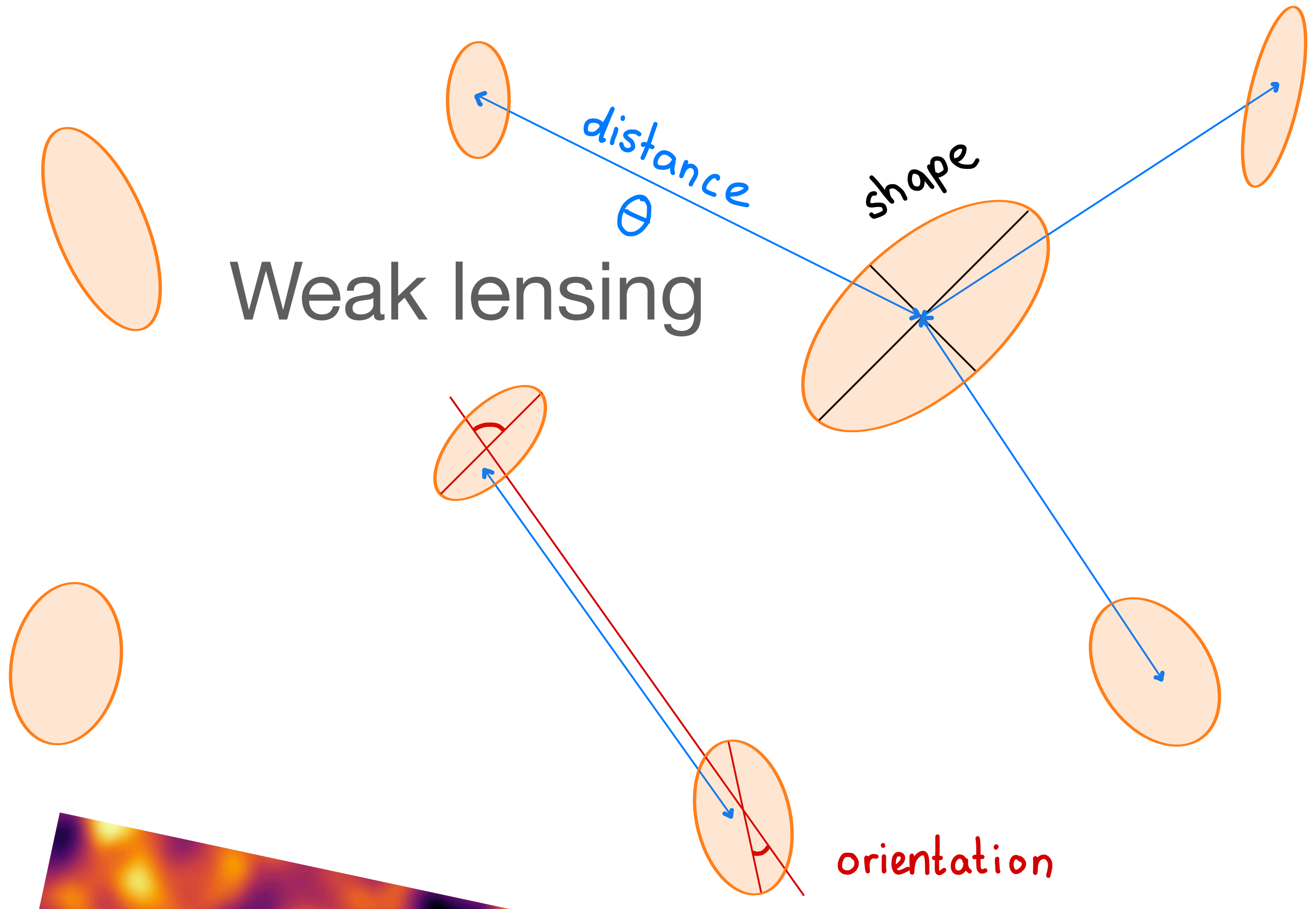
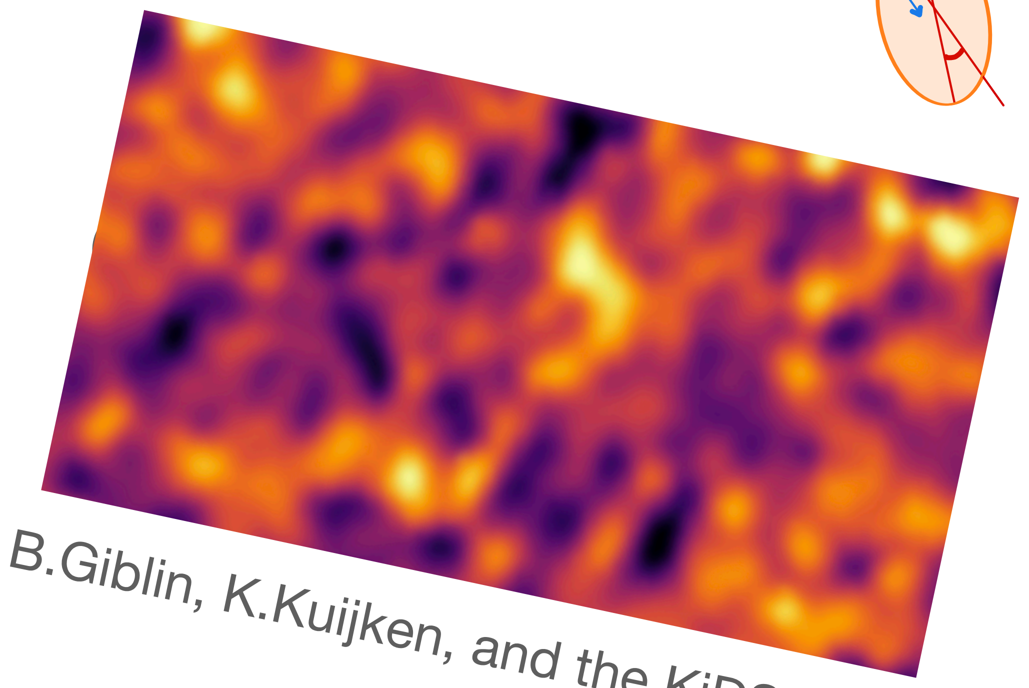
Exact likelihood for correlation functions

Treating masked Gaussian spin-2 fields on the sphere

Veronika Oehl in collaboration with Tilman Tröster

ETH zürich





B.Giblin, K.Kuijken, and the KiDS team

Correlation function:

