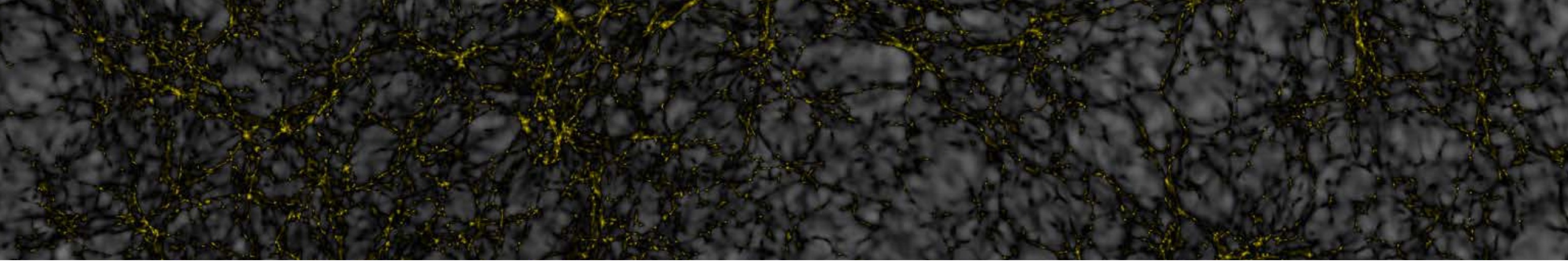


Redshift calibration with angular clustering for DES and Euclid

(multi-surveys, mocks, small scales, tracers)

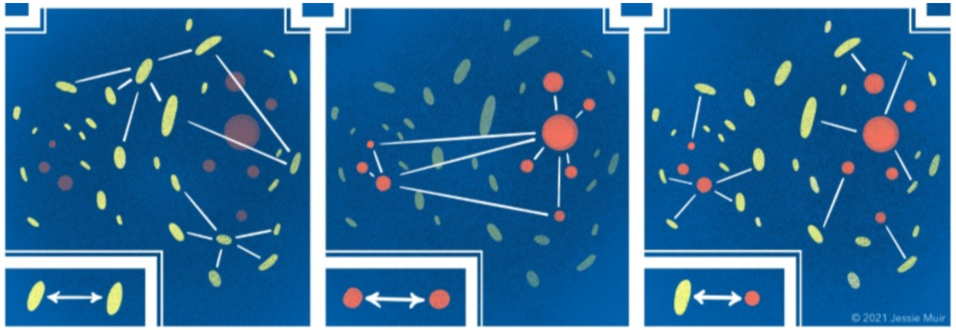
William d'Assignies D.
Cristobal Padilla, Marc Manera
and A. Alarcon, C. Sànchez, G. Bernstein, J. Chavez

Institut de Física d'Altes Energies

THE DARK ENERGY SURVEY

Euclid consortium

The redshift calibration challenge

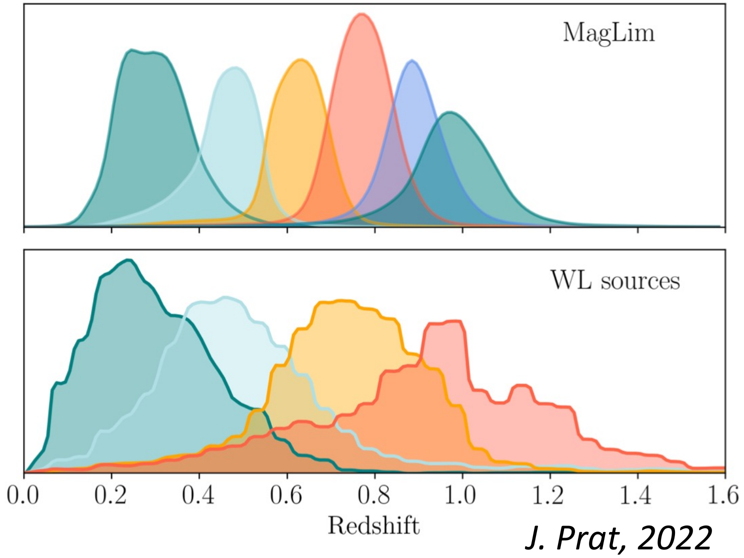


@Jessie Muir

- Photometric surveys: we are measuring a bunch of 2-points correlation functions (3x2) for different bins
- To infer cosmological constrain, we need to model these measurements.
- A key ingredient is the redshift distribution.
- We need to estimate them, their uncertainties, and marginalise to avoid biasing cosmology.

