Variational method in quantum field theory

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based on upcoming work with M. Lájer, G. Mussardo, A. Stampiggi

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Variational approach in quantum mechanic

Let H be a "simple" Hamiltonian, such that its eigenfunctions are known exactly

$$H(g)|\psi(g)\rangle = E_0|\psi(g)\rangle$$

Consider now $\mathcal{H} \sim H$. We use $|\psi(g)\rangle$ as a trial wave functions

$$E(g) \equiv \langle\psi|\mathcal{H}|\psi\rangle$$

Ground state energy E_{GS} of $\mathcal{H}$ will be given by the choice of parameter $g = g^*$, such that

$$\frac{\partial}{\partial g}\langle\psi|\mathcal{H}|\psi\rangle\Big|_{g=g^*} = 0$$

Alternative approach — perturbation theory order by order

Many body systems

S. De Palo *et. al.*, *“Variational Bethe ansatz approach for dipolar one-dimensional bosons”*
P. W. Claeys, *et. al.*, *“Variational method for integrability-breaking Richardson-Gaudin models”*
De Palo *et. al.* — 1D gas with a dipole interaction (T is a kinetic term)

$$\mathcal{H} = T + V_{\text{dipole}} \equiv T + g|\bar{\psi}\psi|^2 + \left(V_{\text{dipole}} - g|\bar{\psi}\psi|^2\right) \equiv \mathcal{H}_{\text{LL}}(g) + \left(V_{\text{dipole}} - g|\bar{\psi}\psi|^2\right)$$

$\mathcal{H}$ is Lieb-Liniger gas (*exactly solvable model*!) with a coupling constant g
Averaging w.r.t eigenvalues of $\mathcal{H}_{\text{LL}}(g)$ and determining $g = g^*$ from the minimization condition we obtain the ground state energy for the dipole gas

Sinh-Gordon model

Integrable massive relativistict model

$$\mathcal{L} = \frac{1}{2}(\partial_\nu \varphi)(\partial^\nu \varphi) + 2\mu \cosh(\beta\varphi)$$

S-matrix is given

$$S(\theta) = \frac{\sinh\theta - i\sin(\pi B)}{\sinh\theta + i\sin(\pi B)}, \qquad B = \frac{\beta^2}{\beta^2 + 1}$$

Model is well defined on range $0 < \beta < 1$

1. Spectrum and TBA are computed by Zamolodchikov, Al. and Zamolodchikov, A.
2. Vacuum expectation values $\langle \dots \rangle$ of *vertex operator* $\exp(\alpha\varphi)$ are computed by Zamolodchikov, Al., Zamolodchikov, A., Fateev, V., Lukyanov, S.
3. Form factor bootstrap established by Babujian, H., Fring, A., Karowski, M., Smirnov F.
4. n-particles form factors are computed by F. Smirnov and Mussardo, G., Simonetti, P.
5. Form factors on a cylinder LeClair, A., Mussardo, G., Takács, G. and Pozsgay, B.

(1+1)D φ^4 theory

$$\mathcal{L} =: \frac{1}{2}(\partial_\nu\varphi)(\partial^\nu\varphi) + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4 :$$

Only one type of divergency

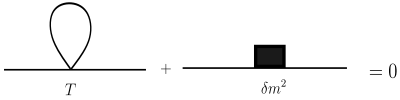


Figure: First order divergent tadpole diagram and corresponding counterterm

Broken symmetry

In strong coupling regime $g = \lambda/m^2 \gg 1$ Z_2 symmetry is broken!

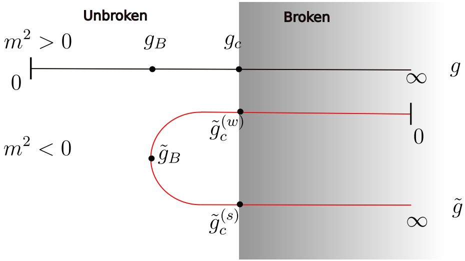


Figure: Phase structure of the φ^4 theory in 2D (Chang, S.-J., 1976). Chang duality in a broken phase

The behaviour of the system is equivalent to the behaviour of the system with negative m^2 term with both small $\tilde{g} = \lambda/\mu^2$ and large $\tilde{g}$

$$\mathcal{L} =: \frac{1}{2}(\partial_\nu \varphi)(\partial^\nu \varphi) - \frac{\mu^2}{4}\varphi^2 + \frac{\lambda}{4!}\varphi^4 :$$

Variational approach in QFT

Variational approach for φ^4: we compute **expectation value of** $\mathcal{H}_{\varphi^4}$ **on vacuum of Sinh-Gordon** theory and use β **as a variational parameter**

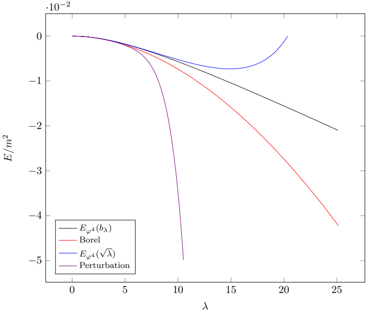
$$\langle \mathcal{H}_{\varphi^4} \rangle_\beta = \mathcal{E}_0(\beta) + \frac{m^2}{2} \langle \varphi^2 \rangle_\beta + \frac{\lambda}{4!} \langle \varphi^4 \rangle_\beta - \mu \langle \cosh(\beta\varphi) \rangle_\beta$$

where $\mathcal{E}_0$ is a bulk energy of Sinh-Gordon

$$\mathcal{E}_0(\beta) \equiv \left\langle \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) + \mu \cosh(\beta\varphi) \right\rangle = \frac{M^2}{8 \sin(\pi B)}$$

Can be compared with a Feynman perturbation theory of φ^4 (built up to 8th order by Serone, M. *et. al.*, Ryckov, S. and Vitale, L. G.)