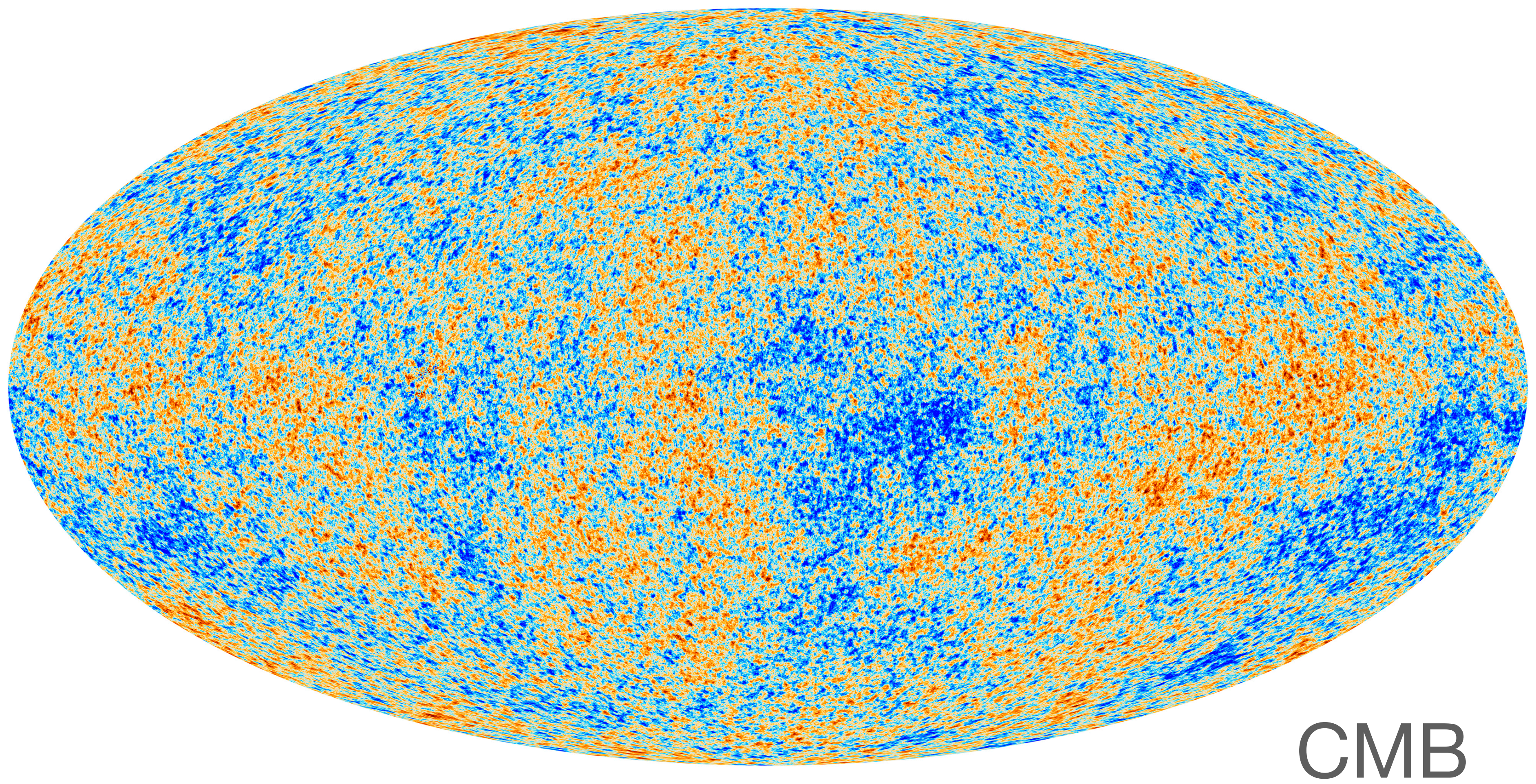
$p(\xi \,|\, \vartheta) \;=\; ?$

Average over realizations of the field

$$\xi^{ij}(\theta) = \left\langle A^i(\boldsymbol{\theta}')\, A^j(\boldsymbol{\theta}' - \boldsymbol{\theta}) \right\rangle$$

Gaussian random field:

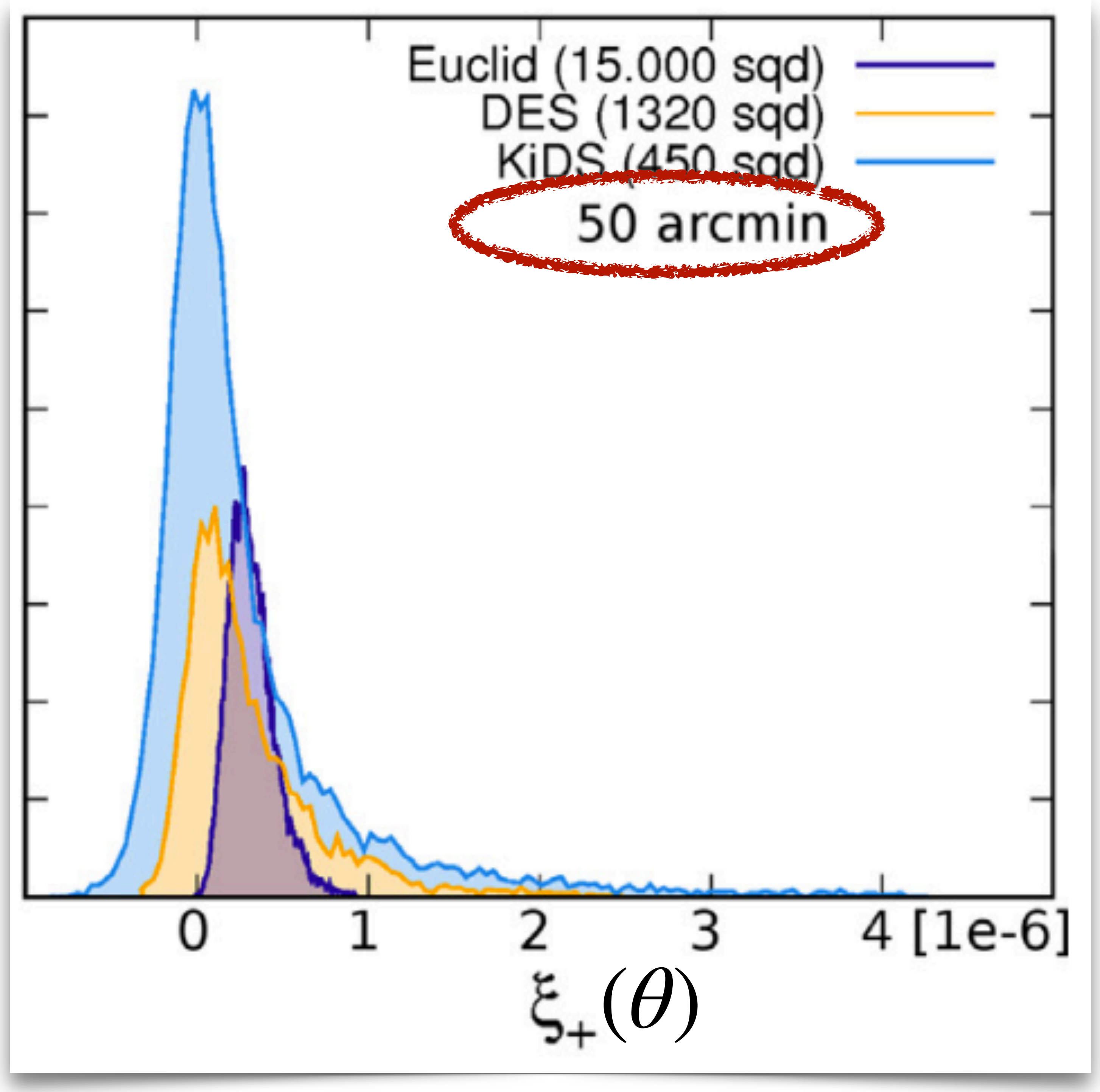
$A^i(\boldsymbol{\theta}) \sim \mathcal{N}(0, \Sigma(\vartheta))$ $\qquad \vartheta = \Omega_m, S_8, H_0, \ldots$



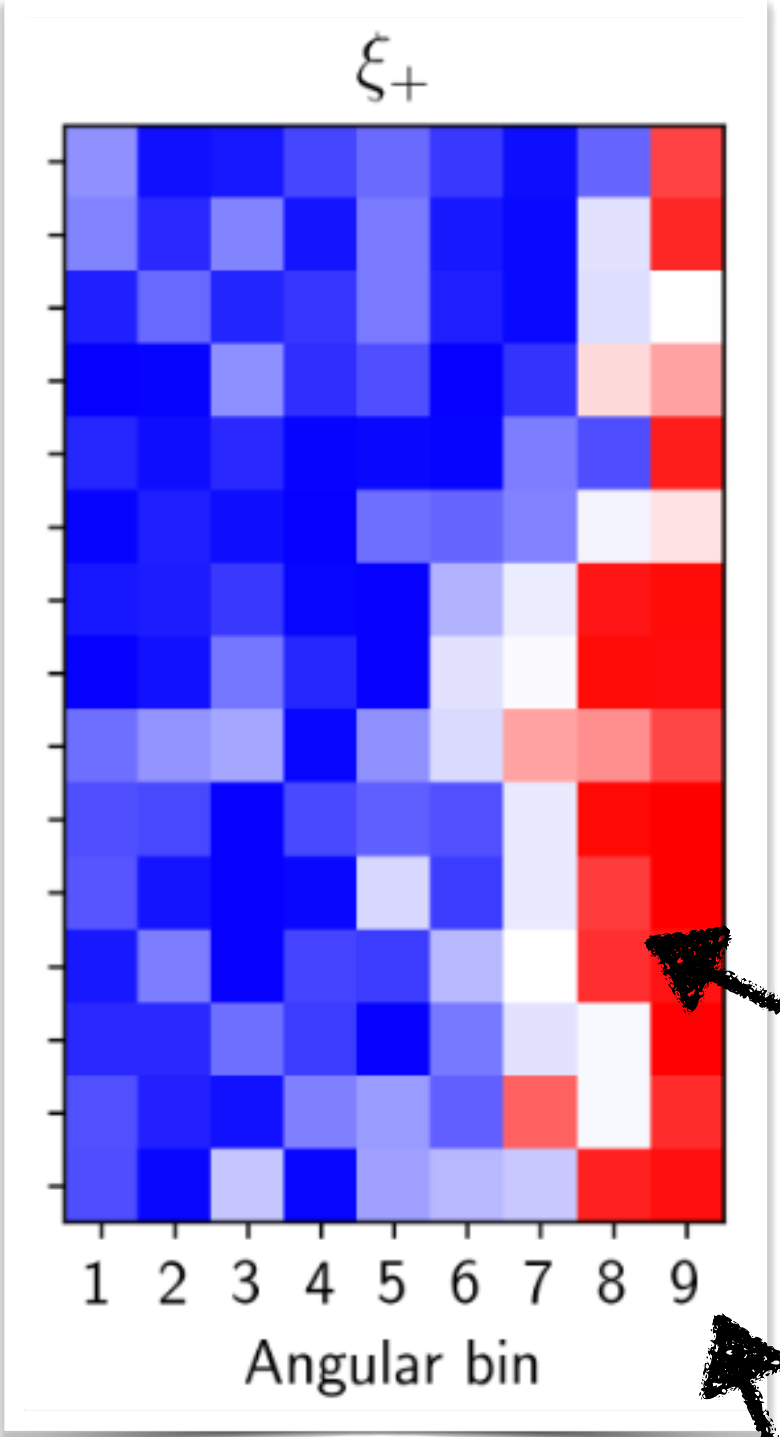
CMB

ESA and the Planck Collaboration

The distribution of $\xi^+(\theta)$...



