On multipoint correlation functions in the Sinh-Gordon 1+1 dimensional quantum field theory

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Goals

Heuristic QFTs: Feynman's path integrals, Wilson's renormalization...
Rigorous constructions of euclidean correlation functions:

- Constructive renormalization: [Glimm, Jaffe, Spencer '70s...] φ^4_2, φ^4_3...
- Probabilistic approaches:
  - Stochastic quantization: [Gubinelli et al. 2020s] φ^4_3, 2d Sine-Gordon; [Hairer et al. 2020s] 3d Yang-Mills-Higgs theory, φ^4_3;
  - Gaussian multiplicative chaos [Vargas et al. 2024]: 2d Sinh-Gordon.

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  - Gaussian multiplicative chaos [Vargas et al. 2024]: 2d Sinh-Gordon.

Output $\to$ correlation functions:

$$\langle \Omega | \mathcal{O}_1(\boldsymbol{x}_1) \dots \mathcal{O}_k(\boldsymbol{x}_k) | \Omega \rangle = \mathbb{E}_\mu[o_1(\boldsymbol{x}_1) \dots o_k(\boldsymbol{x}_k)]$$

Measure μ:

- well-defined $\xi[\mu] = \mu$;
- not explicit, perturbative

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Bootstrap program in 1+1d [Karowski, Weisz '76, Smirnov et al. '80s-90s]:
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2-point functions as an explicit series, e.g. space-like Sinh-Gordon 1+1d:

$$\langle \Omega | \mathcal{O}(\boldsymbol{x}_1) \mathcal{O}^\dagger(\boldsymbol{x}_2) | \Omega \rangle = \sum_{N=0}^{+\infty} \frac{1}{N!} \int_{\mathbb{R}} \mathrm{d}^N \beta \prod_{a=1}^{N} \mathrm{e}^{-mr\cosh(\beta_a)} |\mathcal{F}_N(\beta_N)|^2 \ , \ r = \sqrt{-(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2}$$

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Multipoint correlation functions:

- must satisfy Wightman's axioms → well-defined QFT;
- only partial results for 3,4 points [Niedermaier et al '00, Caselle et al '06, Babujian et al '16].

Here, closed expressions for 1+1d Sinh-Gordon multipoint functions:

$$\langle \Omega | \mathcal{O}_1(\boldsymbol{x}_1) \dots \mathcal{O}_k(\boldsymbol{x}_k) | \Omega \rangle \ , \quad \boldsymbol{x}_i = (t_i, x_i) \in \mathbb{R}^{1,1} \ .$$

Outline

Sinh-Gordon model

Classical Sinh-Gordon equation

$$\left(\partial_t^2 - \partial_x^2\right)\varphi + \frac{m^2}{g}\sinh(g\varphi) = 0\ , \quad m, g > 0\ .$$

Quantum Sinh-Gordon model

- $\mathcal{H} = \bigoplus_{n=0}^{+\infty} L^2(\mathbb{R}^n_>)$, with $\mathbb{R}^n_> = \{\boldsymbol{\alpha}_n = (\alpha_1, \ldots, \alpha_n),\ \alpha_1 > \cdots > \alpha_n\}$;
- Vacuum state $\Omega = (1, 0, 0, \ldots)$;
- Scalar S-matrix [Gryanik, Vergeles '76]:

$$S(u) = \frac{\tanh(u/2 - \mathrm{i}\pi\mathfrak{b})}{\tanh(u/2 + \mathrm{i}\pi\mathfrak{b})}\ , \qquad \mathfrak{b} = \frac{1}{2}\cdot\frac{g^2}{8\pi + g^2}\ .$$

Bootstrap program

State parametrization:

$$\mathcal{H} = \bigoplus_{n=0}^{+\infty} L^2(\mathbb{R}^n_>) \ , \quad f = (f^{(0)}, \ldots, f^{(n)}, \ldots) \in \mathcal{H} \ .$$