Redshift distribution of source bin i

$$W_\kappa^{(i)}(\chi) = \frac{3\Omega_{\mathrm{m0}}}{2\chi_{\mathrm{H}}^2}\frac{\chi}{a(\chi)}\int_\chi^{\chi_{\mathrm{H}}} \mathrm{d}\chi' n_{\mathrm{s}}^{(i)}(\chi')\frac{\chi'-\chi}{\chi'} .$$

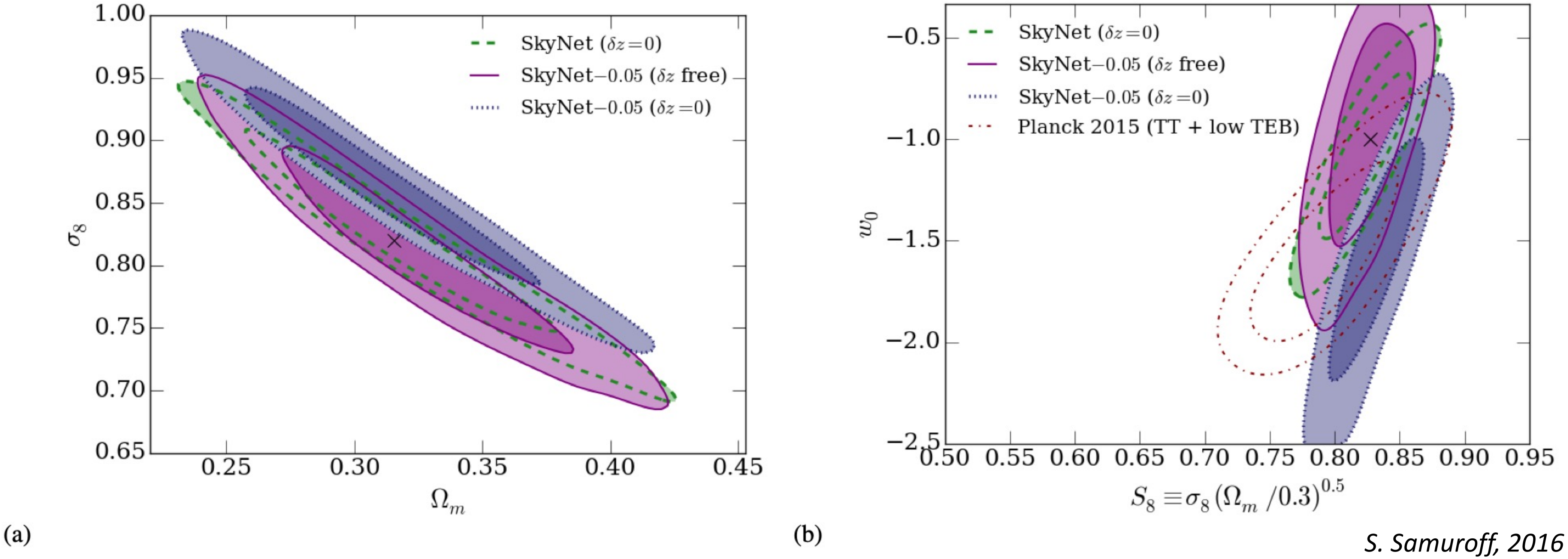


J. Prat, 2022

- Requirement on z-calibration:
  - Euclid: the mean-z of every tomographic bin: $\boldsymbol{\sigma_{\langle z\rangle_i} = 0.002\times(1+z)}$
  - Additional req. on the 2nd moment, but not 'official' req.