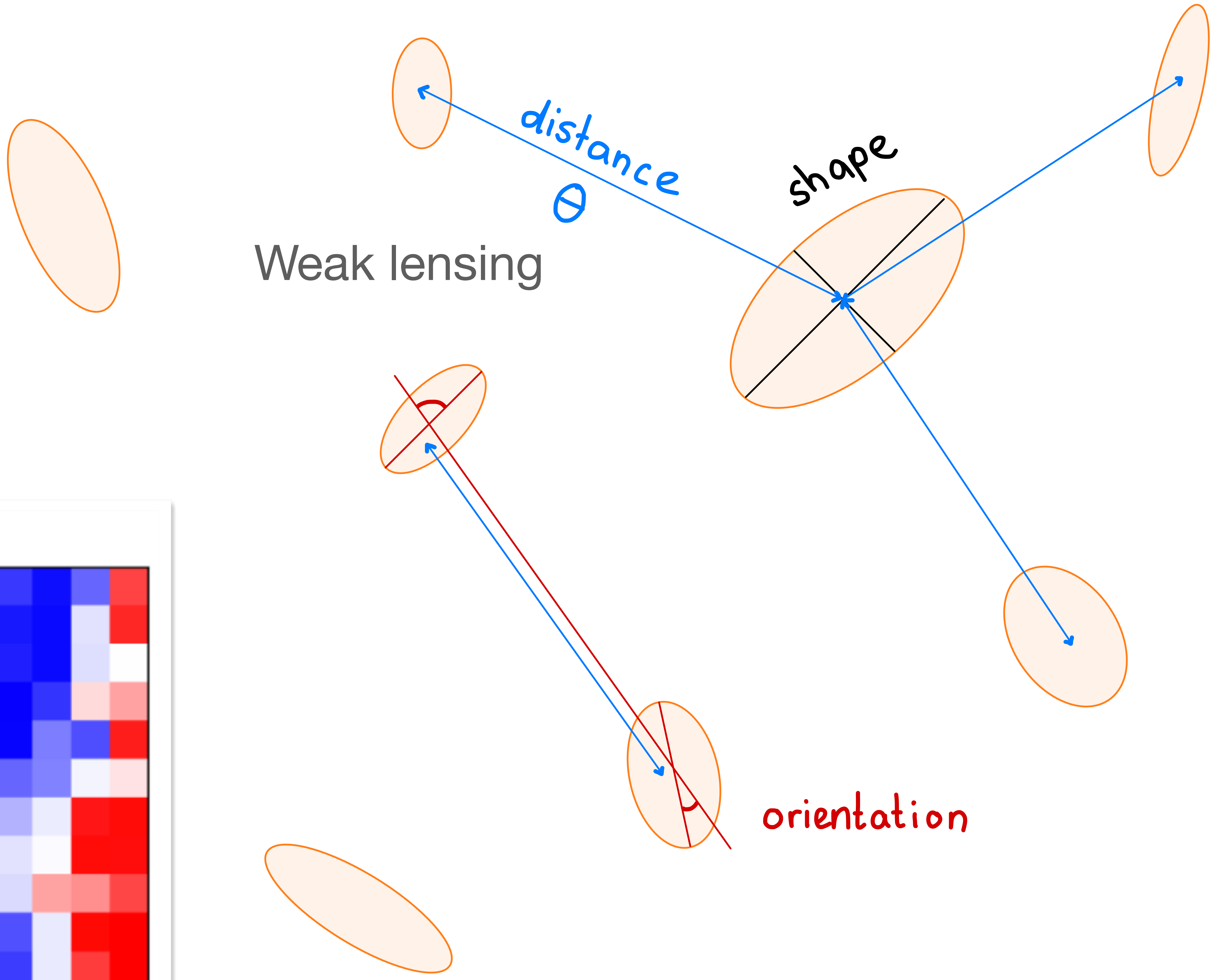
Sellentin et al., 2018 (arXiv:1712.04923)



Joachimi et al., 2021 (arXiv:2007.01844)

p-value < 0.05

$\theta = 5^\circ$

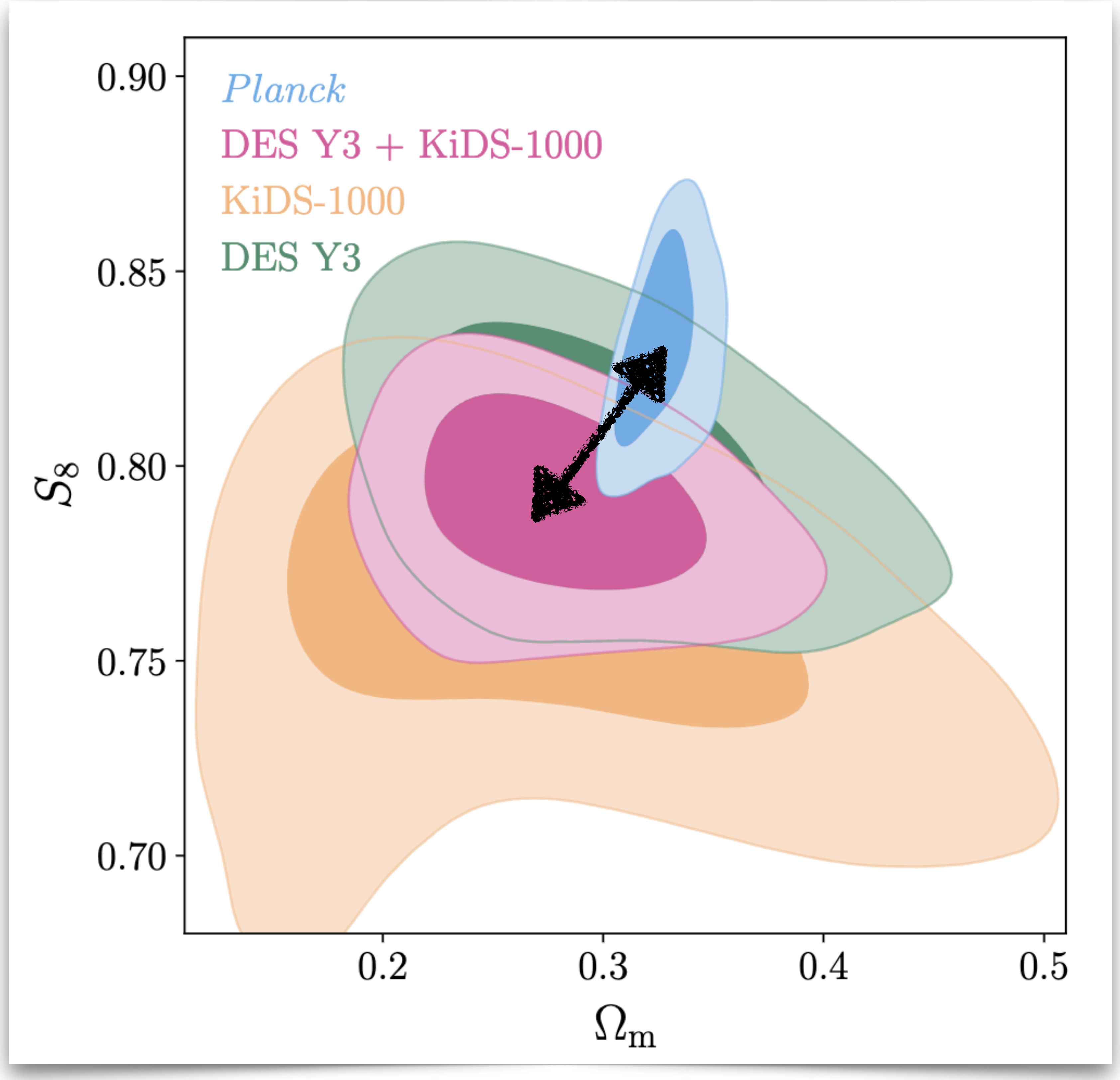


...is non-Gaussian!

