On the recurrence relations for the Bethe vectors

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Abstract and Plan

Different types recurrence relations for the off-shell Bethe vectors in the rational quantum integrable models are discussed. The off-shell Bethe vectors in the $\mathfrak{gl}_N$- and $\mathfrak{o}_{2n+1}$-invariant integrable models are considered. The recurrence relations for these Bethe vectors are based on the hierarchical embedding of the monodromy matrices.

- Introduction and Notations
  - Scalar products and relations for off-shell Bethe vectors
  - R-matrices and generalized model
- Projection method
  - Yangian double and its 'new' realization
  - Coloring arguments and recurrence relations
- Recurrence relations from the projections

ABA for Quantum integrable systems

- $N \times N$ monodromy matrix $((\mathsf{e}_{i,j})_{k,l} = \delta_{i,k}\delta_{j,l})$

$$T(u) = \sum_{i,j}^{N} T_{i,j}(u) \otimes \mathsf{e}_{i,j} \in \mathrm{End}(\mathcal{H} \otimes \mathbb{C}^N)$$

- Physical space of states $\mathcal{H}$ possesses vacuum vector $|\mathrm{vac}\rangle$ with the properties (dual space of states $\mathcal{H}^*$ has left vacuum vector $\langle\mathrm{vac}|$)

$$T_{i,j}(u)|\mathrm{vac}\rangle = 0, \quad i > j, \quad T_{i,i}(u)|\mathrm{vac}\rangle = \lambda_i(u)|\mathrm{vac}\rangle$$

$$\langle\mathrm{vac}|T_{i,j}(u) = 0, \quad i < j, \quad \langle\mathrm{vac}|T_{i,i}(u) = \lambda_i(u)\langle\mathrm{vac}|$$

- Monodromy matrix entries satisfy commutation relations

$$\mathrm{R}(u,v) \cdot (T(u) \otimes \mathbb{1}) \cdot (\mathbb{1} \otimes T(v)) = (\mathbb{1} \otimes T(v)) \cdot (T(u) \otimes \mathbb{1}) \cdot \mathrm{R}(u,v)$$

Algebraic Bethe ansatz for QIS

- Yang-Baxter equation for $\mathrm{R}(u,v)$

$$\mathrm{R}_{12}(u_1,u_2)\mathrm{R}_{13}(u_1,u_3)\mathrm{R}_{23}(u_2,u_3)=\mathrm{R}_{23}(u_2,u_3)\mathrm{R}_{13}(u_1,u_3)\mathrm{R}_{12}(u_1,u_2)$$

- The goal of ABA is to construct off-shell Bethe vectors $\mathbb{B}(\bar t)\in\mathcal{H}$

$$\mathbb{B}(\bar t)=\mathcal{P}\Big(T_{i,j}(t^k_\ell)\Big)_{i\le j}|\mathrm{vac}\rangle \qquad \mathbb{C}(\bar t)=\mathbb{B}(\bar t)^\dagger=\langle\mathrm{vac}|\mathcal{P}^\dagger\Big(T_{i,j}(t^k_\ell)\Big)_{i\ge j}$$

- If the set of Bethe parameters $\bar t$ satisfies so called Bethe equations

$$\mathcal{T}(z)\mathbb{B}(\bar t)=\tau(z;\bar t)\mathbb{B}(\bar t) \qquad \mathbb{C}(\bar t)\mathcal{T}(z)=\tau(z;\bar t)\mathbb{C}(\bar t)$$

$$\mathcal{T}(z)=\sum_{i=1}^N T_{i,i}(z)$$

- Coproduct properties of the Bethe vectors

$\mathfrak{g}$-Invariant R-matrices for $\mathfrak{g} = \mathfrak{gl}_N$ and $\mathfrak{g} = \mathfrak{o}_{2n+1}$

$$\mathrm{R}(u,v) \;=\; \mathbb{1}\otimes\mathbb{1} + \frac{c}{u-v}\,\mathbb{P} \qquad \mathfrak{g} = \mathfrak{gl}_N \qquad \mathrm{rank}(\mathfrak{g}) = N-1$$

$$\mathbb{P} = \sum_{i,j\in I_{\mathfrak{gl}_N}} \mathrm{e}_{i,j}\otimes\mathrm{e}_{j,i} \qquad I_{\mathfrak{g}} = \begin{cases} 1,\dots,N, & \mathfrak{g} = \mathfrak{gl}_N \\ -n,\dots,n, & \mathfrak{g} = \mathfrak{o}_{2n+1} \end{cases}$$

$$\mathrm{R}(u,v) \;=\; \mathbb{1}\otimes\mathbb{1} + \frac{c}{u-v}\,\mathbb{P} - \frac{c}{u-v+c\kappa}\,\mathbb{Q}$$

$$\mathfrak{g} = \mathfrak{o}_{2n+1} \qquad N = 2n+1 \qquad \kappa = N/2 - 1 = n - 1/2 \quad \mathrm{rank}(\mathfrak{g}) = n$$

$$X^{\mathrm{t}}_{i,j} = X_{-j,-i} \qquad \mathbb{Q} = \sum_{i,j\in I_{\mathfrak{o}_{2n+1}}} \mathrm{e}_{i,j}\otimes\mathrm{e}_{-i,-j} = \mathbb{P}^{\mathrm{t}_1} = \mathbb{P}^{\mathrm{t}_2}$$

- A. B. Zamolodchikov, Al .B. Zamolodchikov. *Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models*, Ann. Phys. **120** (1979) 253–291

Notations

- Rational functions

$$f(u,v)=\frac{u-v+c}{u-v},\qquad g(u,v)=\frac{c}{u-v},\qquad h(u,v)=\frac{u-v+c}{c}$$

- Sets of Bethe parameters

$$\bar t^{\ell}=(t^{\ell}_1,\dots,t^{\ell}_{r_\ell}),\qquad |\bar t^{\ell}|=r_\ell$$

- Collections of sets

$$\bar t=(\bar t^{1},\dots,\bar t^{\,\mathrm{rank}(\mathfrak g)}),\qquad \{\bar t^{s}\}_i^j=(\bar t^{i},\dots,\bar t^{j}),\qquad \bar t=\{\bar t^{s}\}_1^{\mathrm{rank}(\mathfrak g)}$$

- Sum over partitions

$$\{\bar t^{\ell}_{\mathrm{I}},\bar t^{\ell}_{\mathrm{II}}\}\vdash\bar t^{\ell},\qquad \bar t^{\ell}_{\mathrm{I}}\cap\bar t^{\ell}_{\mathrm{II}}=\varnothing,\qquad \bar t^{\ell}_{\mathrm{I}}\cup\bar t^{\ell}_{\mathrm{II}}=\bar t^{\ell},\qquad \sum_{\substack{\{\bar t^{\ell}_{\mathrm{I}},\bar t^{\ell}_{\mathrm{II}}\}\vdash\bar t^{\ell}\\ \text{including }\ \bar t^{\ell}_{\mathrm{I},\mathrm{II}}=\varnothing}}$$

- Products

$$\lambda_i(\bar u)=\prod_{u_k\in\bar u}\lambda_i(u_k),\qquad f(\bar u,\bar v)=\prod_{u_j\in\bar u}\ \prod_{v_k\in\bar v}f(u_j,v_k)$$

$$g(\varnothing,\bar t)=h(\bar u,\varnothing)\equiv 1,\qquad \prod_{\ell=a}^{b}X_\ell\equiv 1\quad a>b$$