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Action of local field $\mathcal{O}(\boldsymbol{x})$ = integral operators on $\mathcal{H}$:

$$\Big(\mathcal{O}(\boldsymbol{x}) \cdot f\Big)^{(n)}(\boldsymbol{\alpha}_n) = \sum_{m \geq 0} \mathrm{M}_{\mathcal{O}}^{(n;m)}(\boldsymbol{x} \mid \boldsymbol{\alpha}_n)\big[f^{(m)}\big] \,, \quad \boldsymbol{\alpha}_n = (\alpha_1, \dots, \alpha_n) \,,$$

$$\mathrm{M}_{\mathcal{O}}^{(n;m)}(\boldsymbol{x} \mid \boldsymbol{\alpha}_n)\big[f^{(m)}\big] = \int_{\mathbb{R}^m_>} \mathrm{d}^m\beta \, \mathcal{M}^{(\mathcal{O})}_{n;m}(\boldsymbol{\alpha}_n; \boldsymbol{\beta}_m) \cdot \mathrm{e}^{\mathrm{i}\,\left[\boldsymbol{p}(\boldsymbol{\alpha}_n) - \boldsymbol{p}(\boldsymbol{\beta}_m)\right] \cdot \boldsymbol{x}} \cdot f^{(m)}(\boldsymbol{\beta}_m) \,,$$

$$\boldsymbol{p}(\boldsymbol{\alpha}_n) = m \sum_{k=1}^{n} \big(\cosh \alpha_k, \sinh \alpha_k\big) \in \mathbb{R}^{1,1} \,.$$

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$$\mathtt{M}_{\mathcal{O}}^{(n;m)}(x \mid \boldsymbol{\alpha}_n)\big[f^{(m)}\big] = \int_{\mathbb{R}^m_>} \mathrm{d}^m\beta \, \mathcal{M}_{n;m}^{(\mathcal{O})}(\boldsymbol{\alpha}_n; \boldsymbol{\beta}_m) \cdot \mathrm{e}^{\mathrm{i}\,\big[\boldsymbol{p}(\boldsymbol{\alpha}_n) - \boldsymbol{p}(\boldsymbol{\beta}_m)\big] \cdot x} \cdot f^{(m)}(\boldsymbol{\beta}_m) ,$$

$$\boldsymbol{p}(\boldsymbol{\alpha}_n) = m \sum_{k=1}^{n} \big(\cosh \alpha_k, \sinh \alpha_k\big) \in \mathbb{R}^{1,1} .$$

→ Need to find the generalized functions $\mathcal{M}_{n;m}^{(\mathcal{O})}(\boldsymbol{\alpha}_n; \boldsymbol{\beta}_m) = "\langle \boldsymbol{\alpha}_n | \mathcal{O}(0) | \boldsymbol{\beta}_m \rangle"$.
First look at simple case $n = 0$ related to form factors:

$$\mathcal{M}_{0;m}^{(\mathcal{O})}(\emptyset; \boldsymbol{\beta}_m) = \mathcal{F}_{m,+}^{(\mathcal{O})}(\boldsymbol{\beta}_m) .$$

Bootstrap program

Form factor program

i) $\mathcal{F}_n^{(\mathcal{O})}(\boldsymbol{\beta}_n) \; = \; S(\beta_{a\,a+1}) \cdot \mathcal{F}_n^{(\mathcal{O})}(\boldsymbol{\beta}_n^{(a+1a)})$ where

$$\boldsymbol{\beta}_n^{(a+1a)} \; = \; (\beta_1, \dots, \beta_{a-1}, \beta_{a+1}, \beta_a, \beta_{a+2}, \dots, \beta_n) \;, \qquad \beta_{ab} = \beta_a - \beta_b \;,$$

ii) Let $\boldsymbol{\beta}'_n = (\beta_2, \dots, \beta_n)$ and $\boldsymbol{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$,

$$\mathcal{F}_n^{(\mathcal{O})}(\boldsymbol{\beta}_n + 2\mathrm{i}\pi \boldsymbol{e}_1) \; = \; \mathcal{F}_n^{(\mathcal{O})}(\boldsymbol{\beta}'_n, \beta_1) \; = \; \prod_{a=2}^{n} S(\beta_{a1}) \cdot \mathcal{F}_n^{(\mathcal{O})}(\boldsymbol{\beta}_n) \,,$$

iii) Pole structure:

$$-\mathrm{iRes}\Big(\mathcal{F}_{n+2}^{(\mathcal{O})}(\alpha + \mathrm{i}\pi, \beta, \boldsymbol{\beta}_n) \cdot \mathrm{d}\alpha \,, \; \alpha = \beta \Big) \; = \; \Big\{ 1 - \prod_{a=1}^{n} S(\beta - \beta_a) \Big\} \cdot \mathcal{F}_n^{(\mathcal{O})}(\boldsymbol{\beta}_n) \;,$$

iv) $\mathcal{F}_n^{(\mathcal{O})}(\boldsymbol{\beta}_n + \theta \overline{\boldsymbol{e}}_n) \; = \; \mathrm{e}^{\theta \mathrm{s}_{\mathcal{O}}} \cdot \mathcal{F}_n^{(\mathcal{O})}(\boldsymbol{\beta}_n)$ with $\mathrm{s}_{\mathcal{O}} \in \mathbb{R}$ and $\overline{\boldsymbol{e}}_n = (1, \dots, 1)$.

Bootstrap program

$n = 2$ case $\rightarrow F(\beta) = F(\beta_1 - \beta_2)$.

K-transform approach [Zamolodchikov '90, Lukyanov '95 & '97, Brazhnikov, Lukyanov '98, Babujian, Fring, Karowski, Zapletal '99 & '02]

The n-point form factor is:

$$\mathcal{F}_n^{(\mathcal{O})}(\boldsymbol{\beta}_n) = \prod_{a<b}^{n} \mathrm{F}(\beta_{ab}) \cdot \mathcal{K}_n\left[p_n^{(\mathcal{O})}\right](\boldsymbol{\beta}_n) ,$$

with:

$$\mathcal{K}_n\left[p_n^{(\mathcal{O})}\right](\boldsymbol{\beta}_n) = \sum_{\boldsymbol{\ell}_n \in \{0,1\}^n} (-1)^{\ell_1 + \cdots + \ell_n} \prod_{k<s}^{n} \left\{ 1 - \mathrm{i}\frac{\ell_{ks} \cdot \sin[2\pi \mathfrak{b}]}{\sinh(\beta_{ks})} \right\} \cdot p_n^{(\mathcal{O})}(\boldsymbol{\beta}_n \mid \boldsymbol{\ell}_n) .$$

Bootstrap for $\mathcal{F}_n^{(\mathcal{O})} \longleftrightarrow$ system of eq for the $p_n^{(\mathcal{O})}$.
Example: exponential of the field : $\mathrm{e}^{\mathrm{i}\gamma\phi}$:

$$p_n^{(\gamma)}(\boldsymbol{\beta}_n \mid \boldsymbol{\ell}_n) = K^n \prod_{a=1}^{n} \mathrm{e}^{\frac{2\mathrm{i}\pi \mathfrak{b} \gamma}{g}(-1)^{l_a}} , \quad K \in \mathbb{C} .$$

Operator content

General kernels → 5th axiom:

$$\mathcal{M}^{(\mathcal{O})}_{n;m}(\boldsymbol{\alpha}_n;\boldsymbol{\beta}_m)=\mathcal{M}^{(\mathcal{O})}_{n-1;m+1}(\boldsymbol{\alpha}'_n;(\alpha_1+\mathrm{i}\pi,\boldsymbol{\beta}_m))+2\pi\sum_{a=1}^{m}\delta(\alpha_1-\beta_a)\prod_{k=1}^{a-1}\mathrm{S}(\beta_{ka})\cdot\mathcal{M}^{(\mathcal{O})}_{n-1;m-1}(\boldsymbol{\alpha}'_n;\widehat{\boldsymbol{\beta}_m}^{(a)})$$

with $\boldsymbol{\alpha}'_n=(\alpha_2,\ldots,\alpha_n)$; $\widehat{\boldsymbol{\beta}_m}^{(a)}=(\beta_1,\ldots,\beta_{a-1},\beta_{a+1},\ldots,\beta_m)$.