Veronika Oehl

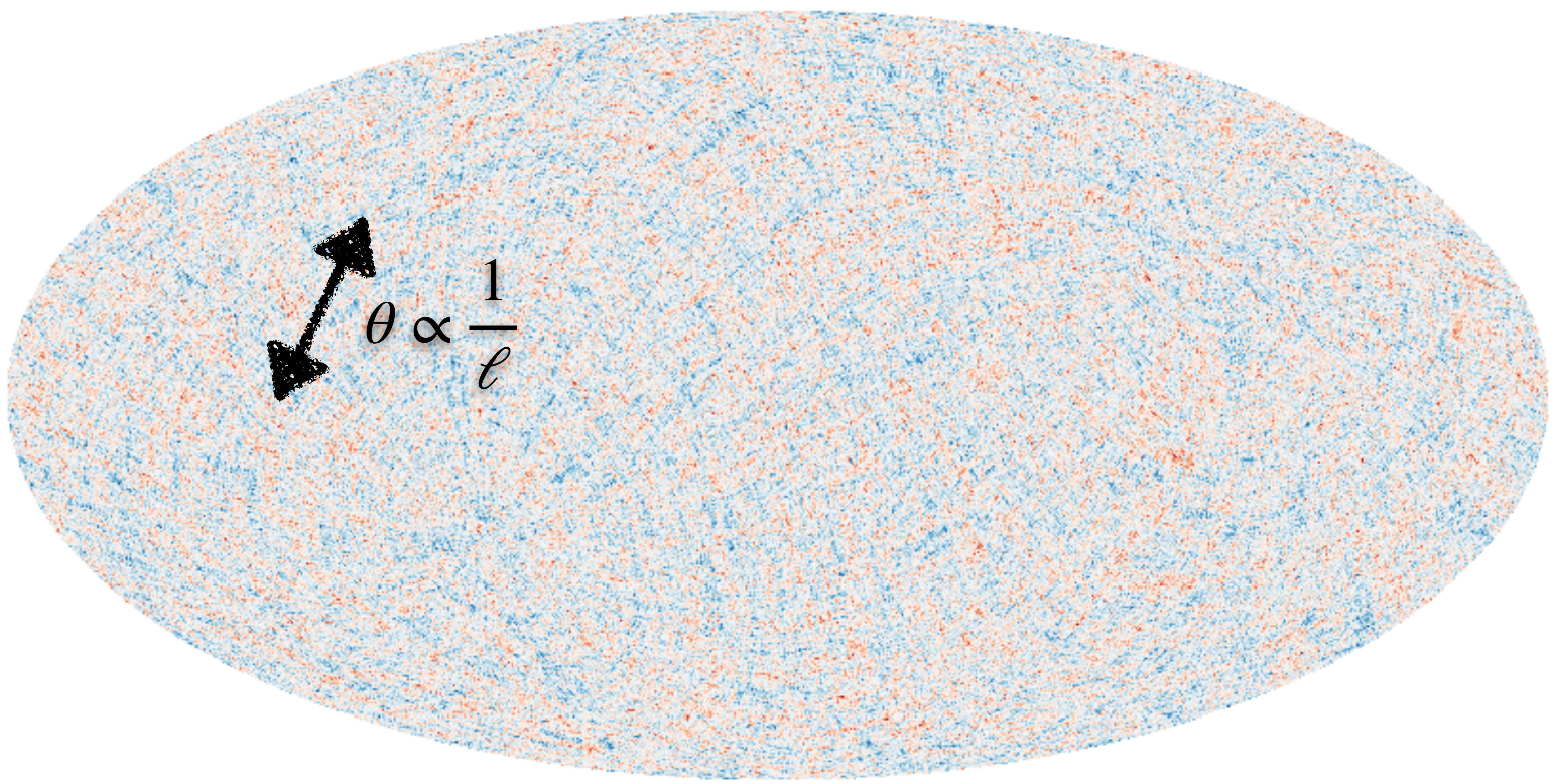
likelihood

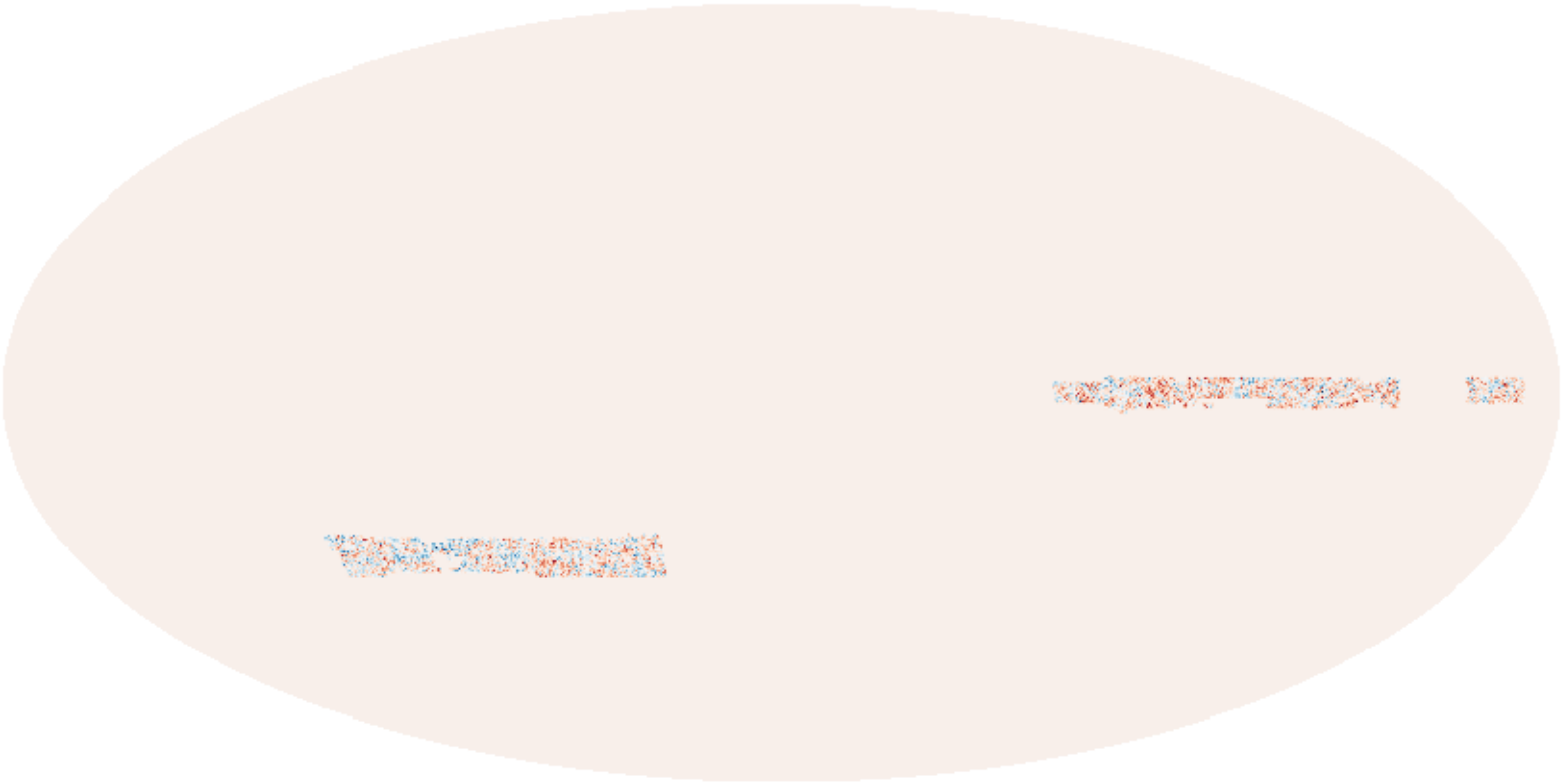
$$p(\vartheta\,|\,\xi) \propto p(\xi\,|\,\vartheta)\; p(\vartheta) \qquad \vartheta = \Omega_m, S_8, H_0, \ldots$$



Kilo-Degree Survey and Dark Energy Survey Collaborations, 2023
(arXiv:2305.17173)

How to calculate this likelihood?