On-shell Bethe vectors ($\mathfrak{g} = \mathfrak{gl}_N$)

$$\mathcal{T}(z)\mathbb{B}(\bar{t}) = \tau(z;\bar{t})\mathbb{B}(\bar{t}), \qquad \mathcal{T}(z) = \sum_{i=1}^{N} T_{i,i}(z)$$

$$\tau(z;\bar{t}) = \sum_{s=1}^{N} \lambda_s(z)\ f(\bar{t}^s, z)\ f(z, \bar{t}^{s-1})$$

$$\alpha_s(z) = \frac{\lambda_s(z)}{\lambda_{s+1}(z)}$$

Bethe equations

$$\alpha_s(\bar{t}^s_{\mathrm{I}}) = \frac{f(\bar{t}^s_{\mathrm{I}}, \bar{t}^s_{\mathrm{II}})}{f(\bar{t}^s_{\mathrm{II}}, \bar{t}^s_{\mathrm{I}})}\ \frac{f(\bar{t}^{s+1}, \bar{t}^s_{\mathrm{I}})}{f(\bar{t}^s_{\mathrm{I}}, \bar{t}^{s-1})}$$

$$\{\bar{t}^s_{\mathrm{I}}, \bar{t}^s_{\mathrm{II}}\} \vdash \bar{t}^s \qquad \bar{t}^0 = \bar{t}^N = \varnothing$$

On-shell Bethe vectors ($\mathfrak{g} = \mathfrak{o}_{2n+1}$)

$$\tau(z;\bar t) = \lambda_0(z)\ f(\bar t^0, z_0)\ f(z,\bar t^0) + \\ + \sum_{s=1}^{n} \Big(\lambda_s(z)\ f(\bar t^s, z)\ f(z,\bar t^{s-1}) + \lambda_{-s}(z)\ f(\bar t^{s-1}, z_{s-1})\ f(z_s,\bar t^s)\Big)$$

$$z_s = z - c(s-1/2), \qquad \mathfrak{f}(u,v) = \frac{u-v+c/2}{u-v}$$

Bethe equations

$$\alpha_s(\bar t^s_{\mathrm{I}}) = \frac{f(\bar t^s_{\mathrm{I}}, \bar t^s_{\mathrm{II}})}{f(\bar t^s_{\mathrm{II}}, \bar t^s_{\mathrm{I}})}\ \frac{f(\bar t^{s+1}, \bar t^s_{\mathrm{I}})}{f(\bar t^s_{\mathrm{I}}, \bar t^{s-1})} \qquad s = 1,\dots,n-1$$

$$\alpha_0(\bar t^0_{\mathrm{I}}) = \frac{\mathfrak{f}(\bar t^0_{\mathrm{I}}, \bar t^0_{\mathrm{II}})}{\mathfrak{f}(\bar t^0_{\mathrm{II}}, \bar t^0_{\mathrm{I}})}\ f(\bar t^1, \bar t^0_{\mathrm{I}})$$

$$T(z) = T^{[1]}(z) \cdot T^{[2]}(z)$$

$$\mathbb{B}(\bar{t}) = \sum_{\text{part}} \Omega_{\mathfrak{g}}(\bar{t}_{\rm I}, \bar{t}_{\rm II})\ \mathbb{B}^{[1]}(\bar{t}_{\rm I}) \cdot \mathbb{B}^{[2]}(\bar{t}_{\rm II}) \prod_{s=1}^{N-1} \alpha_s^{[2]}(\bar{t}_{\rm I}^s)$$

$$\Omega_{\mathfrak{gl}_N}(\bar{t}_{\rm I}, \bar{t}_{\rm II}) = \prod_{s=1}^{N-1} \gamma(\bar{t}_{\rm II}^s, \bar{t}_{\rm I}^s)\ \frac{h(\bar{t}_{\rm I}^s, \bar{t}_{\rm II}^{s-1})}{g(\bar{t}_{\rm II}^{s+1}, \bar{t}_{\rm I}^s)}$$

$$\Omega_{\mathfrak{o}_{2n+1}}(\bar{t}_{\rm I}, \bar{t}_{\rm II}) = \prod_{s=0}^{n-1} \gamma_s(\bar{t}_{\rm II}^s, \bar{t}_{\rm I}^s) \frac{h(\bar{t}_{\rm I}^s, \bar{t}_{\rm II}^{s-1})}{g(\bar{t}_{\rm II}^{s+1}, \bar{t}_{\rm I}^s)}$$

$$\gamma_s(u,v) = \begin{cases} \mathfrak{f}(u,v) = \frac{u-v+c/2}{u-v}, & s=0 \\ \gamma(u,v) = \frac{g(u,v)}{h(v,u)} = \frac{c^2}{(u-v)(v-u+c)}, & s = 1,\dots,n-1 \end{cases}$$

Scalar product

$$S(\bar{u}|\bar{t}) = \mathbb{C}(\bar{u})\mathbb{B}(\bar{t}), \qquad |\bar{u}^{\ell}| = |\bar{t}^{\ell}|$$

$$S(\bar{u}|\bar{t}) = \sum_{\text{part}} W(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}|\bar{t}_{\mathrm{I}}, \bar{t}_{\mathrm{II}}) \prod_{s}^{\mathrm{rank}(\mathfrak{g})} \alpha_s(\bar{u}^s_{\mathrm{I}})\alpha_s(\bar{t}^s_{\mathrm{II}})$$

$$\{\bar{u}^{\ell}_{\mathrm{I}}, \bar{u}^{\ell}_{\mathrm{II}}\} \vdash \bar{u}^{\ell}, \qquad \{\bar{t}^{\ell}_{\mathrm{I}}, \bar{t}^{\ell}_{\mathrm{II}}\} \vdash \bar{t}^{\ell}, \qquad |\bar{u}^{\ell}_{\mathrm{I}}| = |\{\bar{t}^{\ell}_{\mathrm{I}}|$$

$$W(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}|\bar{t}_{\mathrm{I}}, \bar{t}_{\mathrm{II}}) = \Omega(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}})\Omega(\bar{t}_{\mathrm{II}}, \bar{t}_{\mathrm{I}})\bar{Z}(\bar{u}_{\mathrm{I}}|\bar{t}_{\mathrm{I}})Z(\bar{u}_{\mathrm{II}}|\bar{t}_{\mathrm{II}}),$$

$$W(\bar{u}, \varnothing|\bar{t}, \varnothing) = \bar{Z}(\bar{u}|\bar{t}), \qquad W(\varnothing, \bar{u}|\varnothing, \bar{t}) = Z(\bar{u}|\bar{t})$$

Relations for off-shell Bethe vectors ($\mathfrak{g} = \mathfrak{gl}_N$)

Action by the monodromy entry

$$T_{i,j}(z) \cdot \mathbb{B}(\bar{t}) = \sum_{\text{part}:\bar{w}} \Phi^{\text{part}}_{i,j}(\bar{w}_{\mathrm{I}}, \bar{w}_{\mathrm{II}}, \bar{w}_{\mathrm{III}})\ \mathbb{B}(\bar{w}_{\mathrm{II}})$$

partitions $\{\bar{w}^s_{\mathrm{I}}, \bar{w}^s_{\mathrm{II}}, \bar{w}^s_{\mathrm{III}}\} \vdash \bar{w} = \{\bar{t}^s, z\}$ depends on i, j

Recurrence relations

$$\mathbb{B}(\{\bar{t}^s\}_1^{\ell}, \{\bar{t}^{\ell}, z\}, \{\bar{t}^s\}_{\ell+1}^{N-1}) = \sum_{i,j} \sum_{\text{part}:\bar{t}} \Psi^{\text{part}}_{\ell;i,j}(z; \bar{t}_{\mathrm{I}}, \bar{t}_{\mathrm{II}}, \bar{t}_{\mathrm{III}})\ T_{i,j}(z) \cdot \mathbb{B}(\bar{t}_{\mathrm{II}})$$

partitions of $\{\bar{t}^s_{\mathrm{I}}, \bar{t}^s_{\mathrm{II}}, \bar{t}^s_{\mathrm{III}}\} \vdash \bar{t}^s$ depends on i, j

