Long-range to the Rescue of Yang-Baxter

Juan Miguel Nieto García (Universität Hamburg)

Based on: arXiv:2408.03365, with Deniz Bozkurt and Elli Pomoni

Outline

1. The orbifold theory
2. One, two and three excitations
3. Three-Body Long-Range Solution
4. A special solution
5. Yang operators
6. Conclusions

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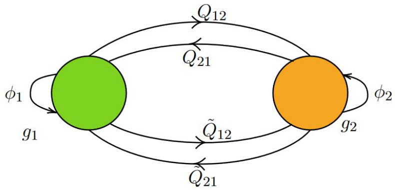
Motivation

It is well known that the computation of the one-loop anomalous dimension of a specific set of operators in $\mathcal{N} = 4$ Super Yang-Mills can be mapped to the computation of energy eigenstates of the Heisenberg spin chain. [Minahan, Zarembo 2002]

As integrability in $\mathcal{N} = 4$ Super Yang-Mills works so well, there has been an interest in testing how much can we generalise the theory and still have integrability help us to get exact results. In this talk, I will consider the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM, dual to strings propagating on $\mathrm{AdS}_5 \times (\mathrm{S}^5/\mathbb{Z}_2)$.

Orbifold theory



For the $\mathbb{Z}_2$ orbifold, we have to consider two copies of the original $SU(N)$ color group. Some of the scalars of $\mathcal{N} = 4$ will transform in the adjoint of one of the color groups (ϕ_i), while other scalars will transform in the fundamental of one and the anti-fundamental of the other (Q_{ij}).

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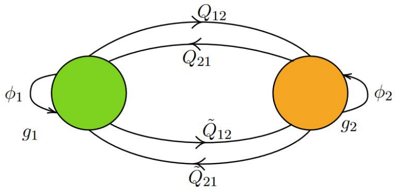

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In principle, both gauge groups have the same coupling constant, but we can introduce a (marginal) deformation to make $\kappa = \frac{g_2}{g_1} \neq 1$.

Closable states, unclosable states

Here I will focus only on operators made of ϕ_i and Q_{ij}, and consider the states

$$\left|\cdots \phi_i \phi_i \phi_i \cdots\right\rangle \,, \quad \text{for} \quad i = 1, 2 \,,$$

as my vacua of the effective spin chain and the Q's as excitations.

Because the color indices have to be contracted appropriately, single-trace operators can only include an even number of bifundamental fields appearing in an alternating fashion (with or without ϕs in between). For example

$$\operatorname{tr}\left(\phi_i^L\right) \,, \qquad \operatorname{tr}\left(\phi_1^{L-A} Q_{12} \phi_2^A Q_{21}\right) \,, \qquad \operatorname{tr}\left(\phi_1^{L-A-B-C} Q_{12} \phi_2^A Q_{21} \phi_1^B Q_{12} \phi_2^C Q_{21}\right) \,.$$

In contrast,

$$\left| {\phi_1}^{a_1}_{a_2} {Q_{12}}^{a_2}_{\bar{a}_1} {\phi_2}^{\bar{a}_1}_{\bar{a}_2} \cdots {\phi_2}^{\bar{a}_{L-1}}_{\bar{a}_L} \right\rangle$$

cannot give rise to a trace operator. However, studying these states is still instructive.

Spin chain Hamiltonian

The Hamiltonian of the effective spin chain that gives us the 1-loop anomalous dimension is

$$\mathcal{H}_{\ell,\ell+1} = \begin{array}{c} \\ \phi_1\phi_1 \\ \phi_1 Q_{12} \\ Q_{12}\phi_2 \\ Q_{12}Q_{21} \\ \phi_2\phi_2 \\ \phi_2 Q_{21} \\ Q_{21}\phi_1 \\ Q_{21}Q_{12} \end{array} \begin{array}{c} \begin{array}{cccccccc} \phi_1\phi_1 & \phi_1 Q_{12} & Q_{12}\phi_2 & Q_{12}Q_{21} & \phi_2\phi_2 & \phi_2 Q_{21} & Q_{21}\phi_1 & Q_{21}Q_{12} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\kappa & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & \kappa & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1/\kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} .$$

We should stress

1. Not all possible field combinations are not allowed due to color index contractions.
2. It has the form of two Temperly-Lieb Hamiltonians.
3. It is invariant under the exchange $1 \leftrightarrow 2$ and $\kappa \leftrightarrow 1/\kappa$.

