# Dualities in the UV

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# Introduction and motivation

Many of the realistic and well motivated (especially supersymmetric) grand unified theories have a Landau pole. They can thus be UV well defined only if they have a non-trivial fixed point in the UV

*Litim, Sannino, '14*

Supersymmetric theories which are free in the IR can exhibit asymptotic safety in the UV only in the non-perturbative regime

How to study such non-perturbative UV fixed points?

1. check all possible constraints that candidates must satisfy;

   *Martin, Wells, '01*

   *Intriligator, Sannino, '15*

   *BB, Sannino, '16*

2. find different descriptions, i.e. dual theories with different degrees of freedom and different interactions, but same physics; if lucky, this dual can be described perturbatively

   *Berkooz, Cho, Kraus, Strassler, '97*

   *Abel, Khoze, '09*

   However both are IR dualities, i.e. these theories are asymptotically UV free

Here I will describe our search for UV dualities among Seiberg-Kutasov type of theories (SQCD with extra adjoints)

Why supersymmetry?

1. in GUTs this is exactly the case where problems with Landau poles appear
2. non-perturbative physics is much easier to study
   - in SCFT the $R$-symmetry is part of the algebra:
   $$R(\phi) = \frac{2}{3} D(\phi)$$
   - dimension $D$ of a composite operator is a sum of dimensions of constituents
   $$D(\phi_1 \phi_2) = D(\phi_1) + D(\phi_2)$$
   - $R$-charges can be determined from superpotential constraints and vanishing of the NSVZ $\beta$-function
3. non-trivial dualities better known here although exceptions

*Francesco's talk yesterday*

Dual theories describe the same physics in different ways.

Example:

QCD vs. chiral perturbation theory ($\chi$PT)

They have the same global symmetries $SU(N_f) \times SU(N_f) \times U(1)_B$

but different gauge groups and fields

Not much can be said on $\chi$PT, it is just the most general expansion with a given global symmetry

Situation is a bit better for supersymmetric theories

Two comments before continuing:

- Here we will consider only $d = 4$, i.e. no extra dimensions

  *Cacciapaglia, Deandrea, Pasechnik, Wang, '23*

- Only field theory, i.e. no gravity (the Landau pole in fact often appears below $M_{Planck}$)

  *Kamila's and Enrico's talks yesterday*

# $\mathcal{N}=1$ supersymmetric dualities in $4d$

Simplest example: SQCD

| ELECTRIC | = | MAGNETIC |
| --- | --- | --- |
| $SU(N_c)$ | | $SU(N_f-N_c)$ |
| $N_f*(Q+\tilde{Q})$ | | $N_f*(q+\tilde{q})+N_f^2*M$ |
| $W=0$ | | $W=\tilde{q}qM$ |

In the conformal window

$$\frac{3}{2}N_c \le N_f \le 3N_c$$

both theories describe the same IR fixed point

*Seiberg, '94*

Many generalisations known.

For SQCD with additional adjoints

- SQCD with one adjoint ($A_k$)

*Kutasov, '95*

*Kutasov and Schwimmer, '95*

- SQCD with two adjoints ($D_{k+2}$ or $E_7$)

*Brodie, '96*

*Intriligator, Wecht, '03*

$k = 2, 3, \ldots$

We will find these dualities later on

Notice that there might be many more fixed points than dualities.

$$
\begin{aligned}
W \quad &\sim \quad X^n + X^p Y^m \\
R(W) = 2 \quad &\to \quad R(X) = \frac{2}{n} \quad , \quad R(Y) = \frac{1}{m}\left(2 - p\frac{2}{n}\right)
\end{aligned}
$$

Not all $n, p, m$ have known duals

*Intriligator, Wecht, '03*

We will constrain ourselves to rational adjoints $R$-charges in the interval between 0 and 2

Global symmetries: $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$

Why do we believe these theories are dual?

Several checks:

- 't Hooft anomaly matching
- same numbers of mesons and baryons
- deformations of the theories leads to new dualities of similar type

A non-trivial unifying description of many checks is the equality of superconformal indices

*Romelsberger, '05*

*Kinney, Maldacena, Minwalla, Raju, '05*

## Superconformal index and dualities

It counts BPS states (short multiplets under the action of $\mathcal{N}=1$ susy generators on $S^3 \times \mathbb{R}$), so it must be the same for the electric and magnetic theories.