Relations for off-shell Bethe vectors ($\mathfrak{g} = \mathfrak{o}_{2n+1}$)

Action by the monodromy entry

$$T_{i,j}(z) \cdot \mathbb{B}(\bar{t}) = \sum_{\text{part}:\bar{w}} \Phi^{\text{part}}_{i,j}(\bar{w}_{\mathrm{I}}, \bar{w}_{\mathrm{II}}, \bar{w}_{\mathrm{III}})\ \mathbb{B}(\bar{w}_{\mathrm{II}})$$

$$\{\bar{w}^s_{\mathrm{I}}, \bar{w}^s_{\mathrm{II}}, \bar{w}^s_{\mathrm{III}}\} \vdash \bar{w} = \{\bar{t}^s, z, z_s = z - c(s - 1/2)\}$$

Recurrence relations

$$\mathbb{B}(\{\bar{t}^s\}_0^{\ell}, \{\bar{t}^{\ell}, z\}, \{\bar{t}^s\}_{\ell+1}^{n-1}) = \sum_{i,j} \sum_{\text{part}:\bar{t}} \Psi^{\text{part}}_{\ell;i,j}(z; \bar{t}_{\mathrm{I}}, \bar{t}_{\mathrm{II}}, \bar{t}_{\mathrm{III}})\ T_{i,j}(z) \cdot \mathbb{B}(\bar{t}_{\mathrm{II}})$$

$$\mathbb{B}(\{\bar{t}^s\}_0^{\ell}, \{\bar{t}^{\ell}, z_{\ell}\}, \{\bar{t}^s\}_{\ell+1}^{n-1}) = \sum_{i,j} \sum_{\text{part}:\bar{t}} \tilde{\Psi}^{\text{part}}_{\ell;i,j}(z; \bar{t}_{\mathrm{I}}, \bar{t}_{\mathrm{II}}, \bar{t}_{\mathrm{III}})\ T_{i,j}(z) \cdot \mathbb{B}(\bar{t}_{\mathrm{II}})$$

$$\{\bar{t}^s_{\mathrm{I}}, \bar{t}^s_{\mathrm{II}}, \bar{t}^s_{\mathrm{III}}\} \vdash \bar{t}^s$$

Yangian double $\mathcal{D}Y(\mathfrak{g})$ (without central extention)

$$\mathcal{D}Y(\mathfrak{g}) = \Big\{ T_{i,j}[\ell], \qquad \ell \in \mathbb{Z}, \qquad i,j \in I_{\mathfrak{g}} \Big\}$$

$$T^{\pm}_{i,j}(u) = \delta_{ij} + \sum_{\substack{\ell \geq 0 \\ \ell < 0}} T_{i,j}[\ell](u/c)^{-\ell-1}$$

$$T^{\pm}(u) = \sum_{i,j \in I_{\mathfrak{g}}} T^{\pm}_{i,j}(u) \otimes \mathrm{e}_{i,j}$$

$$\mathrm{R}(u,v)\,(T^{\mu}(u) \otimes \mathbb{1})\,(\mathbb{1} \otimes T^{\nu}(v)) = (\mathbb{1} \otimes T^{\nu}(v))\,(T^{\mu}(u) \otimes \mathbb{1})\,\mathrm{R}(u,v)$$

- V. G. Drinfeld. *A new realization of Yangians and of quantum affine algebras*, Soviet Math. Dokl. **36** (1988) 212–216

Gaussian decomposition

- J. Ding, I. Frenkel. *Isomorphism of two realizations of quantum affine algebra* $U_q(\mathfrak{gl}(n))$, Comm. Math. Phys. **156** (1993), 277–300

$$T^{\pm}(u) = \mathbf{F}^{\pm}(u)\cdot\mathbf{K}^{\pm}(u)\cdot\mathbf{E}^{\pm}(u)$$

$$\mathbf{F}^{\pm}(u) = \sum_{i,j\in I_{\mathfrak{g}}} \mathrm{F}^{\pm}_{j,i}(u)\otimes \mathsf{e}_{i,j} \qquad \mathbf{K}^{\pm}(u) = \sum_{i\in I_{\mathfrak{g}}} k^{\pm}_{i}(u)\otimes \mathsf{e}_{i,i} \qquad \mathbf{E}^{\pm}(u) = \sum_{i,j\in I_{\mathfrak{g}}} \mathrm{E}^{\pm}_{i,j}(u)\otimes \mathsf{e}_{j,i}$$

$$\mathrm{F}_{i,i}(u) = \mathrm{E}_{i,i}(u) = 1, \qquad \mathrm{F}_{j,i}(u) = \mathrm{E}_{i,j}(u) = 0, \qquad j<i$$

$$T^{\pm}_{i,j}(u) = \sum_{a\in I_{\mathfrak{g}}} \mathrm{F}^{\pm}_{a,i}(u)\, k^{\pm}_{a}(u)\, \mathrm{E}^{\pm}_{j,a}(u)$$

$$F_i(u) = \mathrm{F}^{+}_{i+1,i}(u) - \mathrm{F}^{-}_{i+1,i}(u), \qquad E_i(u) = \mathrm{E}^{+}_{i,i+1}(u) - \mathrm{E}^{-}_{i,i+1}(u)$$

Different types Borel subalgebras

Standard Borel subalgebras and Hopf structure

$$U^{\pm} = \Big\{ \mathrm{F}^{\pm}_{i+1,i}(u), \qquad \mathrm{E}^{\pm}_{i,i+1}(u), \qquad k^{\pm}_i(u), \qquad i \in I_{\mathfrak{g}} \Big\}$$

$$\Delta(T^{\pm}_{i,j}(z)) = \sum_{\ell \in I_{\mathfrak{g}}} T^{\pm}_{\ell,j}(z) \otimes T^{\pm}_{i,\ell}(z)$$

Drinfeld's Borel subalgebras and Hopf structure

$$U_F = \Big\{ F_i(u), \quad k^{+}_i(u), \quad i \in I_{\mathfrak{g}} \Big\}, \qquad U_E = \Big\{ E_i(u), \quad k^{-}_i(u), \quad i \in I_{\mathfrak{g}} \Big\}$$

$$\Delta^{(D)}(F_i(z)) = \mathbf{1} \otimes F_i(z) + F_i(z) \otimes k^{+}_i(z) \left(k^{+}_{i+1}(z)\right)^{-1}$$

$$\Delta^{(D)}(E_i(z)) = E_i(z) \otimes \mathbf{1} + k^{-}_i(z) \left(k^{-}_{i+1}(z)\right)^{-1} \otimes E_i(z)$$

$$\Delta^{(D)}(k^{\pm}_i(z)) = k^{\pm}_i(z) \otimes k^{\pm}_i(z)$$

Cartan-Weyl construction and root ordering

- S. M. Khoroshkin, V. N. Tolstoy. *On Drinfeld realization of quantum affine algebras*. Journal of Geometry and Physics **11** (1993), 101–108

Root Ordering

If α, β, $\gamma \in \Sigma_+$ and $\gamma = \alpha + \beta$ then $\alpha \prec \gamma \prec \beta$ or $\beta \prec \gamma \prec \alpha$