Integrability of the orbifold theory

It was shown that the usual Bethe Ansatz techniques work for the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM, i.e., for the case of $\kappa = 1$. [Beisert, Roiban 2005]

However, when we marginally deformed the theory ($\kappa \neq 1$) these techniques do not work any more and it naively seems like integrability is destroyed. [Gadde, Pomoni, Rastelli 2010]

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One excitation

For the case of one and two Q's in a sea of ϕ_i's, the usual Coordinate Bethe Ansatz can be used.

For the case of one excitation, we have two possible states,

$$|\Psi(p)\rangle_{12} = \sum_{l=-\infty}^{\infty} e^{ipl} \, |\dots \phi_1 \phi_1 Q_{12}(l) \phi_2 \phi_2 \dots \rangle \, ,$$

$$|\Psi(p)\rangle_{21} = \sum_{l=-\infty}^{\infty} e^{ipl} \, |\dots \phi_2 \phi_2 Q_{21}(l) \phi_1 \phi_1 \dots \rangle \, ,$$

both with energy $E_1(p) = \left(\sqrt{\kappa} - \frac{1}{\sqrt{\kappa}}\right)^2 + 4 \sin^2\left(\frac{p}{2}\right)$.

The two states are related by a $1 \leftrightarrow 2$ and $\kappa \leftrightarrow 1/\kappa$ transformation

$$\mathbb{Z}_2 \, |\Psi(p)\rangle_{ij} = |\Psi(p)\rangle_{ji} \, , \qquad \text{for} \qquad ij \in \{12, 21\} \, ,$$

which can be combined with parity to give states with the same fields at infinity

$$\mathbb{Z}_2 \mathcal{P} \, |\Psi(p)\rangle_{ij} = |\Psi(-p)\rangle_{ij} \, .$$

Two excitations

For two excitations, the ansatz

$$
\begin{aligned}
|\Psi(p_1,p_2)\rangle_{ij} &= \sum_{l_1<l_2} (e^{il_1p_1+il_2p_2} + S_{ij}(p_1,p_2)e^{il_1p_2+il_2p_1})\,|\ldots\phi_i Q_{ij}(l_1)\phi_j\ldots\phi_j Q_{ji}(l_2)\phi_i\ldots\rangle \\
&= \sum_{l_1<l_2} (e^{il_1p_1+il_2p_2} + S_{ij}(p_1,p_2)e^{il_1p_2+il_2p_1})\,|l_1,l_2\rangle_{ij} \ ,
\end{aligned}
$$

solves the Schrödinger equation with energy $E_2 = E_1(p_1) + E_1(p_2)$ provided

$$
S_{12} = S_\kappa(p_1,p_2) = -\frac{1+e^{ip_1+ip_2}-2\kappa e^{ip_2}}{1+e^{ip_1+ip_2}-2\kappa e^{ip_1}} \ ,
$$

and $S_{21} = S_{1/\kappa}(p_1,p_2)$.

Three excitations

For three excitations, we can use the ansatz

$$|\varphi(p_1, p_2, p_3)\rangle_{12} = \sum_{l_1<l_2<l_3} \varphi_{12}(p_1, p_2, p_3; l_1, l_2, l_3)\, |l_1, l_2, l_3\rangle_{12} \ ,$$

$$\varphi_{12}(p_1, p_2, p_3; l_1, l_2, l_3) = \left(\sum_{\sigma\in\mathcal{S}_3} A_\sigma \ e^{ip_{\sigma(1)}l_1 + ip_{\sigma(2)}l_2 + ip_{\sigma(3)}l_3} \right) \ .$$