The redshift calibration challenge

- Here, GGL and GC, 3 scenarios:
  - True-z, and no-z-uncertainty (green)
  - Mean-z bias and z-uncertainty (purple)
  - Mean-z bias and no-z-uncertainty (blue)



S. Samuroff, 2016

Two main methods:

- SOM (self organised maps)
- Cz (clustering-redshifts)

Combination of the two methods for DES

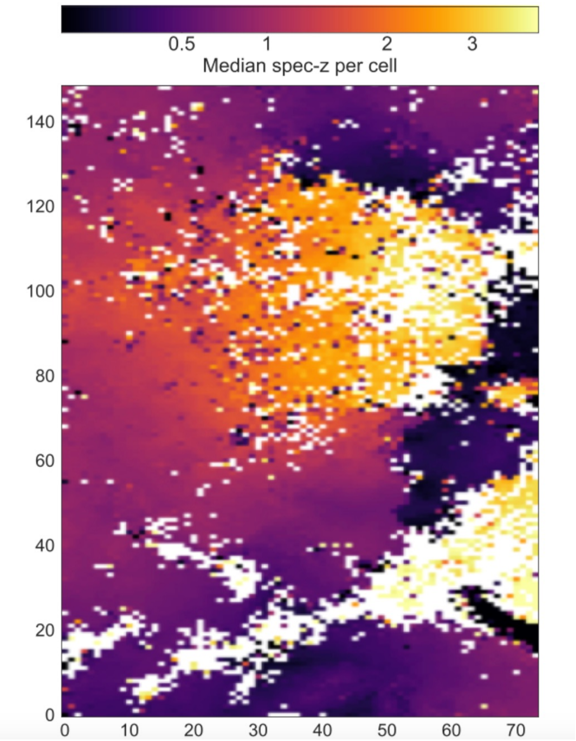
The two methods for Euclid

Two main methods:

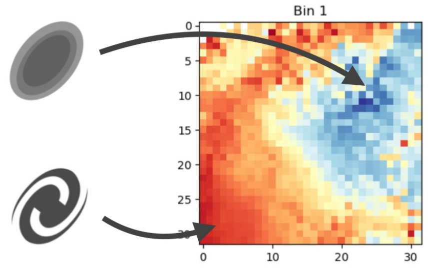
1-Self Organised Map in mag (SOM)

KIDS: H. Hildebrandt 2020, A. Wright 2020
DES: G. Gianninin 2022, A. Alarcon 2022, A. Campos 2024.
Euclid: W. Roster in prep

- **SOM:** a machine learning technique, to categorize N-Dim vectors into a 2D map
- **Photo-z**: dimensions are fluxes, gal. with similar fluxes (and so z) are 'close' in the SOM

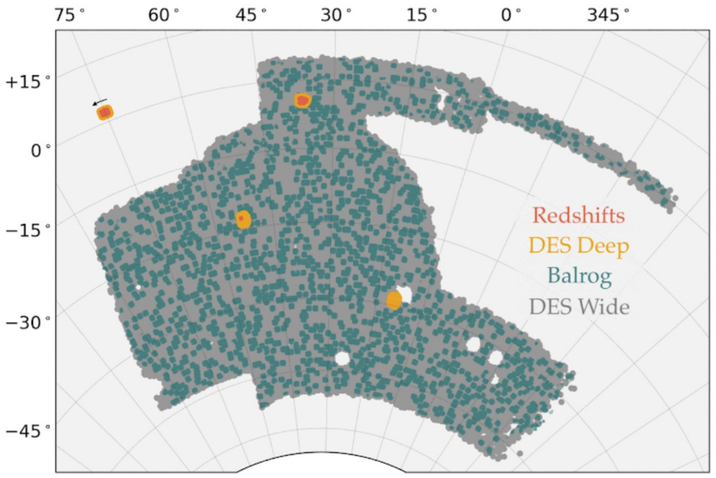


Stanford et al. (2021)