This has been rigorously (mathematically) proven only for special dualities, for example SQCD

*Rains, math.QA/0309252*

Its computation simplifies considerably in the large $N_c$ limit

*Dolan, Osborn, '08*

*Kutasov, Lin, '14*

The electric theory has a gauge group $\mathrm{SU}(N_c)$ with $N_f$ fundamentals ($Q$) plus antifundamentals ($\tilde{Q}$) and $N_A$ adjoints $X_a$

The magnetic theory has a gauge group $\mathrm{SU}(\tilde{N}_c)$ with $N_f$ fundamentals ($q$) plus antifundamentals ($\tilde{q}$), $N_A$ adjoints $X_a$ and $\alpha$ mesons $M_j$

$$\tilde{N}_c = \alpha N_f - N_c$$

The analysis simplify tremendously in the Veneziano limit $N_c, N_f \to \infty$ with

$$x \equiv \frac{N_c}{N_f} \quad \textit{fixed}$$

$R_a$ will be the same for electric and magnetic theory, and the equivalence of the index boils down to ($R_{N_A+1} = 2$)

$$\sum_{j=1}^{\alpha} t^{R_j} = \frac{t^{2\alpha \sum_{a=1}^{N_A+1} (R_a - 1)} - 1}{\sum_{a=1}^{N_A+1} \left( t^{R_a} - t^{2-R_a} \right)}$$

This is our MASTER EQUATION

It must be satisfied for any value of $0 < t < 1$.

Solutions:

- $N_A$ values of $R_a$ (assumed to be rational and $\in ]0, 2[$)
- $\alpha$ values of $R_j$ (mesons $R_{M_j} = 2R_Q + R_j$)

# Dualities in the UV

These dualities are typically valid at the fixed point, no scale is present. Usually they thought are IR dualities. Could we use them as UV dualities as well?

If we look to them as asymptotic relations, i.e. in the far UV or far IR, coming from non-conformal theories, then a further constraints must be satisfied:

$$a_{UV} \geq a_{IR}$$

*Zamolodchikov, '86*

*Cardy, '88; Osborn, "89; Jack, Osborn, '90*

*Komargodski, Schwimmer, '11*

The $a$-central charge must increase towards the ultraviolet: it counts the degrees of freedom

So the flow is irreversible. If we have two connected fixed points, the flow can go only from fixed point with higher central charge to the one with lower central charge.

So for each fixed point with central charge $a$ we check if it is UV or IR by computing

$$a - a_{free}$$

where $a_{free}$ is the central charge of the free theory with all chiral multiplets $R_i = 2/3$

If negative, $a$ is the central charge of IR fixed point (asymptotically free theory)

If positive, $a$ is the central charge of UV fixed point (asymptotically safe theory)

There is a prescription how to calculate this central charge in susy:

$$a = \sum_i a_1(R_i)$$

with

$$a_1(R) = 3(R-1)^3 - (R-1)$$

and $R_i$ the $R$-charge of the superfield $i$

Example: simplest Seiberg duality, electric theory:

$$a = (N_c^2 - 1)a_1(2) + 2N_f N_c a_1(R_Q)$$

From NSVZ: $R_Q = 1 - N_c/N_f$

$$a_{free} = (N_c^2 - 1)a_1(2) + 2N_f N_c a_1(2/3)$$

$a - a_{free} < 0$ always in the conformal window

$\rightarrow$

Seiberg SQCD duality is a IR duality

This test is nontrivial and makes the game difficult

Of course SQCD in the conformal window cannot be UV fixed point also because the theory is asymptotically free (i.e. UV free):

$$b_1 = 3N_c - N_f > 0$$

So the computation of the $a$-central charge is rather a test: to have a UV fixed point duality we need

1. $b_1 < 0$ for both electric and magnetic dual
2. $a - a_{free} > 0$ for at least one among the electric and magnetic duals

## The program

What we thus need is to

1. find a duality solving the master equation where both electric and magnetic theories have $b_1 < 0$
2. check that at least one among electric and magnetic has the $a$-central charge bigger than its free version

Notice that the dual theories have the same central charge by construction, but their free versions do not, i.e.