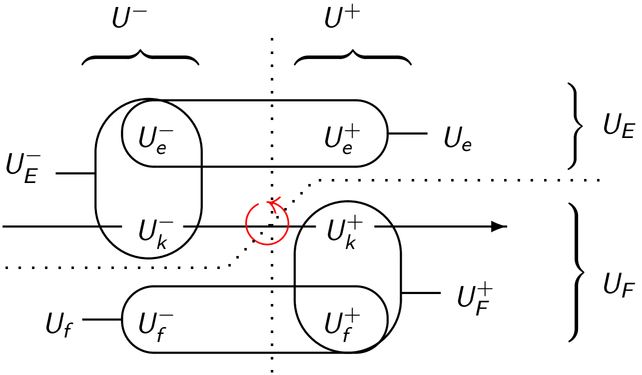
$$e_\gamma := [e_\alpha, e_\beta]_q = e_\alpha e_\beta - q^{-(\alpha,\beta)} e_\beta e_\alpha$$

$$e_{-\gamma} := [e_{-\beta}, e_{-\alpha}]_q = e_{-\beta} e_{-\alpha} - q^{(\alpha,\beta)} e_{-\alpha} e_{-\beta}$$

$$[e_{\pm\alpha}, e_{\pm\beta}]_q = \sum_{\{\gamma_j\},\{n_j\}} C^{\pm\{\gamma_j\}}_{\{n_j\}}(q)\; e^{n_1}_{\pm\gamma_1} e^{n_2}_{\pm\gamma_2} \cdots e^{n_m}_{\pm\gamma_m} \qquad \alpha, \beta \in \Sigma_+$$

$$\alpha \prec \gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_m \prec \beta \qquad \sum_j n_j \gamma_j = \alpha + \beta$$

Different type Borel subalgebras in $\mathcal{D}Y(\mathfrak{g})$



The convex ordering of the Cartan-Weyl generators implies the ordering of the unions of subalgebras $\hat{U}_f^{\pm}$, $\hat{U}_e^{\pm}$ and $\hat{U}_k^{\pm}$ along smallest arcs between starting and ending points of this union. For example, the union of subalgebras $\hat{U}_f^+ \cup \hat{U}_k^+$ or $U_F = \hat{U}_f^- \cup \hat{U}_f^+ \cup \hat{U}_k^+$ or $\hat{U}^+ = \hat{U}_f^+ \cup \hat{U}_k^+ \cup \hat{U}_e^+$ and so on are subalgebras in $\mathcal{D}Y(\mathfrak{g})$

- J. Beck, *Convex bases of PBW type for quantum affine algebras*, Comm. Math. Phys. **165** (1994) 193

Simple fact from the theory of Hopf algebras

For the Hopf algebra $\mathcal{H}$ such that it has subalgebras $\mathcal{H}_1$ and $\mathcal{H}_2$ which satisfy following properties

- the multiplication map $m_{\mathcal{H}} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}$ is a vector space isomorphism,
- $\mathcal{H}_1$ (resp., $\mathcal{H}_2$) is a left (resp., right) coideals of $\mathcal{H}$, i.e.,

$$\Delta_{\mathcal{H}}(\mathcal{H}_1) \subset \mathcal{H} \otimes \mathcal{H}_1, \quad \Delta_{\mathcal{H}}(\mathcal{H}_2) \subset \mathcal{H}_2 \otimes \mathcal{H}\,.$$

Then one can define the linear maps $P_i : \mathcal{H} \to \mathcal{H}_i$ such that

$$P_1(\mathfrak{h}_1 \cdot \mathfrak{h}_2) = \mathfrak{h}_1 \varepsilon_{\mathcal{H}}(\mathfrak{h}_2), \qquad P_2(\mathfrak{h}_1 \cdot \mathfrak{h}_2) = \varepsilon_{\mathcal{H}}(\mathfrak{h}_1)\mathfrak{h}_2$$

for $\mathfrak{h}_i \in \mathcal{H}_i$ where $\varepsilon_{\mathcal{H}}$ is a co-unit map in $\mathcal{H}$. Then the composition of maps

$$m_{\mathcal{H}} \circ (P_1 \otimes P_2) \circ \Delta_{\mathcal{H}} = \mathrm{id}_{\mathcal{H}}$$

is an identity map on $\mathcal{H}$ due to the counit axiom

$$(\varepsilon_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{H}}) \circ \Delta_{\mathcal{H}} = (\mathrm{id}_{\mathcal{H}} \otimes \varepsilon_{\mathcal{H}}) \circ \Delta_{\mathcal{H}} = \mathrm{id}_{\mathcal{H}}$$

Projections onto intersections of different Borel subalgebras

Intersections

$$U_F^- = U_F \cap U^- \qquad U_F^+ = U_F \cap U^+$$

Coideal properties and projections

$$\Delta^{(D)}(U_F^+) = U_F \otimes U_F^+ \qquad \Delta^{(D)}(U_F^-) = U_F^- \otimes U_F$$

$$P_f^+(\mathcal{F}_- \cdot \mathcal{F}_+) = \varepsilon(\mathcal{F}_-)\mathcal{F}_+ \qquad P_f^-(\mathcal{F}_- \cdot \mathcal{F}_+) = \mathcal{F}_-\varepsilon(\mathcal{F}_+) \qquad \mathcal{F}_\pm \in U_F^\pm$$

$\Delta^{(D)}(\mathcal{F}) = \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}$

$$\mathcal{F} = P_f^-\left(\mathcal{F}^{(2)}\right) \cdot P_f^+\left(\mathcal{F}^{(1)}\right) \qquad \mathcal{F} \in \overline{U}_F$$

- B. Enriquez, S. Khoroshkin, S. P., *Weight functions and Drinfeld currents,* Comm. Math. Phys. **276** (2007) 691

Bethe vectors in terms of CW generators for $\mathcal{D}Y(\mathfrak{gl}_N)$

$$\mathcal{F}_i(\bar{u}) = F_i(u_r)F_i(u_{r-1})\cdots F_i(u_2)F_i(u_1)\prod_{a>b}^{r} f(u_a,u_b) \in \overline{U}_F$$

Pre-Bethe vectors

Fix any $\ell \in I_{\mathfrak{gl}_N}$. The pre-Bethe vector $\mathbb{F}(\bar{t})$ does not depend on ℓ

$$\mathbb{F}(\bar{t}) = \prod_{s=1}^{N-1}\frac{1}{h(\bar{t}^s,\bar{t}^s)}\prod_{s=1}^{\ell-1}\frac{1}{g(\bar{t}^{s+1},\bar{t}^s)}\prod_{s=\ell}^{N-2} h(\bar{t}^{s+1},\bar{t}^s)\times$$
$$\times\,\mathcal{F}_\ell(\bar{t}^\ell)\cdots\mathcal{F}_1(\bar{t}^1)\cdot\mathcal{F}_{\ell+1}(\bar{t}^{\ell+1})\cdots\mathcal{F}_{N-1}(\bar{t}^{N-1})$$

Off-shell Bethe vectors via projection

$$\mathbb{B}(\bar{t}) = P_f^+\Big(\mathbb{F}(\bar{t})\Big)|0\rangle$$

- S. Khoroshkin, S. P., *A computation of an universal weight function for the quantum affine algebra $U_q(\mathfrak{gl}(N))$*, J. of Math. of Kyoto Univ., **48** n.2 (2008) 277