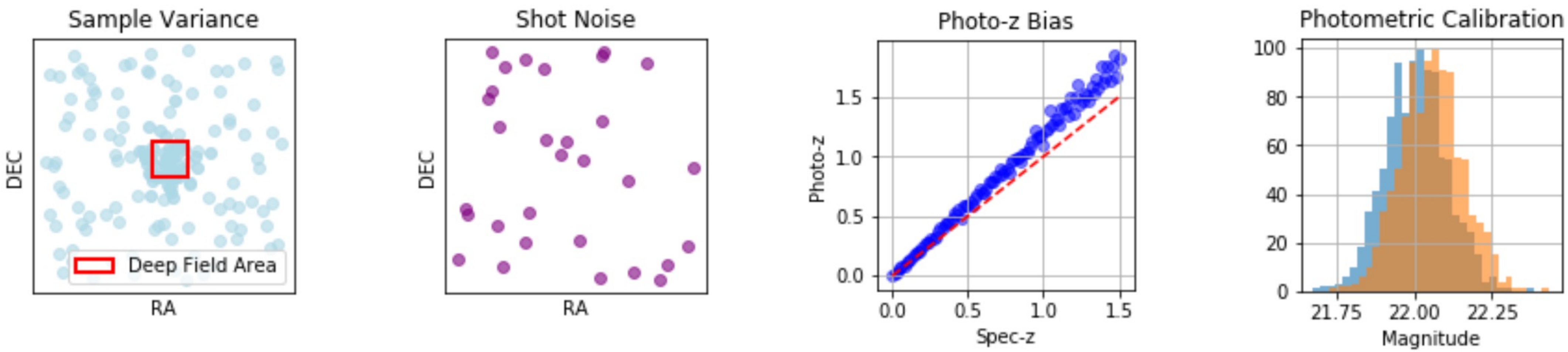
Credit: G. Giannini

- Trained with a calibration sample (spectroscopic, or high quality photo-z like COSMOS)
- These calibration samples are not complete nor representative
- Crucial step of re-weighting this calibration sample with the WL one. (Deep survey)

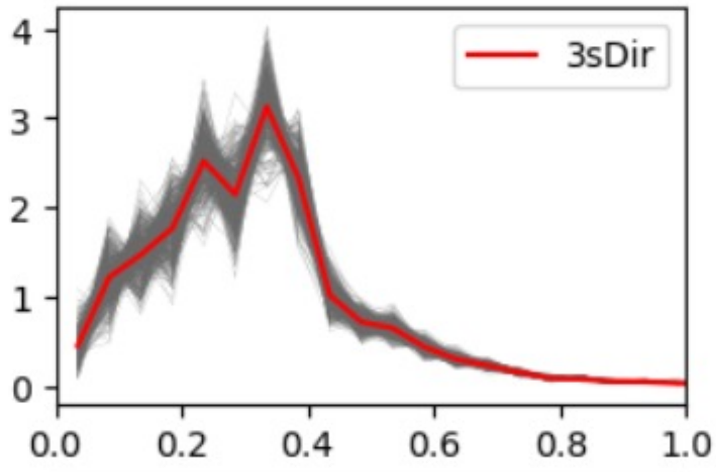


Credit C. Sánchez

Self-Organised-Maps: uncertainties

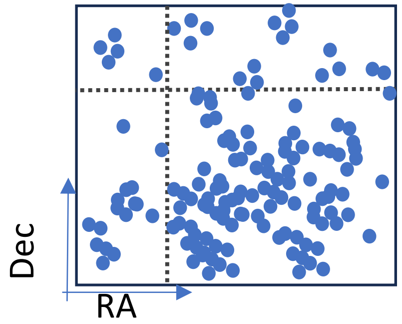


- In DES: The product of the SOM calibration is a **set of n(z) realisations** whose overall variance span all the uncertainties included in the SOMPZ methodology:

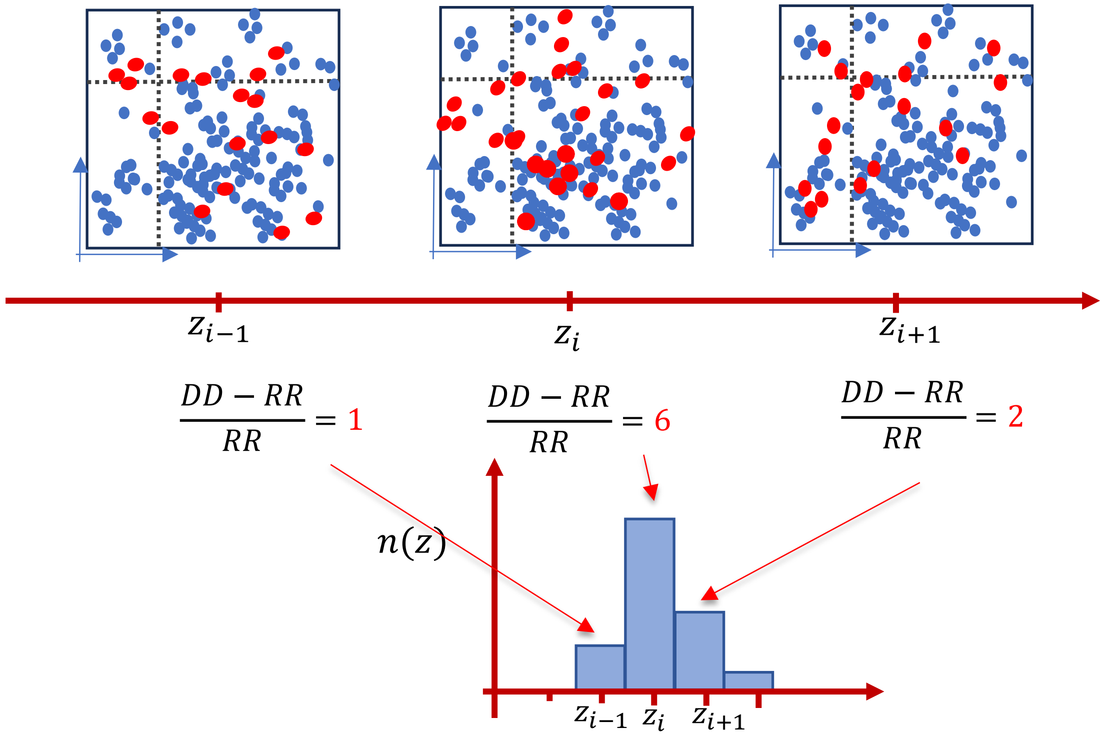


Two main methods:

2- Clustering redshifts



- **A photometric galaxy sample** is spread over a large redshift range
- Idea : Correlate it with **many spectroscopic samples** at different $-z$, compare random expectation (RR)
- Galaxies are tracing the DM field, so we measure the fraction of photo-gal. that are tracing the same DM field as the spec-gal.



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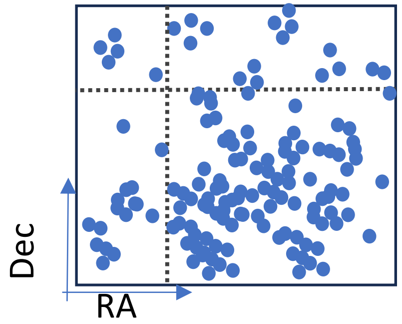