$$a_{electric} = a_{magnetic} \quad but \quad a^{free}_{electric} \neq a^{free}_{magnetic}$$

$$\rightarrow \qquad \Delta a_{electric} \neq \Delta a_{magnetic}$$

# $N_A = 1$

Solution well known

*Kutasov '95*

$b_1^{electric} = 2N_c - N_f < 0$ if

$$N_f > 2N_c$$

$b_1^{magnetic} = 2\tilde{N}_c - N_f = (2\alpha - 1)N_f - 2N_c < 0$ if

$$N_f < 2N_c/(2\alpha - 1)$$

Since the only know solution has $\alpha > 1$ the two constraints are incompatible $\rightarrow$ no solution with $N_A = 1$

# $N_A = 2$

Various solutions known

*Intriligator, Wecht '03, Kutasov, Lin '14, BB '19*

$b_1^{electric} = N_c - N_f < 0$ if

$$N_f > N_c$$

$b_1^{magnetic} = \tilde{N}_c - N_f = (\alpha - 1)N_f - N_c < 0$ if

$$N_f < N_c/(\alpha - 1)$$

Here again all known solutions have $\alpha \geq 2$

$\to$ no solutions with $N_A = 2$

For higher $N_A > 2$ there are no examples in the literature so we need to find new solutions to the master equation.

We will follow the method of cyclotomic polynomials

*B.B., '19*

# How to find solutions of the master equation

Since $R_a$ are rational numbers one can always define a new variable

$$t^2 = y^m$$

so that numerator of r.h.s.

$$t^{2\alpha \sum_{a=1}^{N_A+1}(R_a-1)} - 1 = y^n - 1$$

with $n$ integer.

So our equation

$$\sum_{j=1}^{\alpha} t^{R_j} = \frac{t^{2\alpha \sum_{a=1}^{N_A+1}(R_a-1)} - 1}{\sum_{a=1}^{N_A+1}\left(t^{R_a} - t^{2-R_a}\right)}$$

can be rewritten as

$$\Phi_+(y) = \frac{y^n - 1}{\Phi_-(y)}$$

where

$\Phi_+(y)$ is a polynomial with only positive coefficients

$\Phi_-(y)$ is antipalindromic ($m = p_{N_A+1}$)

$$\Phi_-(y) = \sum_{j=0}^{m} a_j y^j \quad , \quad a_j = -a_{m-j}$$

The classification of possible dualities of our type is reduced to all possible factorisations of

$$y^n - 1$$

into two polynomials with integer coefficients

- one with only positive coefficient $\Phi_+(y)$
- the other anti-palindromic $\Phi_-(y)$

This can be made more systematic with cyclotomic polynomials

# Cyclotomic polynomials

There is a unique way to factorise

$$y^n - 1$$

as products of polynomials with non-negative powers and integer coefficients:

$$y^n - 1 = \prod_{d|n} \Phi_d(y)$$

where $d$ are all the divisors of $n$ and $\Phi_d(y)$ is the $d^{th}$ cyclotomic polynomial

The first few are

$$
\begin{aligned}
\Phi_1(y) &= y - 1 \\
\Phi_2(y) &= y + 1 \\
\Phi_3(y) &= y^2 + y + 1 \\
\Phi_4(y) &= y^2 + 1 \\
\Phi_5(y) &= y^5 + y^4 + y^3 + y^2 + y + 1 \\
\Phi_6(y) &= y^2 - y + 1 \\
&\dots
\end{aligned}
$$

Recursively

$$\Phi_n(y) = \frac{y^n - 1}{\prod_{d|n, d<n} \Phi_d(y)}$$

We can now define sets $\mathcal{D}_n^{\pm}$ so that

$$\mathcal{D}_n^{+} \cup \mathcal{D}_n^{-} = \{d; d|n\} \ \ , \ \ \mathcal{D}_n^{+} \cap \mathcal{D}_n^{-} = \emptyset$$

$$\Phi_{\pm}(y) = \prod_{d \in \mathcal{D}_n^{\pm}} \Phi_d(y)$$

such that

$\Phi_+(y)$ is a product of cyclotomic polynomials which has only non-negative coefficients

$\Phi_-(y) = (y^n - 1)/\Phi_+(y)$

Each of these solutions is a candidate for a duality

From

$$\Phi_-(y) = \sum_{a=1}^{N_A+1} \left(y^{p_a} - y^{p_{N_A+1}-p_a}\right)$$

$$\Phi_+(y) = \sum_{j=1}^{\alpha} y^{q_j}$$

$\rightarrow$

- adjoints R-charges

$$R_a = \frac{2}{p_{N_A+1}} p_a \quad , \quad a = 1, \ldots, N_A$$

- mesons R-charges

$$R_j = \frac{2}{p_{N_A+1}} q_j \quad , \quad j = 1, \ldots, \alpha$$

## $N_A = 3$

Using the above method of cyclotomic polynomials we found two families of solutions with three adjoints $X$, $Y$, $Z$. I will describe here only one, the second one has similar conclusions.

