Integrability of the Inozemtsev spin chain and its generalisation

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Problem

The Inozemtsev quantum spin chain is described by the Hamiltonian (Inozemtsev 1990):

$$\mathscr{H} = \sum_{i<j}^{n} \wp\left(\frac{i-j}{n}\right)(P_{ij} - 1)$$

Here $\wp(z) = \wp(z|1,\tau)$ is the Weierstrass $\wp$-function, and P_{ij} acts by permuting factors of $U^{\otimes n}$ with $U = \mathbb{C}^m$.

Goal: To find elements of $\mathtt{End}_{\mathbb{C}}(U^{\otimes n})$ commuting with $\mathscr{H}$.

Inozemtsev 1990: found $\mathscr{I}_1$, $\mathscr{I}_2$ with $[\mathscr{H}, \mathscr{I}_1] = [\mathscr{H}, \mathscr{I}_2] = 0$.

The fact that $[\mathscr{I}_1, \mathscr{I}_2] = 0$ was proved in [Dittrich–Inozemtsev 2008] by a very long calculation.

Also, in [Inozemtsev 2002] an infnite family $\{\mathscr{I}_k\}$ was constructed, with $[\mathscr{H}, \mathscr{I}_k] = 0$. It is still unknown whether $[\mathscr{I}_k, \mathscr{I}_l] = 0$.

Hints at integrability

- Trigonometric version: $\wp \mapsto \sin^{-2}$,

$$\mathscr{H} = \sum_{i<j}^{n} \frac{1}{\sin^2(x_i - x_j)}(P_{ij} - 1), \qquad x_i = \frac{\pi i}{n}$$

  Haldane–Shastry spin chain (Haldane 1988, Shastry 1988)

- Its integrability linked to quantum groups (Bernard, Gaudin, Haldane, Pasquier 1993).
  Key role: spin quantum Calogero–Moser–Sutherland system

$$\widehat{H} = \sum_{i=1}^{n} \hbar^2 \frac{\partial^2}{\partial x_i^2} - \sum_{i<j}^{n} \frac{2\kappa(\kappa - \hbar P_{ij})}{\sin^2(x_i - x_j)}$$

- Polychronakos 1992: "freezing trick"
  He argued that $\mathscr{H}$ is the "limit" of $\widehat{H}$, when $\kappa \to \infty$. In this limit, particles x_i tend to classical equilibrium positions ("freeze"). It wasn't clear why this would work for higher order Hamiltonians.

Hints at integrability

- For "non-spin" CMS system,

$$H = \sum_{i=1}^{n} \hbar^2 \frac{\partial^2}{\partial x_i^2} - 2 \sum_{i<j}^{n} \frac{\kappa(\kappa - \hbar)}{\sin^2(x_i - x_j)},$$

  its integrability is best understood in terms of Dunkl operators (Dunkl 1989, Heckman 1990, Cherednik 1990). They were used in [BGHP93] to see Yangian symmetry in the spin CMS system. However, even in this case

  (1) it wasn't clear how to use Dunkl operators to implement the freezing trick

  (2) it wasn't known how to construct additional Hamiltonians for spin quantum CMS model

Even less was known in the elliptic case.

Main results

Our results are:

(a) a construction of extra spin Hamiltonians in terms of elliptic Dunkl operators,

(b) a full justification of the freezing trick $\Rightarrow$ integrability of the Inozemtsev chain,

(c) we find an integrable defrormation of the Inozemtsev spin chain.

Elliptic CM system and Dunkl operators

Elliptic CM Hamiltonians are

$$H = \sum_{i=1}^{n} \hbar^2 \frac{\partial^2}{\partial x_i^2} + 2\sum_{i<j}^{n} \kappa(\kappa - \hbar)\wp(x_i - x_j) \qquad \text{(no spins)}$$

$$\widehat{H} = \sum_{i=1}^{n} \hbar^2 \frac{\partial^2}{\partial x_i^2} + 2\sum_{i<j}^{n} \wp(x_i - x_j)\kappa(\kappa - \hbar P_{ij}) \qquad \text{(spin version)}$$

Elliptic Dunkl operators are (Buchstaber–Felder–Veselov 1994)

$$y_i = \hbar\frac{\partial}{\partial x_i} - \kappa \sum_{j\neq i}^{n} \phi_{\lambda_i - \lambda_j}(x_i - x_j) s_{ij}\,, \qquad \phi_\mu(z) = \frac{\sigma(z-\mu)}{\sigma(z)\sigma(-\mu)}$$

$\sigma(z) = \sigma(z|1,\tau)$ is Weierstrass σ,
$\lambda = (\lambda_1, \ldots, \lambda_n)$ are auxiliary “spectral variables”.