Numerical results and comparison



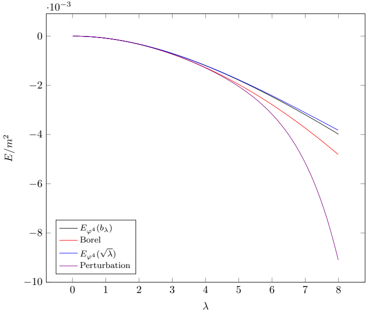


Figure: Energy density of φ^4 as function of λ. **Black line** — minimization of energy, the blue line — the vacuum energy of ShG with $\beta = \sqrt{\lambda}$. Violet line — perturbation theory up to $O(\lambda^8)$. Red line – Borel resummation of the perturbation theory. The energy obtained through the variational principle agrees with the Borel resummation within $2 \cdot 10^{-3}$ for $\lambda \lesssim 8$

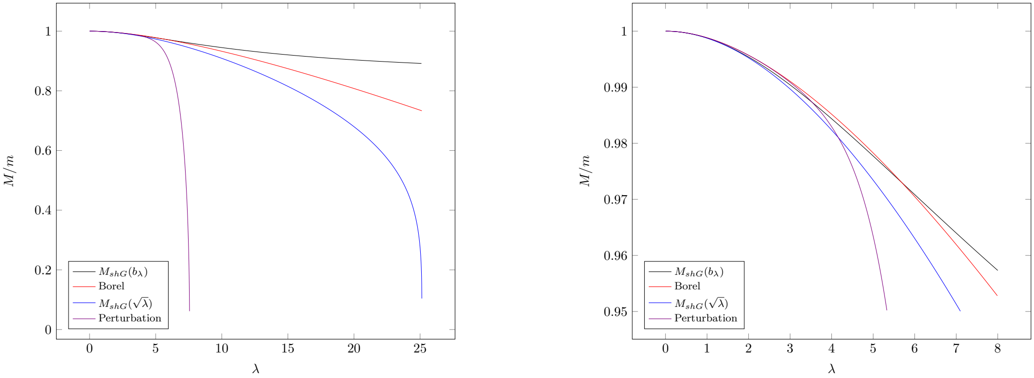


Figure: Mass of $\mathcal{H}_{\varphi^4}$ as function of λ. **Black line** — mass of ShG with the optimization of $\beta = b^*$, blue line — classical solution $\beta = \sqrt{\lambda}$, violet line — perturbation theory up to $O(\lambda^8)$, red line — Borel resummation of the perturbation theory. The mass estimate obtained through the variational principle agrees surprisingly with the Borel resummation within 10^{-2} for $\lambda \lesssim 8$.

Thermodynamic Bethe ansatz (TBA)

On the cylinder of radius R and length $L \to \infty$ *pseudoenergy* of particles is given by *the thermodynamic Bethe ansatz*

$$\varepsilon(\theta) = M\,R\cosh\theta - \int_{-\infty}^{\infty} \frac{du}{2\pi}\phi(\theta - u)\log\left(1 + e^{-\varepsilon(u)}\right)$$

Where integration kernel is given by

$$\varphi(\theta) = -i\frac{d}{d\theta}\log S(\theta)$$

On the cylinder with radius R and length $L \to \infty$ the energy E_0 is given by (Zamolodchikov & Zamolodchikov)

$$E_0 = R\,\mathcal{E}_0 - M \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh\theta \, \log\left(1 + e^{-\varepsilon(\theta)}\right)$$

where $\mathcal{E}_0$ is the bulk energy on the plane

$$\mathcal{E}_0 = \frac{M^2}{8\sin(\pi B)}$$

LeClair-Mussardo theorem

Expaectation value of the operator O on the cylinder is given by

$$\langle O \rangle_R = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\prod_{i=1}^{n} \int \frac{d\theta_i}{(2\pi)} f_\sigma(\theta_i) \right) F_c^O(\theta_1, \ldots, \theta_n)$$

$e(\theta)$ is the one-particle energy (determined by TBA), $\varepsilon(\theta) = e(\theta)R$,

$$f_\sigma(\theta) = \frac{1}{1 + e^{\varepsilon(\theta_i)R}}$$

n-particles *connected form factors* $F_c^O(\theta_1, \ldots, \theta_n)$ are defined by

$$F_c^O(\theta_1, \ldots, \theta_n) = \text{Finite Part} \lim_{\eta_i \to 0} \langle \theta_n, \ldots, \theta_1 | O | \theta_1 + \eta_1, \ldots, \theta_n + \eta_n \rangle$$

Conclusions

- Variational approach is by order simpler and faster then the Feynman perturbation theory
- It is difficult *ad hoc* to estimate however the region in which the approach can be trusted
- The comparison with known results shows that the "radius of convergence" is much smaller than Feynman diagrams combined Borel resummation technique
- Description of the φ^4 in a broken phase (Bullough-Dodd model as a good candidate)
- Different models?
- Finite radius energy