Let $p$, $m$ be positive integers:

$$(R_X, R_Y, R_Z) = \frac{(2, 4p+2, 8pm+2)}{4mp+2p+1}$$

and mesons

$$R_{M_{j,i}} = 2R_Q + \frac{2}{2k+1}\left(4p(j-1)+(i-1)\right)$$

$$j = 1, \ldots, m \quad , \quad i = 1, \ldots, 2p \quad (\alpha = 2pm)$$

Here both electric and magnetic

$$b_1 = -N_f < 0$$

so this is satisfied.

We will now show that $\Delta a < 0$ always, so these UV fixed points cannot be connected with the free theory.

Introduce

$$x = \frac{N_c}{N_f}$$

Let us first prove that

$$\Delta a_{el} < 0$$

for $0 < x < \alpha \ \ (= 2pm)$:

$$\Delta a_{el} = x\left(ax^3 + bx + c\right)$$

with

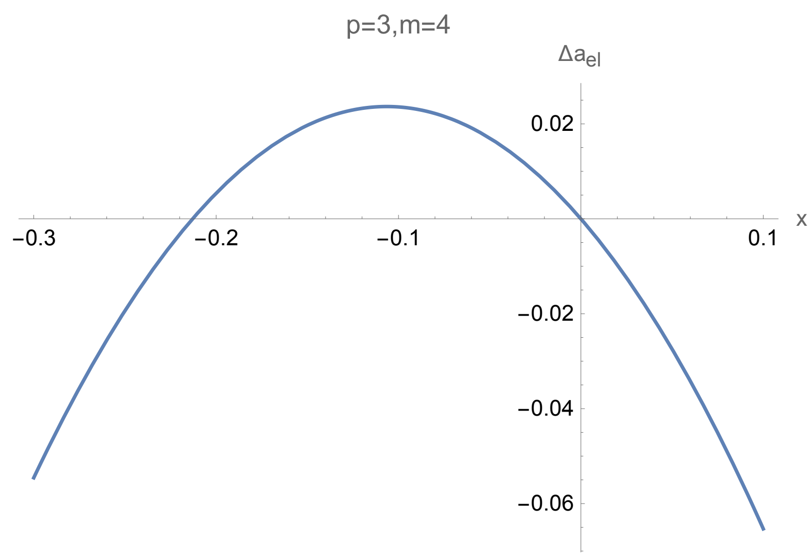
$$\begin{aligned} a &= -\frac{384}{((4m+2)p+1)^3} < 0 \\ b &= \frac{8\left(24\left(8m^2 - m + 2\right)p^2 - 8(2mp+p)^3 + 5\right)}{3((4m+2)p+1)^3} \\ c &= -\frac{4}{9} < 0 \end{aligned}$$

Since

$$\Delta \equiv -4ab^3 - 27a^2c^2 < 0$$

for all integer positive $p$, $m$, there are only two zeros of $\Delta a_{el}$, i.e. for $x = 0$ and $x < 0$ (because $c < 0$)

Example:



Finally, since $a < 0$ we have

$$\lim_{x\to\infty} \Delta a_{el} < 0$$

and so

$\to \Delta a_{el} < 0$ for all positive integer $p$, $m$ and $0 < x < 2pm$

QED

Next, what about $\Delta a_{mag}$ ? Can it be positive for some $x$?

The analysis here is a bit more complicated, but a combination of explicit computation for low parameters $p$, $m$ and an expansion in large $p$, $m$ shows that

$\Delta a_{mag} < 0$ in the whole range $0 < x < 2pm$.

In conclusion this solutions cannot describe UV dualities (because $\Delta a < 0$)

but neither IR dualities (because $b_1 < 0$ - they are IR free)

# Conclusions

- Asymptotic safety is a possible UV completion of theories with perturbative Landau poles (for example in supersymmetric GUTs)
- Supersymmetric safe theories are intrinsically non-perturbative. To be able to to compute something we look for dual formulations of such UV fixed points which will hopefully be perturbative.
- Here we checked a particular family of toy models, following the generalisation of the Kutasov-Seiberg dualities (SQCD with adjoints). The result is so far negative, no such duality in a consistent UV fixed point has been found.