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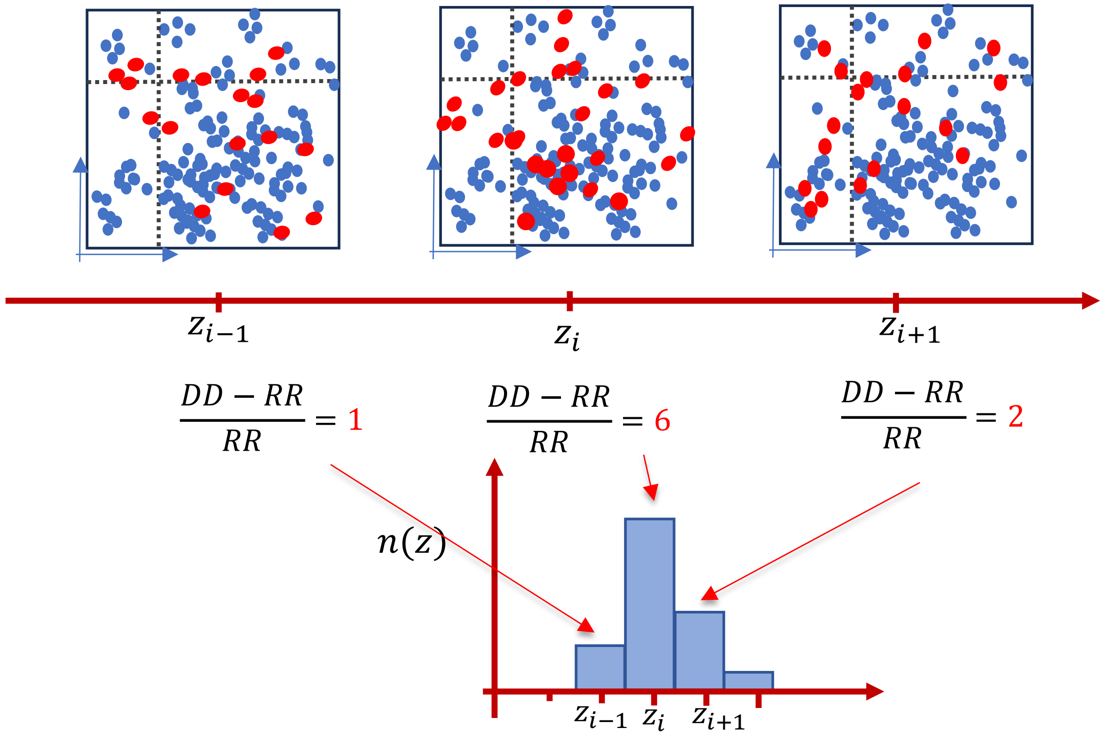