Elliptic CM system and Dunkl operators

Elliptic Dunkl operators $y_i = y_i(\lambda)$ are viewed as λ-dependent elements of $\mathscr{D} * S_n$. The algebra $\mathscr{D} * S_n$ is defined as follows:

- $\mathscr{D}$ is the algebra of partial differential operators in n variables $x_1, \ldots, x_n$ (with meromorphic coefficients)
- S_n acts on $\mathscr{D}$ in a natrual way, $(w.f)(x) = f(w^{-1}x)$, $w.\partial_\xi = \partial_{w\xi} \quad \forall \xi \in \mathbb{C}^n$ (e.g. $s_{ij}.\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_j}$)
- Elements of $\mathscr{D} * S_n$ are written as $a = \sum_{w \in S_n} a_w w$ with $a_w \in \mathscr{D}$
- Multiplication is by $(a_1 w_1)(a_2 w_2) = a_1 (w_1.a_2) w_1 w_2$

The elements of $\mathscr{D} * W$ act on functions of n variables in the obvious way.

Elliptic CM system and Dunkl operators

$$y_i(\lambda) = \hbar \frac{\partial}{\partial x_i} - \kappa \sum_{j \neq i}^{n} \phi_{\lambda_{ij}}(x_{ij}) s_{ij} , \quad \lambda_{ij} = \lambda_i - \lambda_j, \quad x_{ij} = x_i - x_j .$$

Two main properties are commutativity and equivariance:

$$[y_i, y_j] = 0, \qquad w y_\xi(\lambda) w^{-1} = y_{w\xi}(w\lambda)$$

Here $y_\xi := \sum_i \xi_i y_i$ for $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$.
Following [Etingof–Felder–Ma–Veselov 2011], one can construct commuting (non-spin) CM Hamiltonians from y's. For this,
(1) take suitable symmetric combinations of y's,
(2) set $\lambda = 0$, and
(3) restrict the result onto $\mathbb{C}(x)^{S_n}$ to get differential operators.

Example 1: quadratic Hamiltonian

$$y_1^2+\cdots+y_n^2 = \hbar^2 \sum_{i=1}^{n} \partial_i^2 - 2\hbar\kappa \sum_{i<j}^{n} \phi'_{\lambda_{ij}}(x_{ij}) s_{ij} + 2\kappa^2 \sum_{i<j}^{n} \left(\wp(\lambda_{ij}) - \wp(x_{ij})\right)$$

Here $\phi'_\mu(z) = \frac{d}{dz}\phi_\mu(z) \xrightarrow{\lambda\to 0} -\wp(z)$. To regularize at $\lambda = 0$, take

$y_1^2 + \cdots + y_n^2 - 2\kappa^2 \sum_{i<j}^{n} \wp(\lambda_{ij})$, written as $H_2 + \hbar A_2$ with

$$H_2 = \sum_{i=1}^{n} \hat{p}_i^2 - 2\sum_{i<j}^{n} \kappa(\kappa - \hbar)\wp(x_{ij}), \qquad \hat{p}_i := \hbar \frac{\partial}{\partial x_i}$$

$$A_2 = -2\kappa \sum_{i<j}^{n} \left[\phi'_{\lambda_{ij}}(x_{ij}) s_{ij} + \wp(x_{ij})\right] \xrightarrow{\lambda\to 0} 2\kappa \sum_{i<j} \wp(x_{ij})(s_{ij} - 1)$$

Upon restriction onto $\mathbb{C}(x)^{S_n}$, last term disappears, so we get H_2.