However, when we try to solve the Schrödinger equation, we find

$$\frac{A_{213}}{A_{123}} = S_\kappa(p_1, p_2) \ , \qquad \frac{A_{231}}{A_{213}} = S_{1/\kappa}(p_1, p_3) \ , \qquad \frac{A_{321}}{A_{231}} = S_\kappa(p_2, p_3) \ ,$$
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which only has the trivial solution. This happens because

$$S_\kappa(p_2, p_3) S_{1/\kappa}(p_1, p_3) S_\kappa(p_1, p_2) \neq S_{1/\kappa}(p_1, p_2) S_\kappa(p_1, p_3) S_{1/\kappa}(p_2, p_3) \ .$$

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New three excitations ansatz

We can try to fix our problem by introducing contact terms, but we quickly check that it is now enough. However, inspired by that, we write the ansatz

$$|\Psi(p_1,p_2,p_3)\rangle_{12} = \sum_{l_1<l_2<l_3} \Psi(p_1,p_2,p_3;l_1,l_2,l_3)\,|l_1,l_2,l_3\rangle_{12} \,,$$

$$\Psi(p_1,p_2,p_3;l_1,l_2,l_3) = \left(\sum_{\sigma\in S_3} \left(A_\sigma + D_\sigma^{l_2-l_1-1,l_3-l_2-1}\right) e^{i\vec{p}_\sigma\cdot\vec{l}} \right) \,.$$

We have introduced position-dependent corrections that depend on the permutation, σ, the momenta, and the relative position of the excitations.

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We have introduced position-dependent corrections that depend on the permutation, σ, the momenta, and the relative position of the excitations.

So let's now see what happens when we substitute it in the Schrödinger equation.

Well-separated

If the three magnons are well separated, the Schrödinger equation takes the form

$$\gamma(n,m) = \sum_{\sigma\in\mathcal{S}_3} \gamma_\sigma(p_1,p_2,p_3,n,m)\, e^{i(n+1)(p_{\sigma(2)}+p_{\sigma(3)})+i(m+1)p_{\sigma(3)}}\ , \tag{1}$$

where the term associated with the identity permutation takes the form

$$\begin{aligned}\gamma_{123}(n,m) = &-e^{ip_1} D_{123}^{n-1,m} - e^{-ip_1} D_{123}^{n+1,m} - e^{-ip_3} D_{123}^{n,m-1} \\ &+\varepsilon_3 D_{123}^{n,m} - e^{ip_3} D_{123}^{n,m+1} - e^{ip_2} D_{123}^{n+1,m-1} - e^{-ip_2} D_{123}^{n-1,m+1}\,.\end{aligned} \tag{2}$$

Here $\varepsilon_3 = 3\left(\kappa+\frac{1}{\kappa}\right) - E_3 = \sum_{j=1}^{3}\left(e^{ip_j}+e^{-ip_j}\right)$, $n = l_2 - l_1 - 1 > 0$ and $m = l_3 - l_2 - 1 > 0$. The other contributions are obtained by permuting the indices of the D's and the momenta.

Similarly to the CBA, we demand that each γ_σ vanishes independently.

Two close together

Here we have two options. On the one hand, we can have the two magnons on the left close together and the one to the right very far away. In this case, the Schrödinger equation becomes

$$\beta_I(m) = \beta_{I,(12)}(m) e^{i(m+2)p_3} + \beta_{I,(13)}(m) e^{i(m+2)p_2} + \beta_{I,(23)}(m) e^{i(m+2)p_1} \ ,$$

where

$$\begin{aligned}
\beta_{I,(12)}(m) = & \ A_{123} \ a(p_2, p_1, \kappa) + A_{213} \ a(p_1, p_2, \kappa) \\
& - e^{-ip_3} \left(e^{ip_1} D_{213}^{0,m-1} + e^{ip_2} D_{123}^{0,m-1} \right) - e^{ip_3} \left(e^{ip_1} D_{213}^{0,m+1} + e^{ip_2} D_{123}^{0,m+1} \right) \\
& - e^{-ip_1 - ip_2} \left(e^{2ip_1} D_{213}^{1,m} + e^{2ip_2} D_{123}^{1,m} \right) - \left(e^{2ip_1} D_{213}^{1,m-1} + e^{2ip_2} D_{123}^{1,m-1} \right) \\
& + (\varepsilon_3 - 2\kappa) \left(e^{ip_1} D_{213}^{0,m} + e^{ip_2} D_{123}^{0,m} \right) \ .
\end{aligned}$$

Here $a = a(p_1, p_2, \kappa) = 1 + e^{ip_1 + ip_2} - 2\kappa e^{ip_1}$.
Notice that, because $l_2 = l_1 + 1$, we can only separate the equation into three pieces instead of six.