'New' realization of $\mathcal{D}Y(\mathfrak{gl}_N)$

$$k_i^{\pm}(u)F_i(v)k_i^{\pm}(u)^{-1} = f(v,u)\ F_i(v)$$
$$k_{i+1}^{\pm}(u)F_i(v)k_{i+1}^{\pm}(u)^{-1} = f(u,v)\ F_i(v)$$

$$k_i^{\pm}(u)^{-1}E_i(v)k_i^{\pm}(u) = f(v,u)\ E_i(v)$$
$$k_{i+1}^{\pm}(u)^{-1}E_i(v)k_{i+1}^{\pm}(u) = f(u,v)\ E_i(v)$$

$$f(u,v)\ F_i(u)F_i(v) = f(v,u)\ F_i(v)F_i(u)$$

$$f(v,u)\ E_i(u)E_i(v) = f(u,v)\ E_i(v)E_i(u)$$

$$h(v,u)\ F_i(u)F_{i+1}(v) = g(v,u)^{-1}\ F_{i+1}(v)F_i(u)$$

$$g(v,u)^{-1}\ E_i(u)E_{i+1}(v) = h(v,u)\ E_{i+1}(v)E_i(u)$$

$$[E_i(u),F_j(v)] = c\ \delta_{i,j}\ \delta(u,v)\Big(k_i^{+}(v)\cdot k_{i+1}^{+}(v)^{-1} - k_i^{-}(u)\cdot k_{i+1}^{-}(u)^{-1}\Big)$$

$$\operatorname*{Sym}_{v_1,v_2}\Big[F_i(v_1),[F_i(v_2),F_{i\pm1}(u)]\Big] = 0 \qquad \operatorname*{Sym}_{v_1,v_2}\Big[E_i(v_1),[E_i(v_2),E_{i\pm1}(u)]\Big] = 0$$

Bethe equations and coproduct properties

$$T^{+}_{i,j}(z)\cdot\mathbb{B}(\bar t)=\lambda_N(z)\ h(\bar t^1,z)\ h(z,\bar t^{N-1})\sum_{\rm part}\mathbb{B}(\bar w_{\rm II})\ \Phi^{\rm part}_{i,j}(\bar w_{\rm I},\bar w_{\rm II},\bar w_{\rm III})\prod_{p=j}^{N-1}\alpha_p(\bar w^p_{\rm III})$$

$$\bar w^p=\{\bar t^p,z\}\ \text{for}\ p=1,\dots,N-1$$

$$\{\bar w^p_{\rm I},\bar w^p_{\rm II}\}\vdash\bar w^p\ \text{for}\ p=1,\dots,i-1\ \text{such that}\ |\bar w^p_{\rm I}|=1$$

$$\{\bar w^p_{\rm II},\bar w^p_{\rm III}\}\vdash\bar w^p\ \text{for}\ p=j,\dots,N-1\ \text{such that}\ |\bar w^p_{\rm III}|=1$$

$$\Phi^{\rm part}_{i,j}(\bar w_{\rm I},\bar w_{\rm II},\bar w_{\rm III})=\Omega_{\mathfrak{gl}_N}(\bar w_{\rm III},\bar w_{\rm II})\ \Omega_{\mathfrak{gl}_N}(\bar w_{\rm III},\bar w_{\rm I})\ \Omega_{\mathfrak{gl}_N}(\bar w_{\rm II},\bar w_{\rm I})$$

$$\text{Example}:\quad T^{+}_{1,N}(z)\cdot\mathbb{B}(\bar t)=\lambda_N(z)\ h(\bar t^1,z)\ h(z,\bar t^{N-1})\ \mathbb{B}(\bar w)$$

Property of the projection

$$\Delta\Big(P^+_f(\mathcal{F})\Big)|0\rangle\otimes|0\rangle=\Big(P^+_f\otimes P^+_f\Big)\circ\Delta^{(D)}(\mathcal{F})|0\rangle\otimes|0\rangle\qquad\forall\mathcal{F}\in\overline{U}_F$$

Color operators for $\mathcal{D}Y(\mathfrak{gl}_N)$

$$T^+_{i,i}(z) = k^+_i(z) + o(z^{-1}) = \mathbf{1} + T^+_{i,i}[0]\,\frac{c}{z} + o(z^{-1}) = \mathbf{1} + k^+_i[0]\,\frac{c}{z} + o(z^{-1})$$

$$[k^+_i[0], F_i(u)] = -F_i(u), \qquad [k^+_i[0], F_{i-1}(u)] = F_{i-1}(u)$$

$$\mathrm{h}_s \cdot \mathbb{B}(\bar{t}) = r_s\,\mathbb{B}(\bar{t}), \qquad \mathrm{h}_s = \sum_{i=s+1}^{N} k^+_i[0], \qquad s = 1, \dots, N-1$$

$$\Theta(m) = \begin{cases} 1, & \text{for} \quad m \geq 0 \\ 0, & \text{for} \quad m < 0 \end{cases} \qquad \Theta(m) + \Theta(-m-1) \equiv 1$$

$$[\mathrm{h}_s, T_{i,j}(z)] = (\Theta(j-s-1) - \Theta(i-s-1))\,T_{i,j}(z)$$

Color operators for $\mathcal{D}Y(\mathfrak{gl}_N)$

$$\mathbb{B}(\bar{t}^1,\dots,\bar{t}^{\ell-1},\{\bar{t}^\ell,z\},\bar{t}^{\ell+1},\dots,\bar{t}^{N-1}) = \sum_{i,j}\sum_{\text{part}} \Psi^{\text{part}}_{i,j}(z,\bar{t}_{\rm i},\bar{t}_{\rm ii})\ T_{i,j}(z)\cdot\mathbb{B}(\bar{t}_{\rm ii})$$

$$|\bar{t}^s_{\rm i}| = \Theta(j-s-1)-\Theta(i-s-1)-\delta_{s,\ell}$$

$$\text{For } s=\ell: \quad |\bar{t}^\ell_{\rm i}| = -\Theta(\ell-j)-\Theta(i-\ell-1)$$

$$1\le i\le \ell<j\le N \quad \text{and} \quad \bar{t}^\ell_{\rm i}=\varnothing$$

$$\text{For } s<\ell: \quad |\bar{t}^s_{\rm i}| = |\bar{t}^s_{\rm I}| = 1-\Theta(i-s-1) = \begin{cases} 0, & s=1,\dots,i-1 \\ 1, & s=i,\dots,\ell-1 \end{cases}$$

$$\text{For } s>\ell: \quad |\bar{t}^s_{\rm i}| = |\bar{t}^s_{\rm III}| = \Theta(j-s-1) = \begin{cases} 1, & s=\ell+1,\dots,j-1 \\ 0, & s=j,\dots,N-1 \end{cases}$$

Gaussian decomposition for the recursion

$$\mathbb{B}(\{\bar{t}^s\}_1^{\ell-1}, \{z, \bar{t}^\ell\}, \{\bar{t}^s\}_{\ell+1}^{N-1}) = \frac{P_f^+\Big(\mathrm{F}_{\ell+1,\ell}^+(z) k_{\ell+1}^+(z) \cdot \mathbb{F}(\bar{t})\Big)|0\rangle}{\lambda_{\ell+1}(z)\ g(z, \bar{t}^{\ell-1})\ h(z, \bar{t}^\ell)\ h(\bar{t}^\ell, z)\ g(\bar{t}^{\ell+1}, z)}$$

$$\mathrm{F}_{\ell+1,\ell}^+(z)\ k_{\ell+1}^+(z) = \sum_{j=\ell+1}^{N} T_{\ell,j}^+(z)\ \tilde{\mathrm{E}}_{\ell+1,j}^+(z)$$

$$\mathbf{F}^+(u) \cdot \mathbf{K}^+(u) = T^+(u) \cdot \mathbf{E}^+(u)^{-1}$$

$$\mathbf{E}^+(u)^{-1} = \sum_{i<j} \tilde{\mathrm{E}}_{i,j}^+(u) \otimes \mathsf{e}_{j,i}$$

$$\tilde{\mathrm{E}}_{i,j}^+(u) = -\mathrm{E}_{i,j}^+(u) + \sum_{b=1}^{j-i-1} (-)^{\ell+1} \sum_{j>i_b>\cdots>i_1>i} \mathrm{E}_{i_b,j}^+(u) \cdots \mathrm{E}_{i_1,i_2}^+(u) \mathrm{E}_{i,i_1}^+(u)$$