Example 2: cubic Hamiltonian

Here we start with $\sum_{i<j<k} y_i y_j y_k$. The regularised combination is

$$\sum_{i<j<k} y_i y_j y_k + \kappa^2 \sum_{i\neq j\neq k} \wp(\lambda_{ij}) y_k$$

Substituting

$$y_i(\lambda) = \hat{p}_i - \kappa \sum_{j\neq i}^{n} \phi_{\lambda_{ij}}(x_{ij}) s_{ij}\,, \quad \hat{p}_i := \hbar \frac{\partial}{\partial x_i}\,,$$

we get $H_3 + \hbar A_3$ where

$$\begin{aligned} H_3 &= \sum_{i<j<k} \hat{p}_i \hat{p}_j \hat{p}_k + \sum_{i\neq j\neq k} \kappa(\kappa - \hbar)\wp(x_{ij})\hat{p}_k \\ A_3 &= \kappa \sum_{i\neq j\neq k} \left[\phi'_{\lambda_{ij}}(x_{ij}) s_{ij} + \wp(x_{ij})\right] \hat{p}_k \\ &\quad - \kappa^2 \sum_{i\neq j\neq k} (\phi'_{\lambda_{ij}}(x_{ij})\phi_{\lambda_{ki}}(x_{kj}) + \phi'_{\lambda_{ij}}(x_{ik})\phi_{\lambda_{jk}}(x_{jk})) s_{ijk}\,. \end{aligned}$$

Here s_{ijk} denotes the cyclic permutation.

Example 2: cubic Hamiltonian

After taking $\lambda \to 0$, A_3 becomes $A_3 = \sum_{i<j<k} A_{ijk}$ with

$$
\begin{aligned}
A_{ijk} &= \kappa \wp_{ij} \hat{p}_k (1 - s_{ij}) + \kappa \wp_{ik} \hat{p}_j (1 - s_{ik}) + \kappa \wp_{jk} \hat{p}_i (1 - s_{jk}) \\
&\quad + \kappa^2 \left\{ (\wp_{ij} + \wp_{jk} + \wp_{ki})(\zeta_{ij} + \zeta_{jk} + \zeta_{ki}) + \frac{1}{2}(\wp'_{ij} + \wp'_{jk} + \wp'_{ki}) \right\} s_{ijk} \\
&\quad + \kappa^2 \left\{ (\wp_{kj} + \wp_{ji} + \wp_{ik})(\zeta_{kj} + \zeta_{ji} + \zeta_{ik}) + \frac{1}{2}(\wp'_{kj} + \wp'_{ji} + \wp'_{ik}) \right\} s_{kji} \,.
\end{aligned}
$$

Here ζ is the Weierstrass ζ-function, $\zeta_{ij} := \zeta(x_{ij})$, $\wp_{ij} := \wp(x_{ij})$, etc. Upon restriction onto $\mathbb{C}(x)^{S_n}$, each A_{ijk} vanishes so $H_3 + \hbar A_3$ reduces to H_3.

Bosonic restriction

Applying this method to other symmmetric combinations of y's, with suitable regularisation, we obtain commuting scalar Hamiltonians $H_1 = \hat{p}_1 + \cdots + \hat{p}_n$, $H_2, \ldots, H_n$. A similar procedure can be used to construct their spin generalisation.

In previous examples H_2, H_3 were obtained by restriction onto $\mathbb{C}(x)^{S_n}$. This is equivalent to applying the map

$$\mathtt{Res}: \mathscr{D}(V) * S_n \to \mathscr{D}(V), \qquad \sum_{w \in S_n} a_w w \mapsto \sum_{w \in S_n} a_w .$$

Spin Hamiltonians are obtained by applying another map,

$$\widehat{\mathtt{Res}}: \mathscr{D}(V) * S_n \to \mathscr{D}(V) \otimes \mathbb{C}S_n, \qquad \sum_{w \in S_n} a_w w \mapsto \sum_{w \in S_n} a_w \otimes w^{-1} .$$

This can be interpreted as restricting to $\mathbb{C}S_n$-valued functions $f(x)$ such that $f(wx) = wf(x)$.

Principal spin Hamiltonians

This produces n commuting spin CM Hamiltonians $\widehat{H}_1, \ldots, \widehat{H}_n \in \mathscr{D} \otimes \mathbb{C}S_n$ ("principal Hamiltonians"). E.g.,

$$\widehat{H}_2 = \sum_{i=1}^{n} \hat{p}_i^2 - 2\sum_{i<j}^{n} \wp(x_{ij}) \otimes \kappa(\kappa - \hbar s_{ij}),$$

$$\begin{aligned}\widehat{H}_3 = \sum_{i<j<k} \hat{p}_i \hat{p}_j \hat{p}_k + \sum_{i\neq j\neq k} \wp(x_{ij})\hat{p}_k \otimes \kappa(\kappa - \hbar s_{ij}) \\ + \hbar\kappa^2(\wp_{ij} + \wp_{jk} + \wp_{ki})(\zeta_{ij} + \zeta_{jk} + \zeta_{ki}) \otimes (s_{jik} - s_{ijk}) \\ + \frac{1}{2}\hbar\kappa^2(\wp'_{ij} + \wp'_{jk} + \wp'_{ki}) \otimes (s_{jik} - s_{ijk}).\end{aligned}$$

Choosing as a representation of S_n a space $U^{\otimes n}$ with s_{ij} acting by permuting tensor factors, we see that $\widehat{H}_2$ is the same as the spin CM Hamiltonian $\widehat{H}$ introduced earlier.

