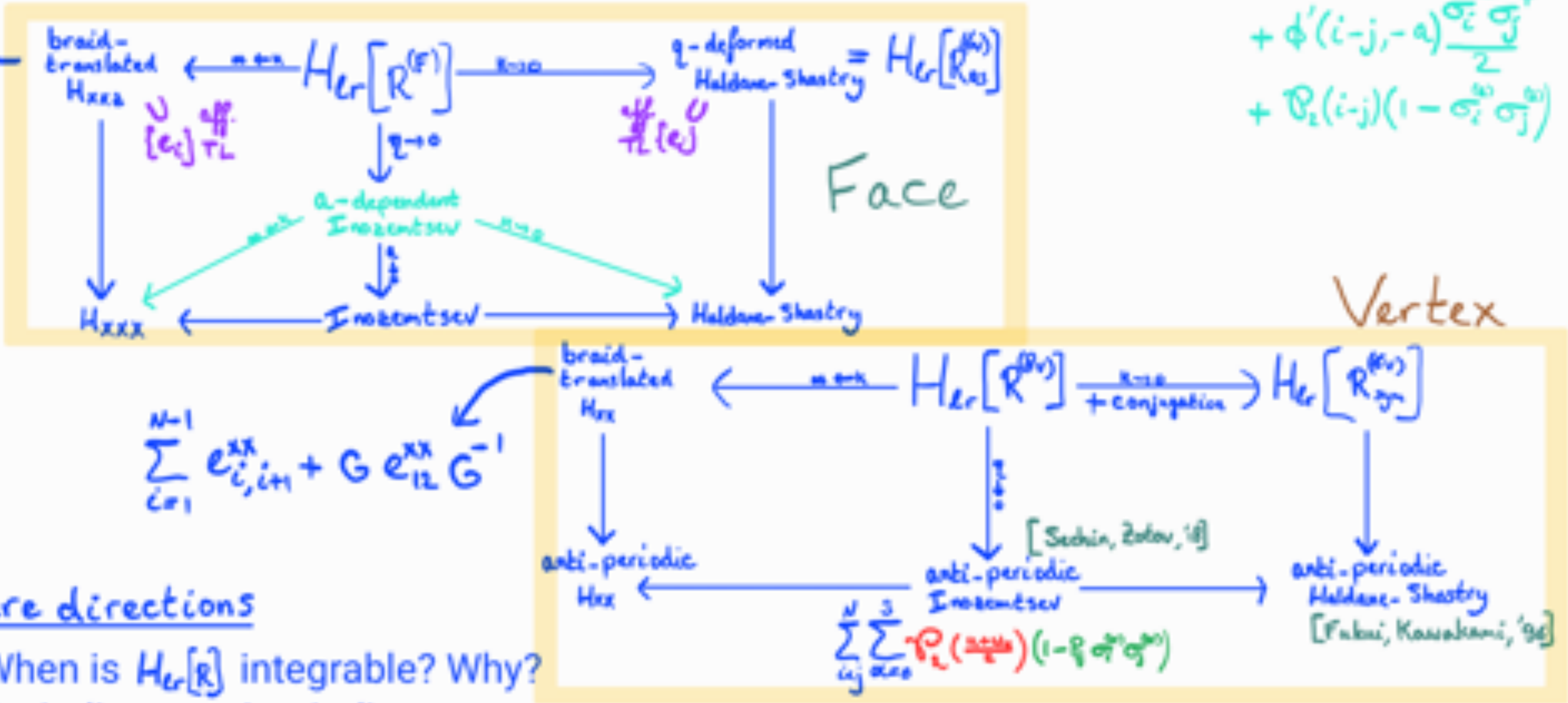
Future directions

* When is $H_{cr}[R]$ integrable? Why?
* Periodic vs antiperiodic
* For $H_{cr}[R^{(F)}]$: generalised ABA, eigenvectors, generalised TL and dAHA
* For $H_{cr}[R^{(8V)}]$: find new solution strategies
* Connection to perturbative framework
* H_{xyz} ?

Conclusions

$$\sum_{i=1}^{N-1} e_{i,i+1}(a) + G(a)\, e_{12}(a)\, G^{-1}(a)$$

$$H_{\mathrm{Ino}}(a) = \frac{1}{2}\sum_{i<j} \phi'(i-j,a)\frac{\sigma_i^+\sigma_j^-}{2} + \phi'(i-j,-a)\frac{\sigma_i^-\sigma_j^+}{2} + \wp_2(i-j)\left(1-\sigma_i^{(z)}\sigma_j^{(z)}\right)$$



Future directions

- When is $H_{\ell r}[R]$ integrable? Why?
- Periodic vs antiperiodic
- For $H_{\ell r}[R^{(F)}]$: generalised ABA, eigenvectors, generalised TL and dAHA
- For $H_{\ell r}[R^{(Bv)}]$: find new solution strategies
- Connection to perturbative framework
- H_{XYZ}?

Thank you