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General kernels $\rightarrow$ 5th axiom:

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with $\boldsymbol{\alpha}'_n = (\alpha_2,\dots,\alpha_n)$; $\widehat{\boldsymbol{\beta}_m}^{(a)} = (\beta_1,\dots,\beta_{a-1},\beta_{a+1},\dots,\beta_m)$.
Recall initialization $\mathcal{M}^{(\mathcal{O})}_{0;m}(\emptyset;\boldsymbol{\beta}_m) = \mathcal{F}^{(\mathcal{O})}_{m,+}(\boldsymbol{\beta}_m)$.
$\rightarrow$ Resolution: $\mathcal{M}^{(\mathcal{O})}_{n;m}(\boldsymbol{\alpha}_n;\boldsymbol{\beta}_m)$ = combinatorial sum of form factors [Kirillov, Smirnov '87];
$\rightarrow$ explicit expression of $\mathcal{O}(\boldsymbol{x})$ as a distribution-valued operator;
$\rightarrow$ possible computation of the correlation function!

Some notations

Local operator: technical issues → introduce smeared operators $\mathcal{O}[g]$:

$$\mathcal{O}[g] = \int_{\mathbb{R}^{1,1}} \mathrm{d}^2\boldsymbol{x} g(\boldsymbol{x})\mathcal{O}(\boldsymbol{x}) \, .$$

Avoid the series problem → truncature parameter $r \in \mathbb{N}$ and truncated operator:

$$\mathcal{O}^{(r)}[g] = \pi_r \circ \mathcal{O}[g]$$

with π_r the canonical projection $\mathcal{H} \mapsto L^2(\mathbb{R}^r_>)$.

Correlation functions

Theorem (Kozlowski, Potaux, S. 2024)

Let $g_1, \dots, g_k$ be smooth, compactly supported. The truncated k-point correlation is an explicit sum:

$$\langle\, \Omega \,| \mathcal{O}_1^{(0)}[g_1] \mathcal{O}_2^{(r_1)}[g_2] \cdots \mathcal{O}_k^{(r_{k-1})}[g_k] |\, \Omega \,\rangle \;=\; \sum_{\boldsymbol{n} \in \mathcal{N}_{\boldsymbol{r}}} \frac{1}{\boldsymbol{n}!} \cdot \mathcal{I}_{\boldsymbol{n}}\big[g_1, \dots, g_k\big] \;,$$

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with the general term:

$$\mathcal{I}_{\boldsymbol{n}}\big[g_1, \ldots, g_k\big] = \prod_{s=1}^{k} \int_{\mathbb{R}^{1,1}} \mathrm{d}\boldsymbol{x}_s \prod_{b>a}^{k} \int_{\mathbb{R}^{n_{ba}}} \mathrm{d}^{n_{ba}}\gamma^{(ba)}\, \mathcal{P}_k[\{\boldsymbol{x}_i, \partial_{\boldsymbol{x}_i}\}; \gamma]\Big(g_1(\boldsymbol{x}_1), \ldots, g_k(\boldsymbol{x}_k)\Big) .$$

Here:

- $\mathcal{P}_k$ *is a differential polynomial with explicit coefficients;*
- $\boldsymbol{n} = (n_{21}, n_{31}, n_{32}, \ldots, n_{kk-1})$ *is a* $\frac{k(k-1)}{2}$ *multi-index of summation;*
- $\boldsymbol{n}! = \prod_{b>a} n_{ba}$*;*
- $\mathcal{N}_{\boldsymbol{r}}$ *is an explicit set of constrains imposed on* $\boldsymbol{n}$*.*

Correlation functions

In some cases: distribution $\longrightarrow$ function

Theorem (Kozlowski, Potaux, S. 2024)

Let $g_1, \dots, g_k$ be smooth, compactly supported such that, for $a < b$,

$$\boldsymbol{x}_{ba}^2 < 0\,, \qquad x_{a;1} > x_{b;1} \qquad \text{for any} \quad \boldsymbol{x}_a = (x_{a;0}, x_{a;1}) \in \mathrm{supp}[g_a]\,.$$

then the truncated k-point function takes the explicit form:

$$\langle\, \Omega \,|\mathcal{O}_1^{(0)}[g_1]\mathcal{O}_2^{(r_1)}[g_2]\cdots\mathcal{O}_k^{(r_{k-1})}[g_k]|\, \Omega\,\rangle = \int_{(\mathbb{R}^{1,1})^k} \prod_{a=1}^{k} \mathrm{d}\boldsymbol{x}_a \cdot g_a(\boldsymbol{x}_a)\cdot \mathcal{W}_{\boldsymbol{r}}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_k)$$

with truncature parameters $\boldsymbol{r} = (r_1, \dots, r_{k-1})$, and $\mathcal{W}_{\boldsymbol{r}}$ is an explicit function.