$\sim$ Wishardt

$$\hat{C}_\ell = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} a_{\ell m} a_{\ell m}^{\;*}$$

$\sim \mathcal{N}(0, C_\ell)$

linear combinations of $a_{\ell m}^{E/B}$

$$\hat{\tilde{C}}_\ell = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \tilde{a}_{\ell m} \tilde{a}_{\ell m}^{\;*}$$

$\sim \mathcal{N}(0, \Sigma)$

$$\xi^+_{ij}\left(\bar{\theta}\right) = \sum_{\ell} K_\ell\left(\bar{\theta}\right) \sum_{m=-\ell}^{\ell} \left(\tilde{a}^{E,i}_{\ell m} \tilde{a}^{E,j}_{\ell m}{}^{*} + \tilde{a}^{B,i}_{\ell m} \tilde{a}^{B,j}_{\ell m}{}^{*} \right)$$

$$= 2\pi \quad \frac{2}{\theta^2_{\max} - \theta^2_{\min}} \int_{\theta_{\min}}^{\theta_{\max}} \mathrm{d}\theta \; \theta B(\theta) d^{\ell}_{22}(\theta)$$

$$\xi^+_{ij}\left(\bar{\theta}\right) = \mathbf{\tilde{a}}^{\mathbf{T}} \mathbf{M}^{\xi^+_{\mathrm{ij}}}\left(\bar{\theta}\right) \mathbf{\tilde{a}}$$

Gaussian distributed,
covariance Σ depends on mask geometry and theory C_ℓ

NxN matrix: $N \propto \ell^2_{max} + \ell_{max}$

$$\xi^{+}(\theta) = \tilde{\mathbf{a}}^{\mathrm{T}}\mathbf{M}^{\xi^{+}}(\theta)\tilde{\mathbf{a}}$$

Quadratic forms have known characteristic function $\varphi(t)$:

covariance matrix of $\tilde{\mathbf{a}}$

eigenvalues

$$\varphi(t) = \prod_j (1 - 2i\lambda_j)^{-1/2}, \qquad \lambda_j \in \lambda\,(t\,\mathbf{M}\,\mathbf{\Sigma})$$

Fourier equivalent of ξ

diagonal matrix

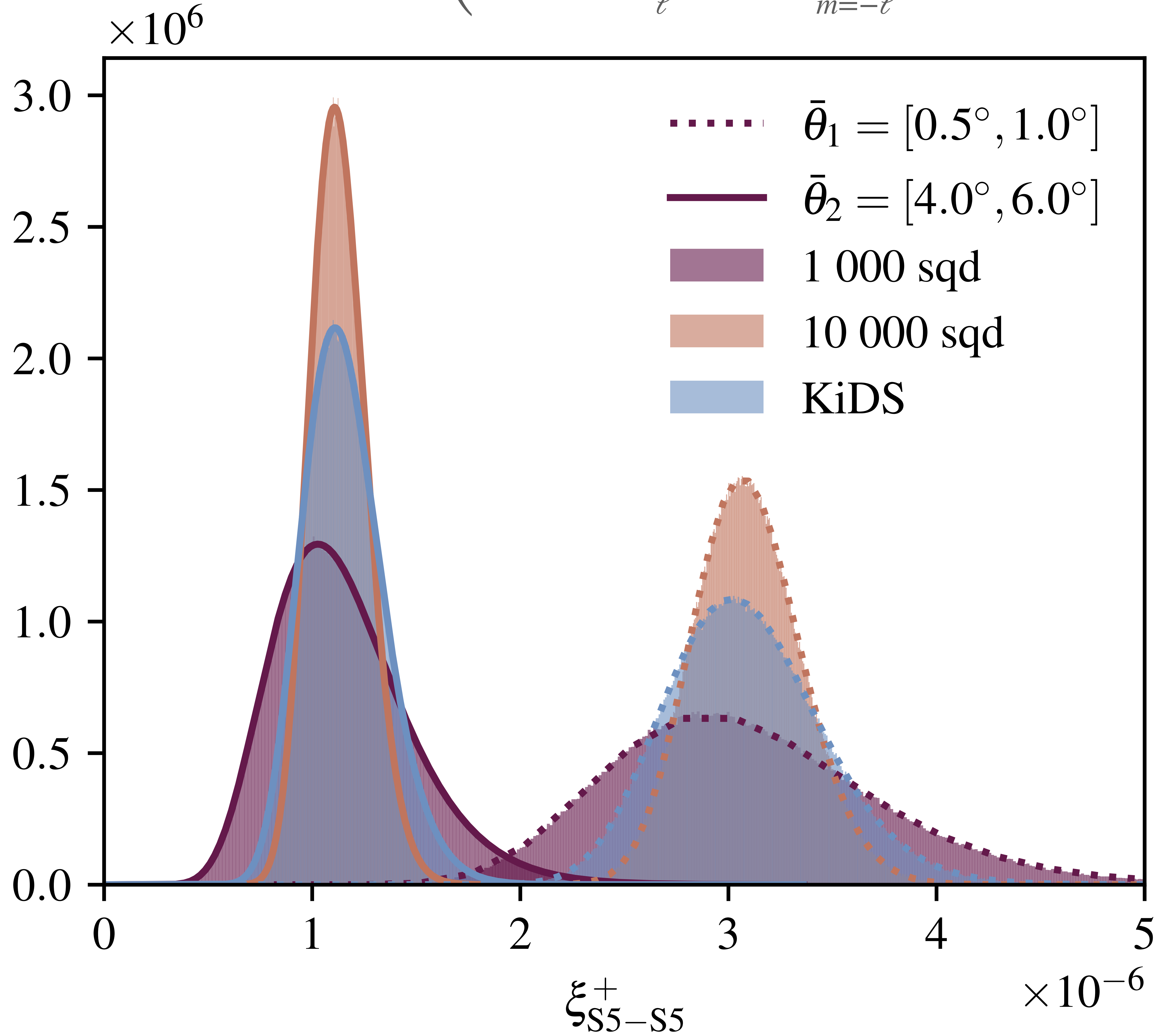
PDF (likelihood)

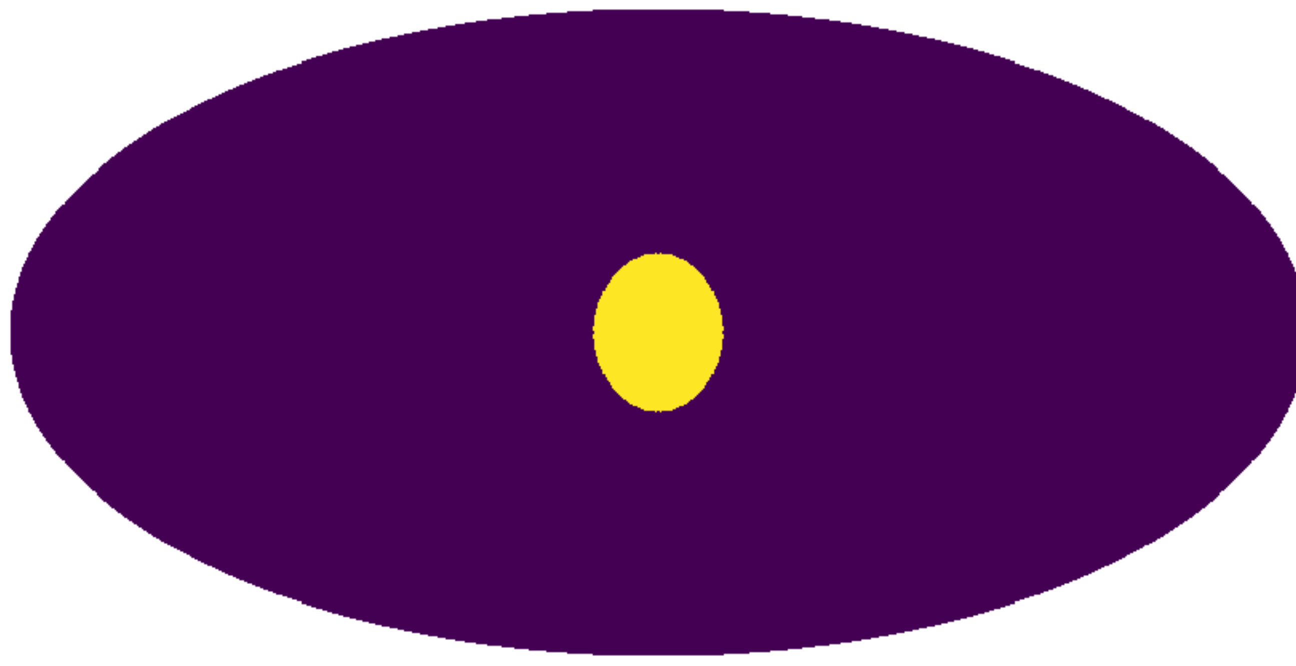
characteristic function

$$p(\xi\,|\,\vartheta) \propto \int \mathrm{d}t\;\; \exp(-i\xi t)\,\varphi(t, \vartheta)$$

Fourier pair: (ξ, t)

$\ell_{\text{exact}} = 30$ $$\left(\xi^{+}_{ij}(\bar{\theta}) = \sum_{\ell} K_{\ell}(\bar{\theta}) \sum_{m=-\ell}^{\ell} \left(\tilde{a}^{E,i}_{\ell m} \tilde{a}^{E,j\,*}_{\ell m} + \tilde{a}^{B,i}_{\ell m} \tilde{a}^{B,j\,*}_{\ell m} \right) \right)$$





+ shape noise

Oehl & Tröster, 2024 (in prep.)