Two close together

On the other hand, we can have the two magnons on the left close together and the one to the right very far away. In this case, the Schrödinger equation has a similar structure, but with

$$\begin{aligned}\beta_{r,(12)}(n) = {} & A_{312}\ a(p_2, p_1, \kappa^{-1}) + A_{321}\ a(p_1, p_2, \kappa^{-1}) \\ & - e^{ip_3}\left(e^{ip_1} D_{321}^{n-1,0} + e^{ip_2} D_{312}^{n-1,0}\right) - e^{-ip_3}\left(e^{ip_1} D_{321}^{n+1,0} + e^{ip_2} D_{312}^{n+1,0}\right) \\ & - \left(e^{2ip_1} D_{321}^{n,1} + e^{2ip_2} D_{312}^{n,1}\right) - e^{-ip_1-ip_2}\left(e^{2ip_1} D_{321}^{n-1,1} + e^{2ip_2} D_{312}^{n-1,1}\right) \\ & + \left(\varepsilon_3 - 2\kappa^{-1}\right)\left(e^{ip_1} D_{321}^{n,0} + e^{ip_2} D_{312}^{n,0}\right)\ .\end{aligned}$$

Again, we can only separate the equation into three pieces instead of six because $l_3 = l_2 + 1$.

Three close together

When the three are close together, we get only one equation

$$\alpha(p_1, p_2, p_3) = \sum_{\sigma\in S_3} \Big[\big(\kappa + \frac{1}{\kappa} - E_3 - e^{ip_{\sigma(3)}} - e^{-ip_{\sigma(1)}}\big)A_\sigma \\ - e^{-ip_{\sigma(1)}} D_\sigma^{1,0} - e^{ip_{\sigma(3)}} D_\sigma^{0,1}\Big] e^{i(p_{\sigma(2)}+2p_{\sigma(3)})} \ .$$

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Thus, the equations γ_σ form a set of recurrence equations on n and m, β_l form a set of recurrence equations on m that act as initial data for γ, similarly for β_r. Finally, α provides initial data for the β's.

These equations are not enough to fix all the D's, so we still have some freedom to play with.

Generating functions

As we have a set of recurrence relations, the best way to solve them is by transforming them into algebraic equations by defining the generating function

$$G_\sigma(x,y) = \sum_{n,m=1}^{\infty} D_\sigma^{n,m}(p_1,p_2,p_3;\kappa)x^n y^m \ .$$

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Then, the equations of type γ_σ can be solved to give

$$G(e^{-ip_{\sigma(1)}}x, e^{ip_{\sigma(3)}}y)_\sigma=\frac{\text{function of initial conditions and } \sigma}{\varepsilon_3 xy-e^{-i\mathcal{P}}x^2-e^{i\mathcal{P}}y^2-x-y-x^2y-y^2x} \ ,$$

where the shift in the arguments makes the denominator independent of the permutation considered. Notice also that the denominator defines an elliptic curve $\rightarrow$ ellipticity of the model?

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The S_κ solution

Let me use the freedom we have to fix

$$\frac{A_{213}}{A_{123}} = S_\kappa(p_1, p_2)\ , \qquad \frac{A_{312}}{A_{132}} = S_\kappa(p_1, p_3)\ , \qquad \frac{A_{321}}{A_{231}} = S_\kappa(p_2, p_3)$$
$$\frac{A_{132}}{A_{123}} = S_\kappa(p_2, p_3)\ , \qquad \frac{A_{231}}{A_{213}} = S_\kappa(p_1, p_3)\ , \qquad \frac{A_{321}}{A_{312}} = S_\kappa(p_1, p_2)\ ,$$
$$\tilde{D}^{n,m}_{213} = S_\kappa(p_1, p_2)\tilde{D}^{n,m}_{123}\ , \quad \tilde{D}^{n,m}_{312} = S_\kappa(p_1, p_3)\tilde{D}^{n,m}_{132}\ , \quad \tilde{D}^{n,m}_{321} = S_\kappa(p_2, p_3)\tilde{D}^{n,m}_{231}\ .$$

where $\tilde{D}^{n,m}_{\sigma} = e^{-inp_{\sigma(1)}+imp_{\sigma(3)}} D^{n,m}_{\sigma}$.