Details of calculations

$$\tilde{\mathrm{E}}^{\pm}_{i,j}(z)\cdot\mathbb{F}(\bar{t})|0\rangle \sim \sum_{\rm part} \mathbb{F}(\{\bar{t}^s\}_1^{\ell},\{\bar{t}^s_{\rm II}\}_{\ell+1}^{j-1},\{\bar{t}^s\}_j^{N-1})|0\rangle \times$$

$$|\bar{t}^p_{\rm III}|=1 \qquad \times\, g(\bar{t}^{\ell+1}_{\rm III},z)\prod_{p=\ell+1}^{j-1}\alpha_p(\bar{t}^p_{\rm III})\gamma(\bar{t}^p_{\rm II},\bar{t}^p_{\rm III})\frac{h(\bar{t}^p_{\rm III},\bar{t}^{p-1}_{\rm II})}{g(\bar{t}^{p+1}_{\rm II},\bar{t}^p_{\rm III})}$$

$$\mathbb{F}(\{\bar{t}\}_1^{N-1})=\sum_{\rm part}\prod_{s=1}^{N-1}\gamma(\bar{t}^s_{\rm I},\bar{t}^s_{\rm II})\frac{h(\bar{t}^{s+1}_{\rm II},\bar{t}^s_{\rm I})}{g(\bar{t}^s_{\rm I},\bar{t}^{s-1}_{\rm II})}\;P^-_f\left(\mathbb{F}(\{\bar{t}_{\rm I}\}_1^{N-1})\right)\cdot P^+_f\left(\mathbb{F}(\{\bar{t}_{\rm II}\}_1^{N-1})\right),$$

$$P^+_f\left(T^+_{\ell,j}(z)\cdot P^-_f\left(\mathbb{F}(\{\bar{t}_{\rm I}\}_1^{N-1})\right)\right)=\begin{cases} g(z,\bar{t}^{\ell-1}_{\rm I})\;T^+_{i,j}(z), & |\bar{t}^s_{\rm I}|=1,\quad s=i,\dots,\ell-1 \\ & i=1,\dots\ell \\ 0,\quad \text{otherwise} & \end{cases}$$

- L. Frappat, S. Khoroshkin, S. Pakuliak, E. Ragoucy. *Bethe Ansatz for the Universal Weight Function*, Ann. H. Poincare 10 (2009) 513
- A. Hutsalyuk, A. Liashyk, S. Z. Pakuliak, E. Ragoucy, N. A. Slavnov, *Current presentation for the double super-Yangian $DY(\mathfrak{gl}(m|n))$ and Bethe vectors*, Russ. Math. Surv. **72**:1 (2017) 33–99

Recurrence relation for $\mathcal{D}Y(\mathfrak{gl}_N)$

$$\mathbb{B}(\{\bar t^s\}_1^{\ell-1},\{\bar t^\ell,z\},\{\bar t^s\}_{\ell+1}^{N-1}) =$$
$$= \sum_{i=1}^{\ell}\sum_{j=\ell+1}^{N}\sum_{\text{part}} \frac{T^{+}_{i,j}(z)\cdot\mathbb{B}(\{\bar t^s_{\mathrm{II}}\}_1^{N-1})}{\lambda_{\ell+1}(z)\ g(z,\bar t^{\ell-1}_{\mathrm{II}})\ h(z,\bar t^\ell)\ h(\bar t^\ell,z)\ g(\bar t^{\ell+1}_{\mathrm{II}},z)}\times$$
$$\times\,\Omega_{\mathfrak{gl}_N}(\bar t_{\mathrm{II}},\bar t_{\mathrm{I}})\Omega_{\mathfrak{gl}_N}(\bar t_{\mathrm{III}},\bar t_{\mathrm{II}})\prod_{p=\ell+1}^{j-1}\alpha_p(\bar t^p_{\mathrm{III}})$$

$$\{\bar t^p_{\mathrm{I}},\bar t^p_{\mathrm{II}}\}\vdash \bar t^p \text{ for } p=i,\dots,\ell-1 \text{ such that } |\bar t^p_{\mathrm{I}}|=1$$

$$\{\bar t^p_{\mathrm{II}},\bar t^p_{\mathrm{III}}\}\vdash \bar t^p \text{ for } p=\ell+1,\dots,j-1 \text{ such that } |\bar t^p_{\mathrm{III}}|=1$$

Embedding of $\mathcal{D}Y(\mathfrak{gl}_{N-1})$ into $\mathcal{D}Y(\mathfrak{gl}_N)$

$$
\begin{array}{c|cccc|}
T_{1,1} & T_{1,2} & * & * & T_{1,N} \\ \hline
T_{2,1} & T_{2,2} & * & * & T_{2,N} \\
* & * & & & * \\
* & * & & & * \\
T_{N,1} & T_{N,2} & * & * & T_{N,N} \\ \hline
\end{array}
\qquad
\begin{array}{|cccc|c}
\hline
T_{1,1} & * & * & T_{1,N-1} & T_{1,N} \\
* & & & * & * \\
* & & & * & * \\
T_{N-1,1} & * & * & T_{N-1,N-1} & T_{N-1,N} \\ \hline
T_{N,1} & * & * & T_{N,N-1} & T_{N,N}
\end{array}
$$

$$T(u) = \mathbf{F}(u) \cdot \mathbf{K}(u) \cdot \mathbf{E}(u) \qquad T(u)^{\mathrm{t}} = \hat{\mathbf{F}}(u)^{\mathrm{t}} \cdot \hat{\mathbf{K}}^{\mathrm{t}}(u) \cdot \hat{\mathbf{E}}^{\mathrm{t}}(u)$$

Yangian double $\mathcal{D}Y(\mathfrak{o}_{2n+1})$

$$T^{\pm}(u) \to \left(\left(T^{\pm}(u)\right)^{-1}\right)^{\mathrm{t}}, \qquad T^{\pm}(u) \to \left(\left(T^{\pm}(u)\right)^{\mathrm{t}}\right)^{-1}$$

$$T^{\pm}(u-c\kappa)^{\mathrm{t}} \cdot T^{\pm}(u) = T^{\pm}(u) \cdot T^{\pm}(u-c\kappa)^{\mathrm{t}} = z^{\pm}(u)\ \mathbb{1}$$

$$\left(\left(T^{\pm}(u)\right)^{\mathrm{t}}\right)^{-1} = T^{\pm}(u+c\kappa), \qquad \left(\left(T^{\pm}(u)\right)^{-1}\right)^{\mathrm{t}} = T^{\pm}(u-c\kappa)$$

$$T^{\pm}(u) = \mathbf{F}^{\pm}(u) \cdot \mathbf{K}^{\pm}(u) \cdot \mathbf{E}^{\pm}(u)$$

$$\mathrm{F}^{\pm}_{-i,-i-1}(u) = -\mathrm{F}^{\pm}_{i+1,i}(u - c(i-1/2)), \qquad i = 0, \ldots, n-1$$

$$\mathrm{E}^{\pm}_{-i-1,-i}(u) = -\mathrm{E}^{\pm}_{i,i+1}(u - c(i-1/2)), \qquad j = 0, \ldots, n$$

$$k^{\pm}_{-j}(u) = \frac{1}{k^{\pm}_{j}(u - c(j-1/2))} \prod_{s=j+1}^{n} \frac{k^{\pm}_{s}(u - c(s-3/2))}{k^{\pm}_{s}(u - c(s-1/2))}$$