Remark: One can understand that in some sense

$$\mathcal{W}_{\boldsymbol{r}}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_k) = \langle\, \Omega \,|\mathcal{O}_1^{(0)}(\boldsymbol{x}_1)\mathcal{O}_2^{(r_1)}(\boldsymbol{x}_2)\dots\mathcal{O}_k^{(r_{k-1})}(\boldsymbol{x}_k)|\, \Omega\,\rangle$$

Vectorial notations

Let $\boldsymbol{\alpha}_n = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and $\boldsymbol{\beta}_m = (\beta_1, \ldots, \beta_m) \in \mathbb{C}^m$. Denote:

- reverse $\overleftarrow{\boldsymbol{\alpha}_n} = (\alpha_n, \ldots, \alpha_1)$;
- $\overline{\boldsymbol{e}} = (1, \ldots, 1)$ with "fitting size", i.e. $\boldsymbol{\alpha}_n + \mathrm{i}\pi\overline{\boldsymbol{e}} = (\alpha_1 + \mathrm{i}\pi, \ldots, \alpha_n + \mathrm{i}\pi)$;
- concatenate $\boldsymbol{\alpha}_n \cup \boldsymbol{\beta}_m = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$;
- S-matrix for vectors:

$$\mathcal{F}(\boldsymbol{\alpha}_n \cup \boldsymbol{\beta}_m) = S(\boldsymbol{\alpha}_n \cup \boldsymbol{\beta}_m | \boldsymbol{\beta}_m \cup \boldsymbol{\alpha}_n) \cdot \mathcal{F}(\boldsymbol{\beta}_m \cup \boldsymbol{\alpha}_n)$$

Correlation functions

General summand

$$\mathcal{W}_{\boldsymbol{r}}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_k) = \sum_{\boldsymbol{n}\in\mathcal{N}_{\boldsymbol{r}}} \frac{1}{\boldsymbol{n}!}\cdot\prod_{b>a}^{k} \int\limits_{\left\{\mathbb{R}+\mathrm{i}\eta^{(ba)}\right\}^{n_{ba}}} \mathrm{d}^{n_{ba}}\boldsymbol{\gamma}^{(ba)} \cdot \prod_{b>a}^{k} \mathrm{e}^{\mathrm{i}\boldsymbol{p}(\boldsymbol{\gamma}^{(ba)})\cdot\boldsymbol{x}_{ba}}$$

$$\times\, \mathcal{T}(\boldsymbol{\gamma})\cdot\prod_{p=1}^{k}\mathcal{F}^{(\mathcal{O}_p)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(pp-1)}}\cup\dots\cup\overleftarrow{\boldsymbol{\gamma}^{(p1)}}+\mathrm{i}\pi\overline{\boldsymbol{e}},\boldsymbol{\gamma}^{(kp)}\cup\dots\cup\boldsymbol{\gamma}^{(p+1p)}\Big)\,.$$

Here $0<\eta^{(21)}<\eta^{(31)}<\dots<\eta^{(kk-1)}$ are small deformation parameters and:

$$\mathcal{T}(\boldsymbol{\gamma}) = \prod_{\substack{v>p\\ p\geq 3}}^{k-1}\prod_{u>s}^{p-1} S\big(\boldsymbol{\gamma}^{(vu)}\cup\boldsymbol{\gamma}^{(ps)}\,|\,\boldsymbol{\gamma}^{(ps)}\cup\boldsymbol{\gamma}^{(vu)}\big)\,.$$

Remark: the deformation parameters allow $\mathrm{e}^{-\cosh\gamma_p^{(ba)}}$ decays, ensuring the convergence of the integrals.

Local commutativity

Assuming absolute convergence,

$$\left\langle \Omega \left| \mathcal{O}_1[g_1] \cdots \mathcal{O}_k[g_k] \right| \Omega \right\rangle = \sum_{\boldsymbol{n} \in \mathbb{N}^{\frac{k(k-1)}{2}}} \frac{1}{\boldsymbol{n}!} \cdot \mathcal{I}_{\boldsymbol{n}}\left[g_1, \ldots, g_k\right] .$$

Most difficult Wightman's axioms: local commutativity.