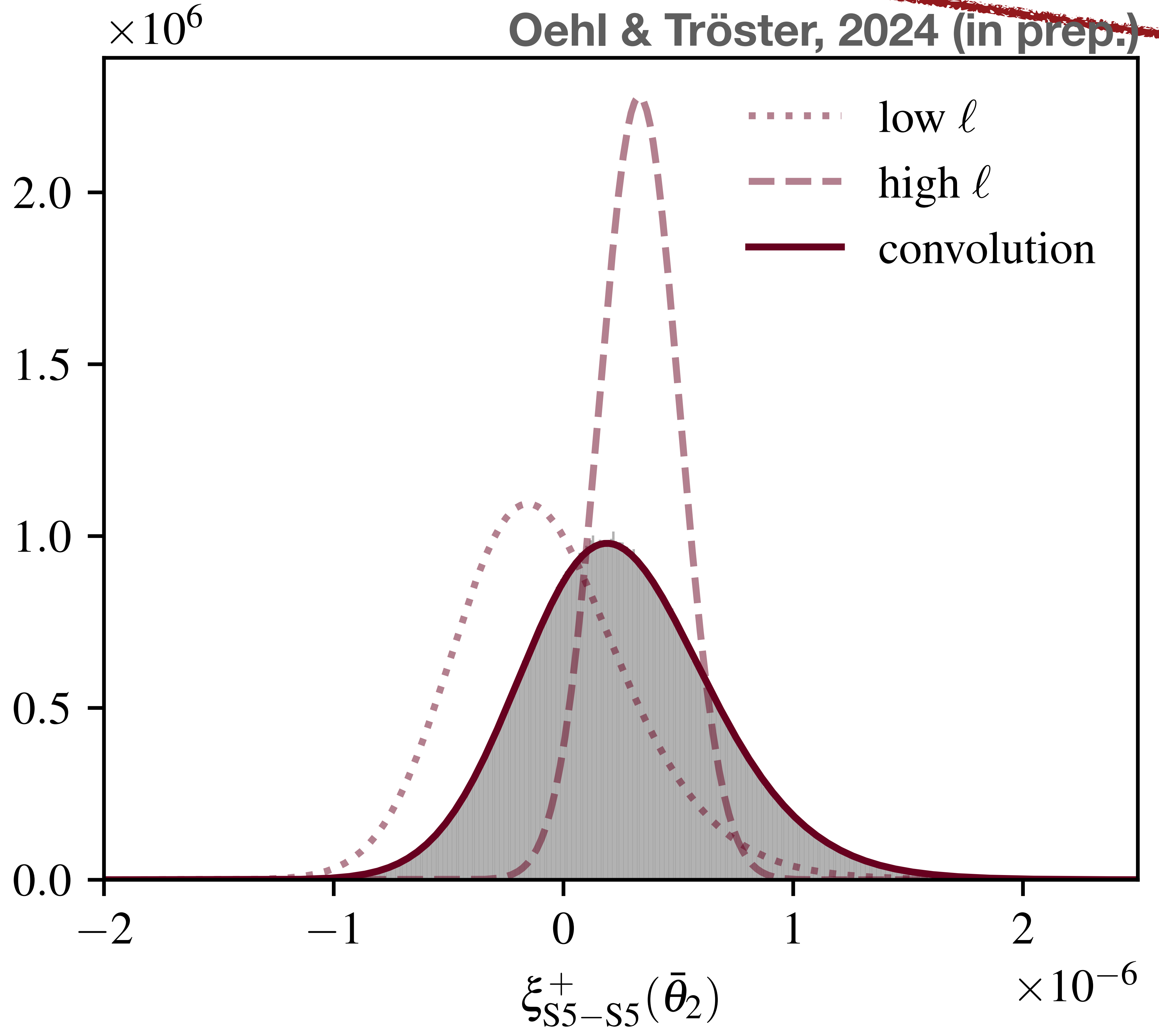
$\bar{\theta}_2 = [4.0^\circ, 6.0^\circ]$ $\bar{\theta}_1 = [0.5^\circ, 1.0^\circ]$

$$\xi^{+}_{ij}\left(\bar{\theta}\right) = \sum_{\ell} K_{\ell}\left(\bar{\theta}\right) \sum_{m=-\ell}^{\ell} \left(\tilde{a}^{E,i}_{\ell m}\tilde{a}^{E,j\,*}_{\ell m} + \tilde{a}^{B,i}_{\ell m}\tilde{a}^{B,j\,*}_{\ell m}\right)$$

low ℓ

exact calculation

$$\varphi_{\mathrm{exact}}(t) = \prod_{j} (1 - 2i\lambda_j)^{-1/2}$$



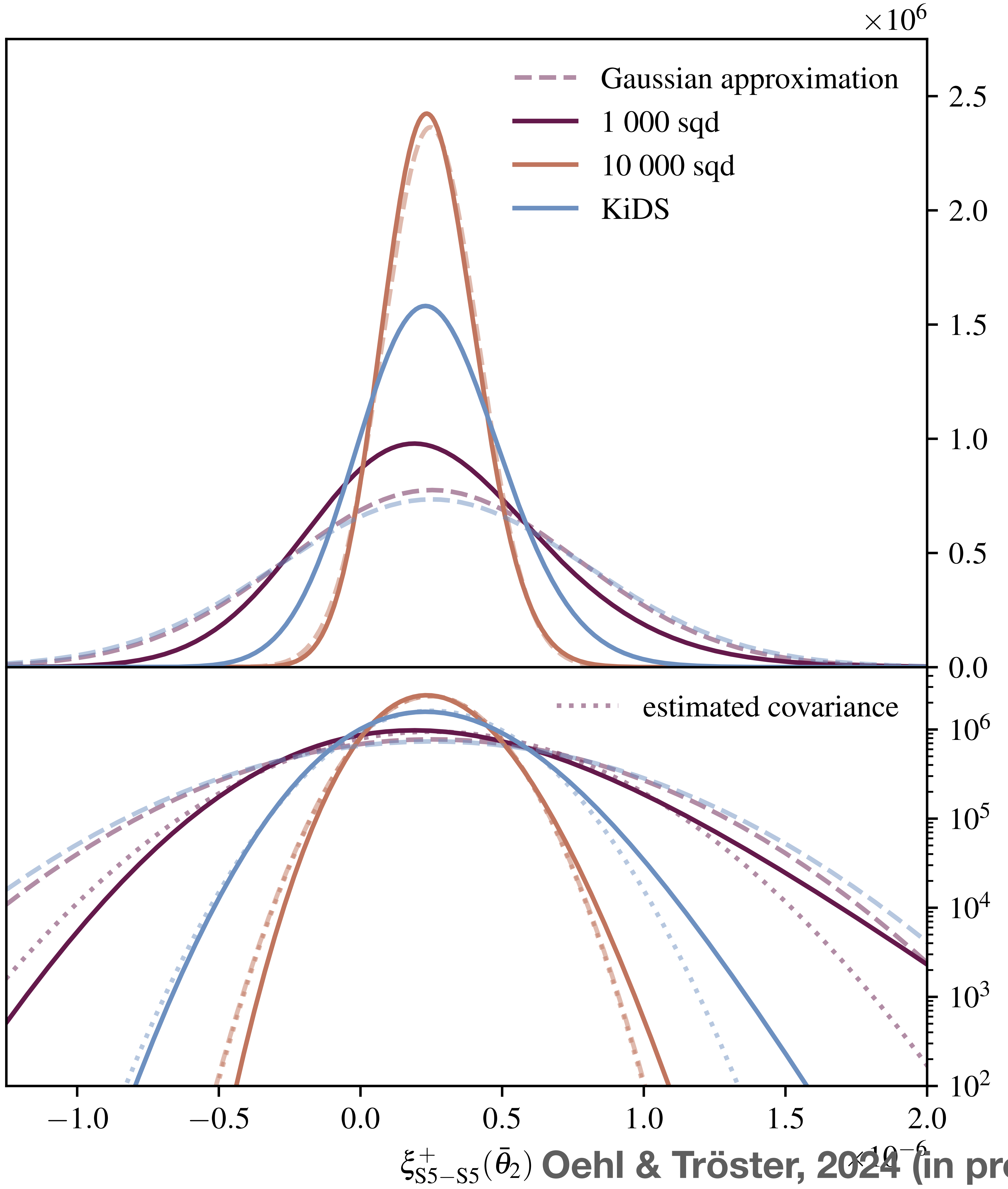
high ℓ

Gaussian approximation

$$\varphi_{\mathrm{Gauss}}(\mathbf{t}) = e^{i\mathbf{t}^{\mathrm{T}}\boldsymbol{\mu} - \frac{1}{2}\mathbf{t}^{\mathrm{T}}\boldsymbol{\Sigma}_{\mathrm{Gauss}}\mathbf{t}}$$

$$\varphi(\mathbf{t}) = \varphi_{\mathrm{exact}}(\mathbf{t})\varphi_{\mathrm{Gauss}}(\mathbf{t}) \qquad p(\xi\,|\,\vartheta) \propto \int \mathrm{d}t\ \exp(-i\xi t)\ \varphi(t)$$