Since the spin system has more degrees of freedom, one expects additional conserved quantities.

Additional spin Hamiltonians

The principal spin Hamiltonians were obtained by taking (regularised) symmmetric combinations of y's.

Extra commuting Hamiltonians can be constructed by using other kinds of symmetric combinations. The allowed combinations are functions $f(\lambda, y) \in \mathbb{C}(y, \lambda)^{S_n}$ that belong to the *classical rational spherical Cherednik algebra*, B_κ. The algebra B_κ is a rational limit of the *spherical DAHA* of type GL_n (with $q = 1$).

Proposition: For any $f \in B_\kappa$, $f(0, y) := \lim_{\lambda \to 0} f(\lambda, y)$ exists.

Define additional spin Hamiltonians $\widehat{I_f}$ by

$$\widehat{I_f} := \widehat{\mathtt{Res}}(f(0, y)), \qquad f \in B_\kappa.$$

By construction, $[\widehat{I_f}, \widehat{I_g}] = 0$ and $[\widehat{I_f}, \widehat{H_i}] = 0$.

Example 3: an extra Hamiltonian

As an example, let us consider the following combination:

$$f(\lambda, y) = \sum_{i\neq j\neq k} \lambda_i \left(y_j y_k + \frac{\kappa^2}{\lambda_{jk}^2} \right) \in B_\kappa \,.$$

Once we know that $f(\lambda, y)$ has a limit as $\lambda \to 0$, it is easy to calculate that $\widehat{I_f} =: \widehat{\mathtt{Res}}(f(0, y))$ has the form

$$\widehat{I_f} = -\,\kappa \sum_{i\neq j\neq k} \hat{p}_i \otimes s_{jk} \\ + \kappa^2 \sum_{i<j<k} \{\zeta(x_{ij}) + \zeta(x_{jk}) + \zeta(x_{ki})\} \otimes (s_{ijk} - s_{kji})$$

This coincides with the operator I_1 in [Dittrich-Inozemtsev 2009].

Commuting Hamiltonians for the Inozemtsev chain

For the Inozemtsevf chain, there will also be two types of commuting Hamiltonians, principal and additional.

Proposition ([C. 2019]); The principal spin CM Hamiltonians $\widehat{H}_i$, $i = 1, \dots, n$, as elements of $\mathscr{D} \otimes \mathbb{C}S_n$, admit decomposition $\widehat{H}_i = H_i \otimes 1 + \hbar \widehat{A}_i$ where H_i is the scalar (i.e. non-spin) quantum CM Hamiltonian.

Definition 1. Define principal spin-chain Hamiltonians $\mathscr{H}_i \in \mathbb{C}S_n$ by taking classical limit of $\widehat{A}_i$ (i.e. $x \mapsto x$, $\hat{p} \mapsto p$, $\hbar \mapsto 0$) followed by evaluation at $(x, p) = (x^*, p^*)$ with $p^* = 0$ and $(x^*)_i = i/n$, $i = 1, \dots, n$.

Definition 2. Define additional spin-chain Hamiltonians $\mathscr{I}_f \in \mathbb{C}S_n$, $f \in B_\kappa$ by taking classical limit of $\widehat{I}_f$ followed by evaluation at $(x, p) = (x^*, p^*)$.

Theorem. The spin-chain Hamiltonians $\mathscr{H}_i$, $i = 1, \dots, n$ and $\mathscr{I}_f$, $f \in B_\kappa$ pairwise commute as elements of $\mathbb{C}S_n$..