The S_κ solution

After some effort, and fixing more freedom, surprisingly we find that

$$
\begin{aligned}
D_{123}^{n,m} &= f(p_1, p_2, p_3; n, m; \kappa) \, , \\
D_{132}^{n,m} &= S_\kappa(p_2, p_3) f(p_1, p_3, p_2; n, m; \kappa) \, , \\
D_{213}^{n,m} &= S_\kappa(p_1, p_2) f(p_2, p_1, p_3; n, m; \kappa) \, , \\
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D_{321}^{n,m} &= S_\kappa(p_2, p_3) S_\kappa(p_1, p_3) S_\kappa(p_1, p_2) f(p_3, p_2, p_1; n, m; \kappa) \, ,
\end{aligned}
$$

which means that, although the D's naively break the permutational symmetry of the excitations, it can be restored.

The S_κ solution

When we substitute this result into our ansatz, it takes the form

$$|\xi(p_1,p_2,p_3)\rangle_\kappa = \sum_{l_1<l_2<l_3} \left(\sum_{\sigma\in\mathcal{S}_3} A_\sigma \; \big(1+f\left(p_{\sigma(1)},p_{\sigma(2)},p_{\sigma(3)};n,m;\kappa\right)\big)\, e^{ip_\sigma \cdot l} \right) |l_1,l_2,l_3\rangle \; .$$

Thus, what we are actually doing is transforming the plane-wave factor into a different basis of functions, but the relative coefficients are the same as the ones we would expect from the naive CBA.

The $S_{1/\kappa}$ solution

But the solution we have computed favors the S_κ coefficients over the $S_{1/\kappa}$ coefficients. Is there a deep meaning behind that?

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and obtained another solution $|\xi(p_1, p_2, p_3)\rangle_{1/\kappa}$.

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and obtained another solution $|\xi(p_1, p_2, p_3)\rangle_{1/\kappa}$.

In fact, the combinations $|\xi(p_1, p_2, p_3)\rangle_\kappa \pm |\xi(p_1, p_2, p_3)\rangle_{1/\kappa}$ are even and odd under the combination $\mathbb{Z}_2\mathcal{P}$.

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$$\Psi(l_1, \cdots, l_N|\tau) = \sum_{\sigma \in \mathcal{S}_N} \mathcal{A}_{\tau|\sigma}(p_1, \cdots, p_N) e^{i\vec{p}_\sigma \cdot \vec{l}_\tau} \, , \tag{3}$$

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for the ordering $l_{\tau(1)} < \cdots < l_{\tau(N)}$.

If we combine the coefficients $\mathcal{A}_{\tau|\sigma}$ for the same τ in a column vector $\Phi(\tau)$, the Yang operator is defined as

$$\Phi(\alpha_j \tau) = Y_j(\alpha_j)\Phi(\tau) .$$

Furthermore, properties of permutations make it fulfil a Yang-Baxter-like equation

$$Y_j(p_2, p_3) Y_{j+1}(p_1, p_3) Y_j(p_1, p_2) = Y_{j+1}(p_1, p_2) Y_j(p_1, p_3) Y_{j+1}(p_2, p_3) .$$