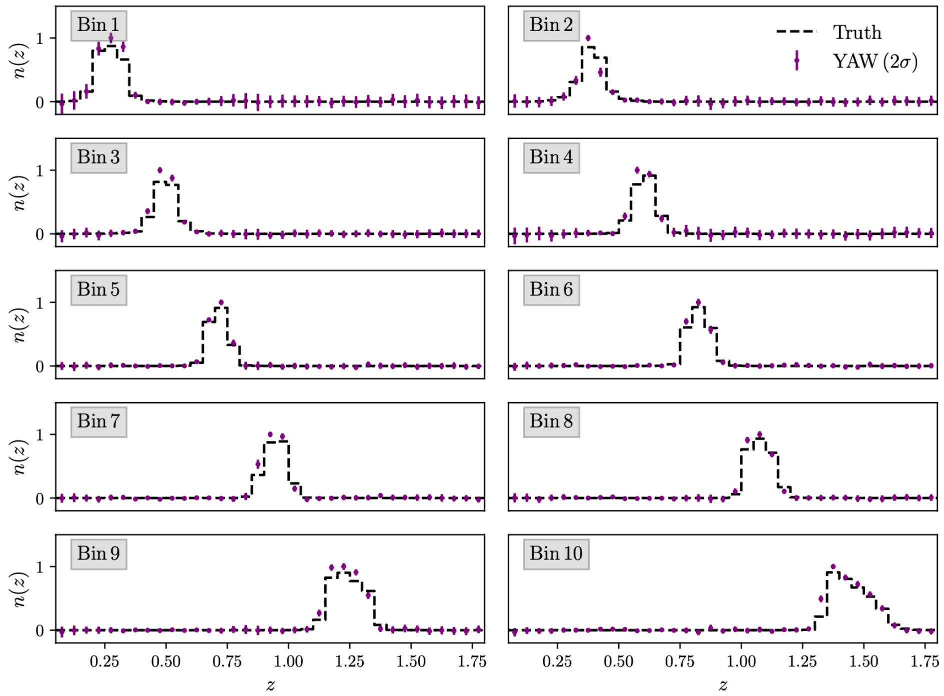

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- SOM (self organised maps)
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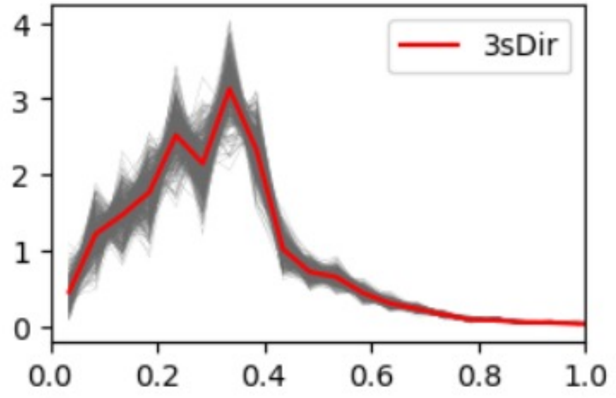
Combination of the two methods for DES

The two methods for Euclid

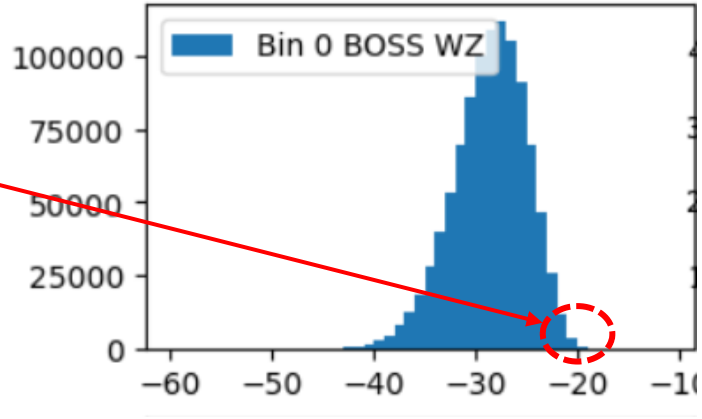
SOM x Cz

G. Bernstein, M. Troxel, A. Amon, B. Yin, A. Alarcon, W. d'Assignies

- In DES: SOM calibration → a **set of n(z) realisations** whose overall variance span all the uncertainties in the SOMPZ methodology:



- Idea of SOMx Cz: Assign to each of these realisations a Cz likelihood, and select the best !



$\log(L(CZ))$

Big set of SOM $n(z)$ — $\{n_i(z)\}_i$ ⟶ $\mathcal{L}\,(n_i|\,Cz)$ ⟶ $\{n_j(z)\}_j$ — Subset of SOM $n(z)$ with high-Cz L

Evaluate and select the best subset

Cz Likelihood

G. Bernstein, M. Troxel, A. Amon,
B. Yin, A. Alarcon, W. d'Assignies

THE DARK ENERGY SURVEY

- One can model Cz :

$$\omega_{\mathrm{mod}}(z_i) = n_p(z_i)\, b_{\mathrm{s}}(z_i)\, b_{\mathrm{p}}(z_i)\, \omega_{\mathrm{dm}}(z_i) \left(1 + \sum_k \mathrm{Sys}_{z_i,\,k} \cdot s_k\right) + \sum_k M_{z_i,k} \cdot p_k$$

Systematics

Magnification

Cz Likelihood

G. Bernstein, M. Troxel, A. Amon,
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Systematics   Magnification

- One can write the Likelihood of our Cz given a $n_p(z)$:

$$\mathcal{L}(\mathrm{C}z|n_p) = \int d\mathbf{s}\, d\mathbf{p}\, \exp\left(-\frac{1}{2}(\omega_{\mathrm{sp}} - \omega_{\mathrm{mod}})^\top \Sigma_{\mathrm{sp}}(\omega_{\mathrm{sp}} - \omega_{\mathrm{mod}})\right) p(s)\, p(p)$$

Marginalisation over sys. and magn.