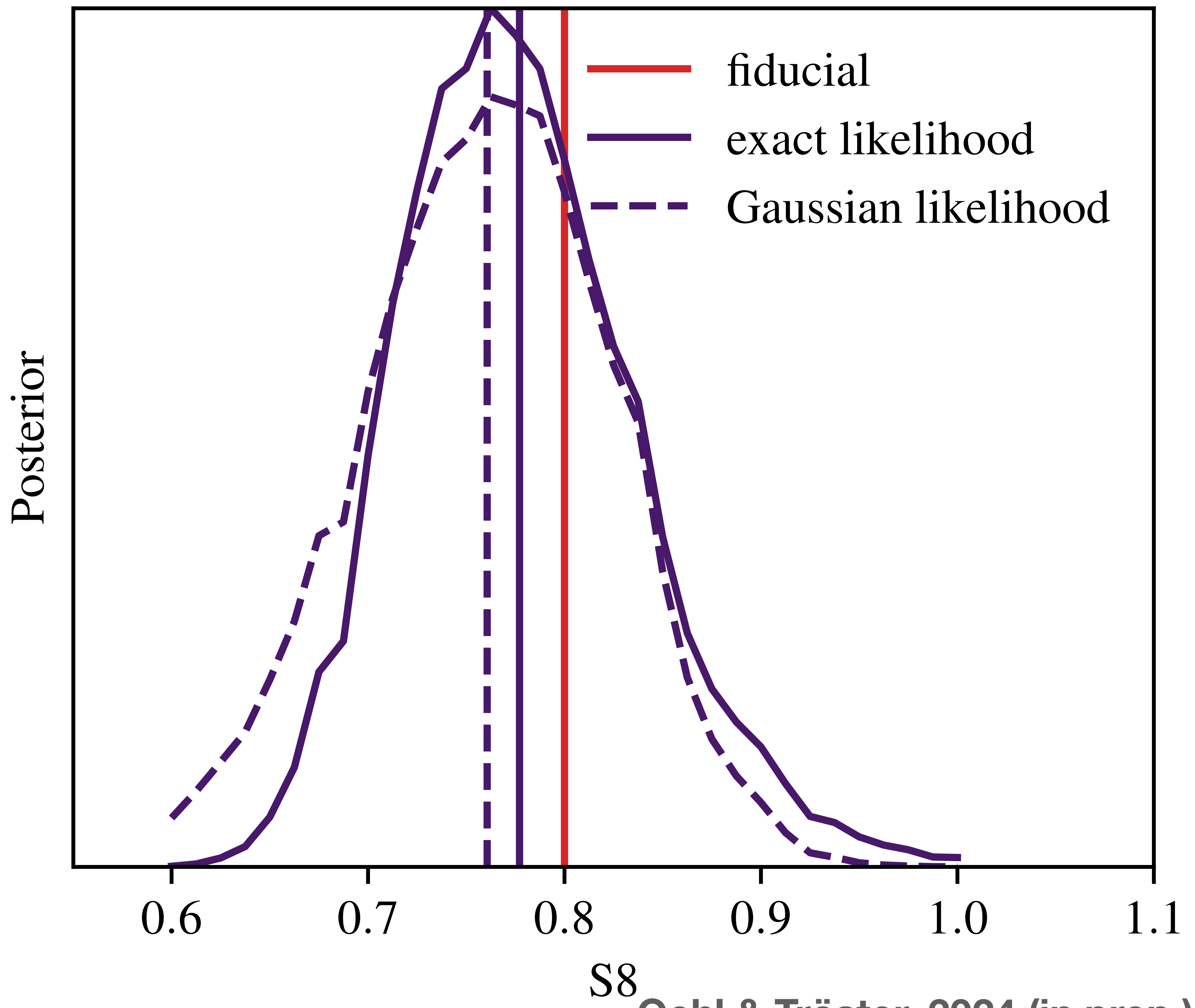
Comparison to Gaussian approximation



Oehl & Tröster, 2024 (in prep.)

Toy Model: Comparing S_8 Posteriors

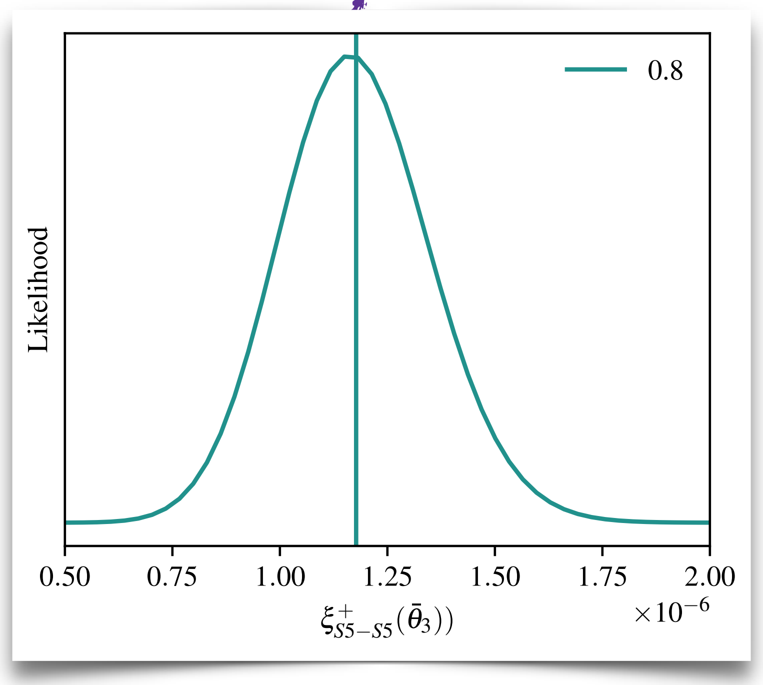
$$p(S_8 \,|\, \xi^{+}_{S5-S5}(\bar{\theta}_3)) \propto p(S_8)\; p(\xi^{+}_{S5-S5}(\bar{\theta}_3) \,|\, S_8)$$



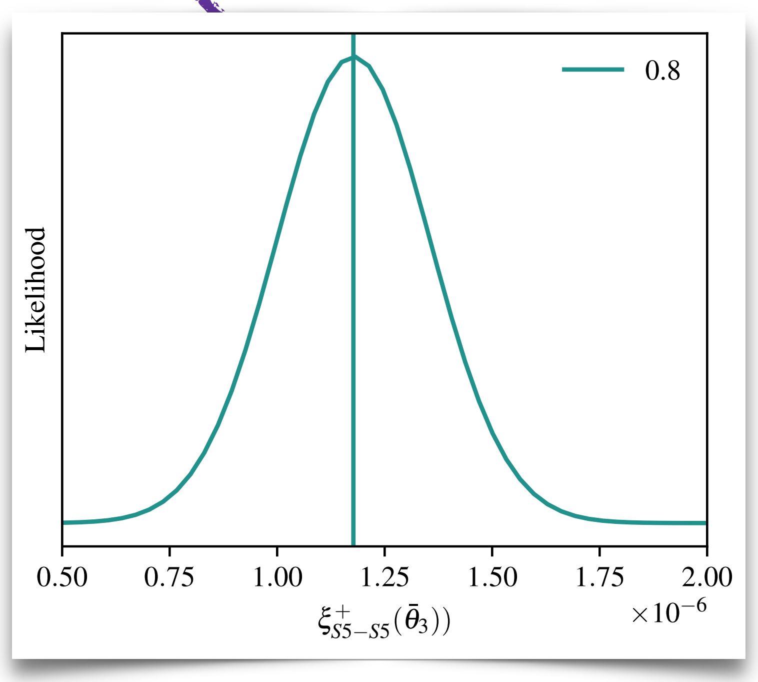
Oehl & Tröster, 2024 (in prep.)