Embedding of $\mathcal{D}Y(\mathfrak{o}_{2n-1})$ into $\mathcal{D}Y(\mathfrak{o}_{2n+1})$

$$
\left[\begin{array}{c|ccccc|c}
T_{-n,-n} & T_{-n,-n+1} & * & * & * & T_{-n,n-1} & T_{-n,n} \\
\hline
T_{-n+1,-n} & T_{-n+1,-n+1} & * & * & * & T_{-n+1,n-1} & T_{-n+1,n} \\
* & * & & & & * & * \\
* & * & & & & * & * \\
* & * & & & & * & * \\
T_{n-1,-n} & T_{n-1,-n+1} & * & * & * & T_{n-1,n-1} & T_{n-1,n} \\
\hline
T_{n,-n} & T_{n,-n+1} & * & * & * & T_{n,n-1} & T_{n,n}
\end{array}\right]
$$

Theorem. (N. Jing, M. Liu, A. Molev) The commutation relations in the Yangian $Y(\mathfrak{o}_{2n-1})$ are implied by the commutation relations in the Yangian $Y(\mathfrak{o}_{2n+1})$

- N. Jing, M. Liu, A. Molev. *Isomorphism between the R-matrix and Drinfeld presentations of Yangian in types B, C and D*, Comm. Math. Phys. **361** (2018) 827–872

$\mathcal{D}Y(\mathfrak{o}_{2n+1})$ off-shell Bethe vectors

$$F_i(u) = \mathrm{F}^+_{i+1,i}(u) - \mathrm{F}^-_{i+1,i}(u), \qquad E_i(u) = \mathrm{E}^+_{i,i+1}(u) - \mathrm{E}^-_{i,i+1}(u), \qquad i = 0, 1, \ldots, n-1$$

$$\mathcal{F}_i(\bar{t}^i) = \; F_i(t^i_{r_i}) F_i(t^i_{r_i-1}) \cdots F_i(t^i_2) F_i(t^i_1) \prod_{a>b}^{r_i} f_i(t^i_a, t^i_b) \; \in \overline{U}_F$$

$$f_i(u,v) = \begin{cases} \mathfrak{f}(u,v) = \frac{u-v+c/2}{u-v}, & i = 0 \\ f(u,v) = \frac{u-v+c}{u-v}, & i = 1, \ldots, n-1 \end{cases}$$

Pre-Bethe vectors for $\mathcal{D}Y(\mathfrak{o}_{2n+1})$

$$\mathbb{F}(\{\bar{t}^s\}_0^{n-1}) = \prod_{s=1}^{n-1} \frac{1}{h(\bar{t}^s, \bar{t}^s)} \prod_{s=1}^{\ell} \frac{1}{g(\bar{t}^s, \bar{t}^{s-1})} \prod_{s=\ell+1}^{n-1} h(\bar{t}^s, \bar{t}^{s-1}) \times$$
$$\times \mathcal{F}_\ell(\bar{t}^\ell) \cdots \mathcal{F}_0(\bar{t}^0) \cdot \mathcal{F}_{\ell+1}(\bar{t}^{\ell+1}) \cdots \mathcal{F}_{n-1}(\bar{t}^{n-1})$$

Off-shell Bethe vectors for $\mathcal{D}Y(\mathfrak{o}_{2n+1})$

$$\mathbb{B}(\{\bar{t}^s\}_0^{n-1}) = P_f^+\left(\mathbb{F}(\{\bar{t}^s\}_0^{n-1})\right)|0\rangle$$

Color operators for $\mathcal{D}Y(\mathfrak{o}_{2n+1})$

$$\mathsf{h}_s = \sum_{\ell=s+1}^{n} T^{+}_{\ell,\ell}[0] = -\sum_{\ell=-n}^{-s-1} T^{+}_{\ell,\ell}[0], \quad s = 0, 1, \ldots, n-1\,.$$

$$[\mathsf{h}_s, T_{i,j}(z)] = \big(\Theta(j-s-1)-\Theta(i-s-1)+\Theta(-i-s-1)-\Theta(-j-s-1)\big)\, T_{i,j}(z)$$

$$T_{\ell,\ell}[0] \cdot \mathbb{B}(\{\bar{t}\}_0^{n-1}) = (r_{\ell-1} - r_{\ell})\; \mathbb{B}(\{\bar{t}\}_0^{n-1}) \qquad \ell = 1, \ldots, n$$

$$\mathsf{h}_s \cdot \mathbb{B}(\{\bar{t}\}_0^{n-1}) = r_s\; \mathbb{B}(\{\bar{t}\}_0^{n-1})$$

Restrictions for the recurrence relations

$$\mathbb{B}(\{\bar{t}^s\}_0^\ell, \{\bar{t}^\ell, z\}, \{\bar{t}^s\}_{\ell+1}^{n-1}) = \sum_{i=-n}^{\ell} \sum_{j=\ell+1}^{n} \sum_{\text{part}:\bar{t}} \Psi^{\text{part}}_{\ell;i,j}(z; \bar{t}_{\rm I}, \bar{t}_{\rm II}, \bar{t}_{\rm III})\ T_{i,j}(z) \cdot \mathbb{B}(\bar{t}_{\rm II})$$

$$\begin{aligned}
&\text{for} \quad s < \ell: \quad |\bar{t}^s_{\rm I}| = \Theta(s-i) + \Theta(-i-s-1), \quad |\bar{t}^s_{\rm III}| = 0\\
&\text{for} \quad s = \ell: \quad |\bar{t}^\ell_{\rm I}| = \Theta(-i-\ell-1), \quad |\bar{t}^s_{\rm III}| = 0\\
&\text{for} \quad s > \ell: \quad |\bar{t}^s_{\rm I}| = \Theta(-i-s-1), \quad |\bar{t}^s_{\rm III}| = \Theta(j-s-1)
\end{aligned}$$

$$\mathbb{B}(\{\bar{t}^s\}_0^\ell, \{\bar{t}^\ell, z_\ell\}, \{\bar{t}^s\}_{\ell+1}^{n-1}) = \sum_{i=-n}^{-\ell-1} \sum_{j=-\ell}^{n} \sum_{\text{part}:\bar{t}} \tilde{\Psi}^{\text{part}}_{\ell;i,j}(z; \bar{t}_{\rm I}, \bar{t}_{\rm II}, \bar{t}_{\rm III})\ T_{i,j}(z) \cdot \mathbb{B}(\bar{t}_{\rm II})$$

$$\begin{aligned}
&\text{for} \quad s < \ell: \quad |\bar{t}^s_{\rm I}| = 0, \quad |\bar{t}^s_{\rm III}| = \Theta(j-s-1) + \Theta(s+j)\\
&\text{for} \quad s = \ell: \quad |\bar{t}^s_{\rm I}| = 0, \quad |\bar{t}^\ell_{\rm III}| = \Theta(j-\ell-1)\\
&\text{for} \quad s > \ell: \quad |\bar{t}^s_{\rm I}| = \Theta(-i-s-1), \quad |\bar{t}^s_{\rm III}| = \Theta(j-s-1)
\end{aligned}$$

Different expression for off-shell BVs

$$F_i(u) = F_{-i-1}(u + c(i - 1/2)) \qquad 0 \le i \le n-1$$

$$\bar\tau^i = \{\tau^i_1, \dots, \tau^i_{r_i}\} = \{t^i_1 + c(i-1/2), \dots, t^i_{r_i} + c(i-1/2)\}$$

$$g(\bar t^s, \bar t^{s-1}) = (-1)^{r_s r_{s-1}} h(\bar\tau^{s-1}, \bar\tau^s)^{-1}, \quad h(\bar t^s, \bar t^{s-1}) = (-1)^{r_s r_{s-1}} g(\bar\tau^{s-1}, \bar\tau^s)^{-1}$$

$$\mathbb{F}(\{\bar t^s\}_0^{n-1}) = \tilde{\mathbb{F}}(\{\bar\tau^s\}_0^{n-1})$$

$$\mathbb{B}(\{\bar t^s\}_0^{n-1}) = \tilde{\mathbb{B}}(\{\bar\tau^s\}_0^{n-1}) = P_f^+\Big(\tilde{\mathbb{F}}(\{\bar\tau^s\}_0^{n-1})\Big)|0\rangle$$

$$\mathbb{B}(\{\bar t^s\}_0^{\ell-1}, \{\bar t^\ell, z_\ell\}, \{\bar t^s\}_{\ell+1}^{n-1}) = \tilde{\mathbb{B}}(\{\bar\tau^s\}_0^{\ell-1}, \{\bar\tau^\ell, z\}, \{\bar\tau^s\}_{\ell+1}^{n-1})$$