Naive Yang operator

If we write the two-magnon solution as

$$\begin{pmatrix} \langle x_1, x_2 | \Psi(p_1, p_2) \rangle_{12} \\ \langle x_2, x_1 | \Psi(p_1, p_2) \rangle_{21} \end{pmatrix} = \begin{pmatrix} 1 & S_\kappa(p_1, p_2) \\ S_{1/\kappa}(p_1, p_2) & 1 \end{pmatrix} \begin{pmatrix} \exp(ip_1 x_1 + ip_2 x_2) \\ \exp(ip_2 x_1 + ip_1 x_2) \end{pmatrix} ,$$

we immediately see that the Yang operator has the form

$$Y_1(p_1, p_2) = \begin{pmatrix} S_\kappa(p_1, p_2) & 0 \\ 0 & S_{1/\kappa}(p_2, p_1) \end{pmatrix} .$$

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Now, if we consider the case of N magnons, and using the fact that a excitation Q_{12} always has to be followed by a Q_{21}, whose interaction is obtained by swapping $\kappa \leftrightarrow 1/\kappa$, we can check that

$$Y_{j+1}(p_1, p_2) Y_j(p_1, p_3) Y_{j+1}(p_2, p_3) \neq Y_j(p_2, p_3) Y_{j+1}(p_1, p_3) Y_j(p_1, p_2) .$$

Modified Yang operator

However, in our case we have to make some modifications (specially due to color contraction)

$$\begin{pmatrix} \langle x_1, x_2, x_3 | \Psi(p_1, p_2, p_3) \rangle_{12} \\ \langle x_3, x_2, x_1 | \Psi(p_1, p_2, p_3) \rangle_{12} \end{pmatrix} = (\Phi(123), \cdots, \Phi(321)) \begin{pmatrix} \exp(ip_1 x_1 + ip_2 x_3 + ip_3 x_3) \\ \cdots \\ \exp(ip_3 x_1 + ip_2 x_3 + ip_1 x_3) \end{pmatrix} ,$$

giving us

$$Y_j^{n,m}(p_{\tau(j)}, p_{\tau(j+1)}; p_k) = \begin{pmatrix} \frac{A_\sigma + D_\sigma^{n,m}}{A_{\alpha_j \sigma} + D_{\alpha_j \sigma}^{n,m}} & 0 \\ 0 & \frac{A_{\sigma^r} + D_{\sigma^r}^{m,n}}{A_{\alpha_{j+1}\sigma^r} + D_{\alpha_{j+1}\sigma^r}^{m,n}} \end{pmatrix} ,$$

$$Y_j^{n,m}(p_1, p_2; p_3) = \begin{pmatrix} S_\kappa(p_1, p_2) \frac{1 + f(p_2, p_1, p_3; n, m; \kappa)}{1 + f(p_1, p_2, p_3; n, m; \kappa)} & 0 \\ 0 & S_\kappa(p_2, p_1) \frac{1 + f(p_3, p_1, p_2; m, n; \kappa)}{1 + f(p_3, p_2, p_1; m, n; \kappa)} \end{pmatrix} ,$$

which fulfils an infinite tower of Yang-Baxter equations

$$Y_j^{n,m}(p_2, p_3; p_1) Y_{j+1}^{n,m}(p_1, p_3; p_2) Y_j^{n,m}(p_1, p_2; p_3) = Y_{j+1}^{n,m}(p_1, p_2; p_3) Y_j^{n,m}(p_1, p_3; p_2) Y_{j+1}^{n,m}(p_2, p_3; p_1) .$$

Outline

1. The orbifold theory
2. One, two and three excitations
3. Three-Body Long-Range Solution
4. A special solution
5. Yang operators
6. Conclusions

Summary

We found an explicit three-magnon solution of the spin chain associated to $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM that

- Has additive energy $E_3(p_1, p_2, p_3) = E_1(p_1) + E_1(p_2) + E_1(p_3)$.
- Is long-range due to the non-vanishing D's.
- Reduces to the usual CBA solution when $\kappa \to 1$.
- Its associated Yang operators fulfil an infinite tower of Yang-Baxter equations.

Future work

- Generalise from $\mathbb{Z}_2$ to $\mathbb{Z}_N$.
- Generalise to 4 magnons [nearly done].
- Understand the elliptic structure behind the model.
- Dual understanding?
- Plenty of integrability questions: what does the tower of YB equations means? R-matrix? Algebraic approach? Higher charges?...

Thanks for your attention!