Proof of the theorem

1. The point (x^*, p^*) is a joint equilibrium point of the principal scalar classical CM Hamiltonians (in the "centre-of-mass" frame).
2. By Proposition, we have $\widehat{H}_i = H_i \otimes 1 + \hbar \widehat{A}_i$, for each principal spin CM Hamiltonian $\widehat{H}_i$. By $[\widehat{H}_i, \widehat{H}_j] = 0$,

   $$0 = [H_i + \hbar\widehat{A}_i, H_j + \hbar\widehat{A}_j] \overset{[H_i, H_j]=0}{=} \hbar[\widehat{A}_i, H_j] + \hbar[H_i, \widehat{A}_j] + \hbar^2[\widehat{A}_i, \widehat{A}_j].$$

   Picking terms of order $\hbar^2$ and evaluating at the equilibrium gives $[\widehat{A}_i^c, \widehat{A}_j^c] = 0$, i.e. $[\mathscr{H}_i, \mathscr{H}_j] = 0$.
3. From $0 = [\widehat{H}_i, \widehat{I}_f] = [H_i + \hbar\widehat{A}_i, \widehat{I}_f]$ we get, by picking terms of order $\hbar$, that $[\widehat{A}_i^c, \widehat{I}_f^c] = 0$, i.e. $[\mathscr{H}_i, \mathscr{I}_f] = 0$.
4. Finally, $[\widehat{I}_f, \widehat{I}_g] = 0$ implies $[\widehat{I}_f^c, \widehat{I}_g^c] = 0$, even without evaluation at (x^*, p^*). Hence, $[\mathscr{I}_f, \mathscr{I}_g] = 0$. $\square$

Some examples

Looking back at the calculations done in Examples 1-3, we get some explicit expressions.

1. From $\widehat{H}_2$, we get

$$\mathscr{H}_2 = \sum_{i<j} \wp_{ij}(s_{ij} - 1)\,, \quad \wp_{ij} := \wp\left(\frac{i-j}{n}\right)$$

2. From $\widehat{H}_3$, we get

$$\begin{aligned}\mathscr{H}_3 =& (\wp_{ij} + \wp_{jk} + \wp_{ki})(\zeta_{ij} + \zeta_{jk} + \zeta_{ki})(s_{jik} - s_{ijk}) \\ &+ \frac{1}{2}(\wp'_{ij} + \wp'_{jk} + \wp'_{ki})(s_{jik} - s_{ijk})\end{aligned}$$

3. From $\widehat{I}_f$ we get

$$\mathscr{I}_f = \sum_{i<j<k} \{\zeta_{ij} + \zeta_{jk} + \zeta_{ki}\}\,(s_{ijk} - s_{kji})\,.$$

These are $\mathscr{H}, \mathscr{I}_1, \mathscr{I}_2$ from [Inozemtsev 1990].

Integrable deformation

We can modify the above constructions to avoid taking $\lambda \to 0$.
Write $\mathscr{D} = \mathscr{D}_\lambda$ to emphasize that differentail operators are allowed to depend on the aixiliary spectral variables $\lambda = (\lambda_1, \dots, \lambda_n)$.
We have two S_n-actions on $\mathscr{D}_\lambda$: one on $x_i, \partial/\partial x_i$ (as before), one on λ_i. Hence, we have two crossed products, denoted $\mathscr{D}_\lambda * S_n$ and $\mathscr{D}_\lambda * S_n^\vee$, respectively.
We can now repeat all the above constructions, replacing the bosonic restriction with the map

$$\mathtt{Res}^\vee : \mathscr{D}_\lambda * S_n \to \mathscr{D}_\lambda * S_n^\vee , \qquad \sum_{w \in S_n} a_w w \mapsto \sum_{w \in S_n} a_w (w^{-1})^\vee .$$

This leads to the deformed spin/spin-chain Hamiltonians,

$$H_q^\vee = \sum_{i=1}^n \hat{p}^2 - 2\kappa^2 \sum_{i<j}^n \wp(x_{ij}) - 2\hbar\kappa \sum_{i<j}^n \phi'_{\lambda_{ij}}(x_{ij}) s_{ij}^\vee \in \mathscr{D}_\lambda * S_n^\vee ,$$

$$\mathscr{H}_q^\vee = \sum_{i<j}^n \phi'_{\lambda_{ij}} \left(\frac{i-j}{n} \right) s_{ij}^\vee \in \mathbb{C}(\lambda) * S_n^\vee .$$

Interpretation of the deformed spin chain

A concrete realisation, as a spin chain model, of

$$\mathscr{H}_q^\vee = \sum_{i<j}^{n} \phi'_{\lambda_{ij}} \left(\frac{i-j}{n} \right) s_{ij}^\vee \in \mathbb{C}(\lambda) * S_n^\vee$$

depends on a choice of a $\mathbb{C}(\lambda) * S_n^\vee$-module.