likelihood

exact

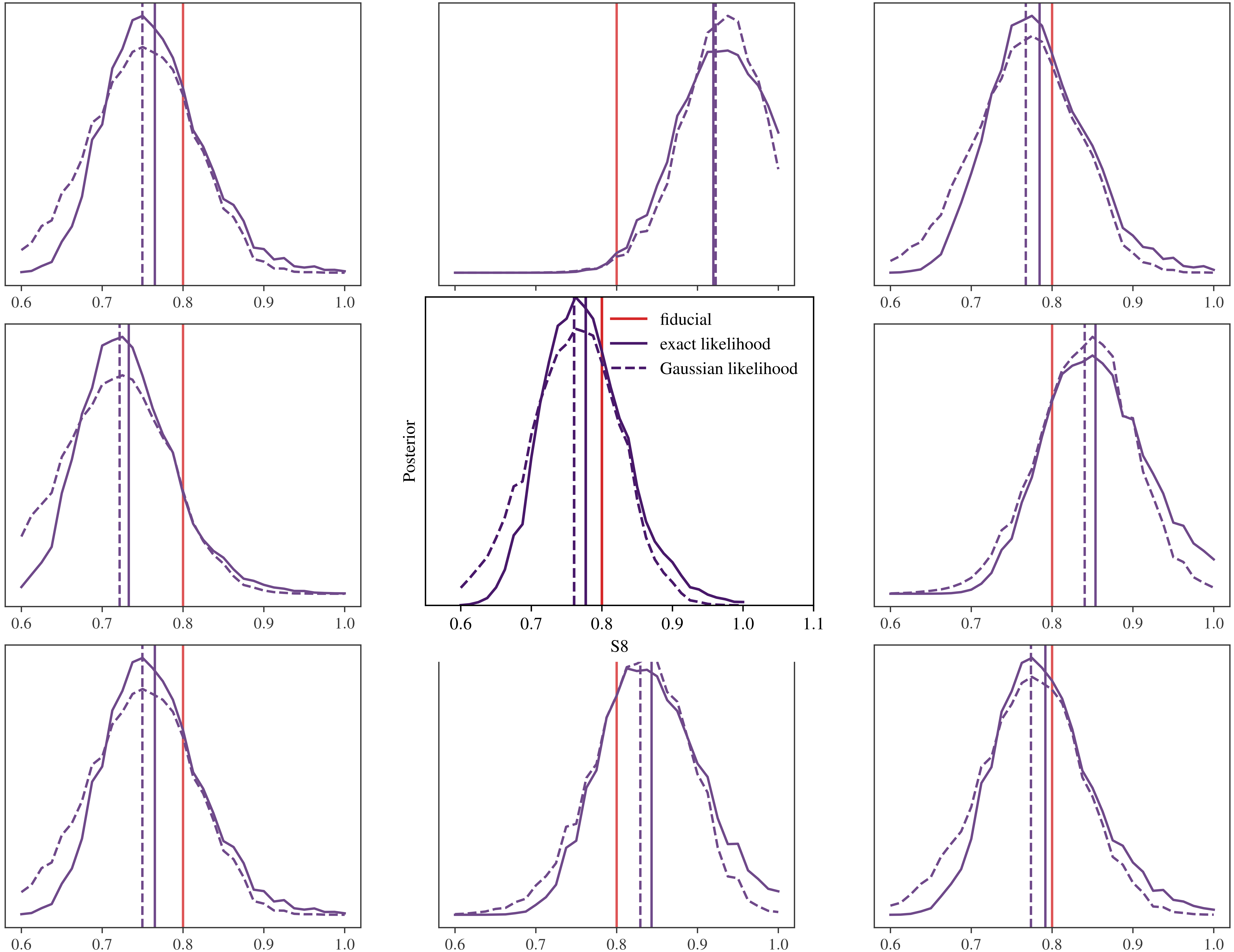


Gaussian



Measurement:

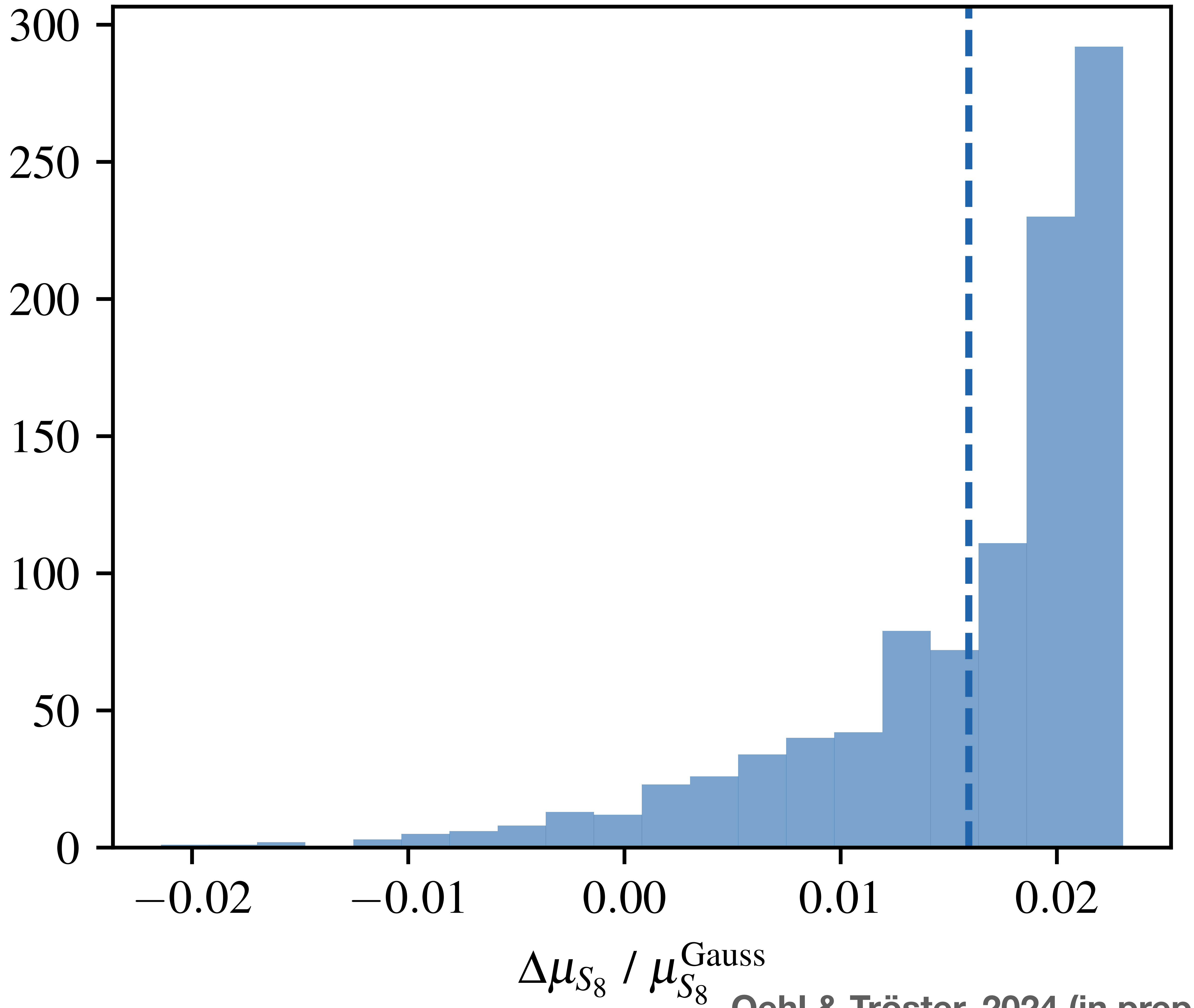
$$\xi^{+}_{S5-S5}(\bar{\theta}_3)) = 1.08 \times 10^{-6}$$



Veronika Oehl

Differences in the posterior mean

$$\Delta\mu_{S_8} = \mu_{S_8}^{\text{exact}} - \mu_{S_8}^{\text{Gauss}}$$



Oehl & Tröster, 2024 (in prep.)

Summary & Outlook

- Analytic expression for the nD likelihood of (cross-)correlation functions on the sphere
- Application to weak lensing setup
- 1D posterior prediction, 2D likelihoods possible

- How to make this computationally feasible?
- Evaluate impact on parameter constraints (for stage IV lensing surveys)