Recurrence relations for $\ell > 0$

$$\mathbb{B}(\{\bar{t}^s\}_0^{\ell-1}, \{z, \bar{t}^\ell\}, \{\bar{t}^s\}_{\ell+1}^{n-1}) = \frac{P_f^+\Big(\mathrm{F}^+_{\ell+1,\ell}(z) k^+_{\ell+1}(z) \cdot \mathbb{F}(\bar{t})\Big)|0\rangle}{\lambda_{\ell+1}(z)\ g(z,\bar{t}^{\ell-1})\ h(z,\bar{t}^\ell)\ h(\bar{t}^\ell, z)\ g(\bar{t}^{\ell+1}, z)}$$

$$\mathrm{F}^+_{\ell+1,\ell}(z) \cdot k^+_{\ell+1}(z) = \sum_{j=\ell+1}^{n} T^+_{\ell,j}(z) \cdot \tilde{\mathrm{E}}^+_{\ell+1,j}(z)$$

$$\mathbb{B}(\{\bar{t}^s\}_0^{\ell-1}, \{z_\ell, \bar{t}^\ell\}, \{\bar{t}^s\}_{\ell+1}^{N-1}) = \frac{P_f^+\Big(\mathrm{F}^+_{-\ell,-\ell-1}(z) k^+_{-\ell}(z) \cdot \mathbb{F}(\bar{t})\Big)|0\rangle}{\lambda_{-\ell}(z)\ g(z,\bar{t}^{\ell-1})\ h(z,\bar{t}^\ell)\ h(\bar{t}^\ell, z)\ g(\bar{t}^{\ell+1}, z)}$$

$$\mathrm{F}^+_{-\ell,-\ell-1}(z) \cdot k^+_{-\ell}(z) = \sum_{j=-\ell}^{n} T^+_{-\ell-1,j}(z) \cdot \tilde{\mathrm{E}}^+_{-\ell,j}(z)$$

Recurrence relations for $\ell > 0$

$$\mathbb{B}(\{\bar{t}^s\}_0^{\ell-1}, \{\bar{t}^\ell, z\}, \{\bar{t}^s\}_{\ell+1}^{n-1}) =$$

$$\times \sum_{i=-n}^{\ell} \sum_{j=\ell+1}^{n} \sum_{\text{part}} \frac{\sigma_{i+1} \; T^{+}_{i,j}(z) \cdot \mathbb{B}(\{\bar{t}^s_{\mathrm{II}}\}_0^{n-1})}{\lambda_{\ell+1}(z) \; g(z, \bar{t}^{\ell-1}_{\mathrm{II}}) \; h(\bar{t}^{\ell}_{\mathrm{II}}, z) \; h(z, \bar{t}^{\ell}) \; g(\bar{t}^{\ell+1}_{\mathrm{I},\mathrm{II}}, z)} \times$$

$$\times \frac{\Omega_{\mathfrak{o}_{2n+1}}(\bar{t}_{\mathrm{II}}, \bar{t}_{\mathrm{I}}) \; \Omega_{\mathfrak{o}_{2n+1}}(\bar{t}_{\mathrm{III}}, \bar{t}_{\mathrm{II},\mathrm{I}})}{g(\bar{t}^{j}_{\mathrm{III}}, \bar{t}^{j-1}_{\mathrm{III}})} \prod_{s=\ell+1}^{j-1} \alpha_s(\bar{t}^s_{\mathrm{III}})$$

$$\text{for} \quad s < \ell: \quad |\bar{t}^s_{\mathrm{I}}| = \Theta(s-i) + \Theta(-i-s-1), \quad |\bar{t}^s_{\mathrm{III}}| = 0$$

$$\text{for} \quad s = \ell: \quad |\bar{t}^\ell_{\mathrm{I}}| = \Theta(-i-\ell-1), \quad |\bar{t}^s_{\mathrm{III}}| = 0$$

$$\text{for} \quad s > \ell: \quad |\bar{t}^s_{\mathrm{I}}| = \Theta(-i-s-1), \quad |\bar{t}^s_{\mathrm{III}}| = \Theta(j-s-1)$$

$$\sigma_i = 2\Theta(i-1) - 1 = \begin{cases} 1, & i > 0 \\ -1, & i \le 0 \end{cases}$$

Known results for $\ell = n-1$ (arXiv:2008.03664)

$$\mathbb{B}\big(\{\bar t^s\}_0^{n-2},\{\bar t^{n-1},z\}\big)=\frac{1}{\lambda_n(z)\ h(z,\bar t^{n-1})}\times$$

$$\times\sum_{i=-n}^{n-1}\sum_{\text{part}}\Omega_{\mathfrak{o}_{2n+1}}(\bar t_{\rm II},\bar t_{\rm I})\frac{\sigma_{i+1}\ \ T^{+}_{i,n}(z)\cdot\mathbb{B}(\bar t_{\rm II})}{g(z,\bar t^{n-2}_{\rm II})h(\bar t^{n-1}_{\rm II},z)}$$

$$\text{for}\quad s<n-1:\quad |\bar t^s_{\rm I}|=\Theta(s-i)+\Theta(-i-s-1)$$
$$\text{for}\quad s=n-1:\quad |\bar t^{n-1}_{\rm I}|=\Theta(-i-n)$$

$$\mathbb{B}\big(\{\bar t^s\}_0^{n-2},\{\bar t^{n-1},z_{n-1}\}\big)=\frac{1}{\lambda_{-n+1}(z)\ h(\bar t^{n-1},z_{n-1})}\times$$

$$\times\sum_{j=-n+1}^{n}\sum_{\text{part}}\Omega_{\mathfrak{o}_{2n+1}}(\bar t_{\rm III},\bar t_{\rm II})\frac{\sigma_{-j+1}\ (-1)^{\delta_{j,n}}\,T^{+}_{-n,j}(z)\cdot\mathbb{B}(\{\bar t^s_{\rm II}\}_0^{n-1})}{g(z_{n-2},\bar t^{n-2}_{\rm II})\ h(z_{n-1},\bar t^{n-1}_{\rm II})}\prod_{p=0}^{n-1}\alpha_p(\bar t^p_{\rm III})$$

$$\text{for}\quad s<n-1:\quad |\bar t^s_{\rm III}|=\Theta(s+j)+\Theta(j-s-1)$$
$$\text{for}\quad s=n-1:\quad |\bar t^{n-1}_{\rm III}|=\Theta(j-n)$$

Conclution

- It was found that projection method is a powerful tool to calculate the different relations for off-shell Bethe vectors in various quantum integrable models
- Further plans
  - Extend these results for the integrable models associated to C and D series
  - Do the same for the trigonometric case using R-matrices classified by M. Jimbo
  - Apply these and other relations to find recurrence relations for the highest coefficients in the Bethe vectors scalar products
  - Try the cases of the exceptional algebras
  - ...

Thanks

Je suis sincèrement reconnaissant au LAPTh de l'opportunité de poursuivre ce projet qui a commencé il y a vingt ans

I am sincerely grateful to LAPTh for the opportunity to continue working on this project that began twenty years ago