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Most difficult Wightman's axioms: local commutativity.
Set $\boldsymbol{x}_i$ and $\boldsymbol{x}_{i+1}$ s.t. $(\boldsymbol{x}_i - \boldsymbol{x}_{i+1})^2 < 0$ and g_i, g_{i+1} test functions separating these points. Following commutational heuristics from [Kirillov, Smirnov '87, Khamitov '88], one more generally can prove:

$$\langle \Omega | \mathcal{O}_1[g_1] \cdots \Big[\mathcal{O}_i[g_i], \mathcal{O}_{i+1}[g_{i+1}]\Big] \ldots \mathcal{O}_k[g_k] | \Omega \rangle = 0 \, .$$

$\longrightarrow$ Local commutativity!

Conclusion and open problems

Explicit expressions for k-point functions in 1+1d Sinh-Gordon:

- as distributions in the general case;
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Explicit expressions for k-point functions in 1+1d Sinh-Gordon:

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Open problems:

- can one represent the distribution as a function in all cases?
- convergence of the series?
- apply this approach to Sine-Gordon?

Additional expressions

Local commutativity result

$$\left\langle \Omega \left| \mathcal{O}_1[g_1] \cdots \mathcal{O}_{i+1}[g_{i+1}] \mathcal{O}_i[g_i] \ldots \mathcal{O}_k[g_k] \right| \Omega \right\rangle = \sum_{\boldsymbol{n} \in \mathbb{N}^{\frac{k(k-1)}{2}}} \frac{1}{\boldsymbol{n}!} \cdot \mathcal{I}_{\sigma(\boldsymbol{n})}\left[g_1, \ldots, g_k\right] \ ,$$

with $\sigma(\boldsymbol{n})$ = fixed permutation of $\boldsymbol{n} = (n_{21}, \ldots, n_{kk-1})$

$$\begin{aligned} n_{ij} &\longleftrightarrow n_{i+1j} \ , \quad j < i \\ n_{ji} &\longleftrightarrow n_{ji+1} \ , \quad j > i+1 \end{aligned}$$

Additional expressions

Matrix elements [Kirillov, Smirnov '87]

Let $\boldsymbol{\alpha}_n \in \mathbb{R}^n$ and $\boldsymbol{\beta}_m \in \mathbb{R}^m$ and $A = \{\alpha_a\}_1^n$, $B = \{\beta_a\}_1^m$. The general matrix element is:

$$\mathcal{M}_{n;m}^{(\mathcal{O})}(\boldsymbol{\alpha}_n; \boldsymbol{\beta}_m) = \sum_{A=A_1 \cup A_2} \sum_{B=B_1 \underset{1}{\cup} B_2} \Delta(\boldsymbol{A}_1 \mid \boldsymbol{B}_1) \cdot S\left(\overleftarrow{\boldsymbol{A}} \mid \overleftarrow{\boldsymbol{A}}_2 \cup \overleftarrow{\boldsymbol{A}}_1\right) \cdot S(\boldsymbol{B} \mid \boldsymbol{B}_1 \cup \boldsymbol{B}_2) \times \mathcal{F}_{|A_2|+|B_2|}^{(\mathcal{O})}\left(\overleftarrow{\boldsymbol{A}}_2 + \mathrm{i}\pi \overline{\boldsymbol{e}}_{|A_2|}; \boldsymbol{B}_2\right) ,$$

where the sum is under the constrain $|A_1| = |B_1|$ while B_1 is not ordered;

$$\Delta(\boldsymbol{A}_1 \mid \boldsymbol{B}_1) = \prod_{k=1}^{|A_1|} 2\pi\delta(A_{1,k} - B_{1,k}) ;$$

and the S functions defined on a vector $\boldsymbol{u}$ and a permutation $\boldsymbol{u}_\sigma$ of its variables:

$$\mathcal{F}_{|\boldsymbol{u}|}^{(\mathcal{O})}(\boldsymbol{u}) = S(\boldsymbol{u} \mid \boldsymbol{u}_\sigma) \cdot \mathcal{F}_{|\boldsymbol{u}|}^{(\mathcal{O})}(\boldsymbol{u}_\sigma) .$$