Here is one possible interpretation (for another, see [C. 2024]).

Consider a sufficiently small S_n-invariant neighbourhood V of $\lambda = 0$, and denote by $\mathcal{O}$ the algebra of holomorphic functions on V, viewed as a $S_n^\vee$-module. We will view $\mathscr{H}^\vee$ as an element of $\mathcal{O} * S_n^\vee$, and will make a choice of a $\mathcal{O} * S_n^\vee$-module $\mathscr{U}$.

Hilbert space $\mathscr{U}$

Choose a finite $S_n^\vee$-orbit $\Sigma \subset V$, and denote by $I(\Sigma) \subset \mathcal{O}$ the ideal of functions vanishing on Σ. The quotient $W := \mathcal{O}/I(\Sigma)$ is a $\mathcal{O} * S_n^\vee$-module, in a natural way, identified with $\mathbb{C}^r$ where $r = |\Sigma|$. Our choice of a $\mathcal{O} * S_n^\vee$-module is then

$$\mathscr{U} = W \otimes U^{\otimes n}, \qquad U = \mathbb{C}^m,$$

with $f(\lambda) \in \mathcal{O}$ acting by multiplication on W, and with s_{ij} acting simaltaneously on W and $U^{\otimes n}$. The space $\mathscr{U}$ is a sum of r copies of $U^{\otimes n}$, labelled by points $\lambda \in \Sigma$. The "factorised" states are of the form

$$u_1^{(\lambda_1)} \otimes u_2^{(\lambda_2)} \otimes \cdots \otimes u_n^{(\lambda_n)}, \quad u_i \in U, \quad \lambda = (\lambda_1, \ldots, \lambda_n) \in \Sigma$$

The summand $\phi'_{\lambda_{ij}}\left(\frac{i-j}{n}\right) s_{ij}^\vee$ in $\mathscr{H}^\vee$ acts on such a state by

$$(\ldots u_i^{(\lambda_i)} \otimes \cdots \otimes u_j^{(\lambda_j)} \ldots) \to \phi'_{\lambda_{ij}}\left(\frac{i-j}{n}\right)(\ldots u_j^{(\lambda_j)} \otimes \cdots \otimes u_i^{(\lambda_i)} \ldots)$$

Special case

Choose Σ to be the orbit of a point $(\epsilon, 0, \dots, 0)$, so $|\Sigma| = n$. In a factorised state, spin at one site is labeled by ϵ ("charged"), other labels are 0 ("zero charge"). On such a state, the summand $\phi'_{\lambda_{ij}}\left(\frac{i-j}{n}\right) s_{ij}^{\vee}$ acts as follows, depending on whether one of the ith or jth spins is charged or not:

$$(\dots u_i^{(0)} \otimes \cdots \otimes u_j^{(0)} \dots) \to -\wp\left(\frac{i-j}{n}\right)(\dots u_j^{(0)} \otimes \cdots \otimes u_i^{(0)} \dots)$$

$$(\dots u_i^{(\epsilon)} \otimes \cdots \otimes u_j^{(0)} \dots) \to \phi'_{\epsilon}\left(\frac{i-j}{n}\right)(\dots u_j^{(0)} \otimes \cdots \otimes u_i^{(\epsilon)} \dots)$$

$$(\dots u_i^{(0)} \otimes \cdots \otimes u_j^{(\epsilon)} \dots) \to \phi'_{-\epsilon}\left(\frac{i-j}{n}\right)(\dots u_j^{(\epsilon)} \otimes \cdots \otimes u_i^{(0)} \dots)$$

When $\epsilon \to 0$, we recover the Inozemtsev spin chain.

Outlook

1. Exactly the same method works for CM systems and associated spin chains for all root systems.

2. Further work (with Jules Lamers):

   a) A similar appraoch works for the systems of Calogero–Moser type defined by R-matrix Lax pairs (Levin–Olshanetsky–Zotov 2014) and the corresponding spin chains (Sechin–Zotov 2018). To appear.

   (b) This can be extended to Zotov–Matsushko relativistic models and spin chains. Work in progress.

   (c) Extension to other root systems seems possible.

3. Can Dunkl operators shed some lights on symmetries/spectrum of the model?

Thank you!