Cz Likelihood

G. Bernstein, M. Troxel, A. Amon, B. Yin, A. Alarcon, W. d'Assignies

THE DARK ENERGY SURVEY

- One can model Cz :

$$\omega_{\mathrm{mod}}(z_i) = n_p(z_i)\, b_{\mathrm{s}}(z_i)\, b_{\mathrm{p}}(z_i)\, \omega_{\mathrm{dm}}(z_i) \left(1 + \sum_k \mathrm{Sys}_{z_i,\,k} \cdot s_k\right) + \sum_k M_{z_i,k} \cdot p_k$$

Systematics Magnification

- One can write the Likelihood of our Cz given a $n_p(z)$:

$$\mathcal{L}(\mathrm{Cz}|n_{\mathrm{p}}) = \int \mathrm{d}\mathbf{s}\, \mathrm{d}\mathbf{p}\, \exp\left(-\frac{1}{2}(\omega_{\mathrm{sp}} - \omega_{\mathrm{mod}})^{\top} \Sigma_{\mathrm{sp}} (\omega_{\mathrm{sp}} - \omega_{\mathrm{mod}})\right) p(s)\, p(p)$$

Marginalisation over sys. and magn.

- Since sys and magn parameters act linearly, one can solve analytically
- Numerically very fast.

$$\mathcal{L}(\mathrm{Cz}|n_{\mathrm{p}}) \propto |\alpha|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\gamma - \beta^{\top} \cdot \alpha^{-1} \cdot \beta)\right)$$

$$\mathrm{Sys}(z,\, s) = \sum_k a_k\, \mathcal{P}_k(u(z))\, s_k$$

$$q = (s,\, p),$$

$$c_i = \omega_{wz}(z_i) - n(z_i)\, b_r(z_i)\, \omega_{\mathrm{DM}},$$

$$A = \left(\quad (n(z_i)\, b_r(z_i)\, \omega_{\mathrm{DM}}(z_i)\, a_k\, \mathcal{P}_k(u(z_i)))_{i,k} \quad | \quad M_{z_i,\,k} \quad \right)$$

$$\alpha = A^{\top} \Sigma_{\mathrm{wz}}\, A + \Sigma_q$$

$$\beta = A^{\top} \Sigma_{\mathrm{wz}} \cdot c + \Sigma_q \cdot \mu_q$$

$$\gamma = c^{\top} \cdot \Sigma_{\mathrm{wz}} \cdot c + \mu_q^{\top} \cdot \Sigma_q \cdot \mu_q.$$

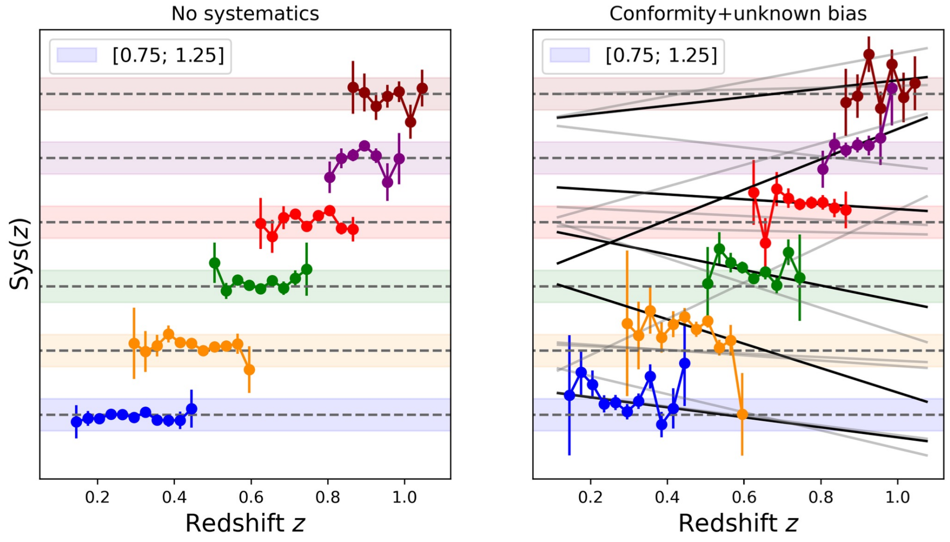
Cz Likelihood

G. Bernstein, M. Troxel, A. Amon, B. Yin, A. Alarcon, W. d'Assignies

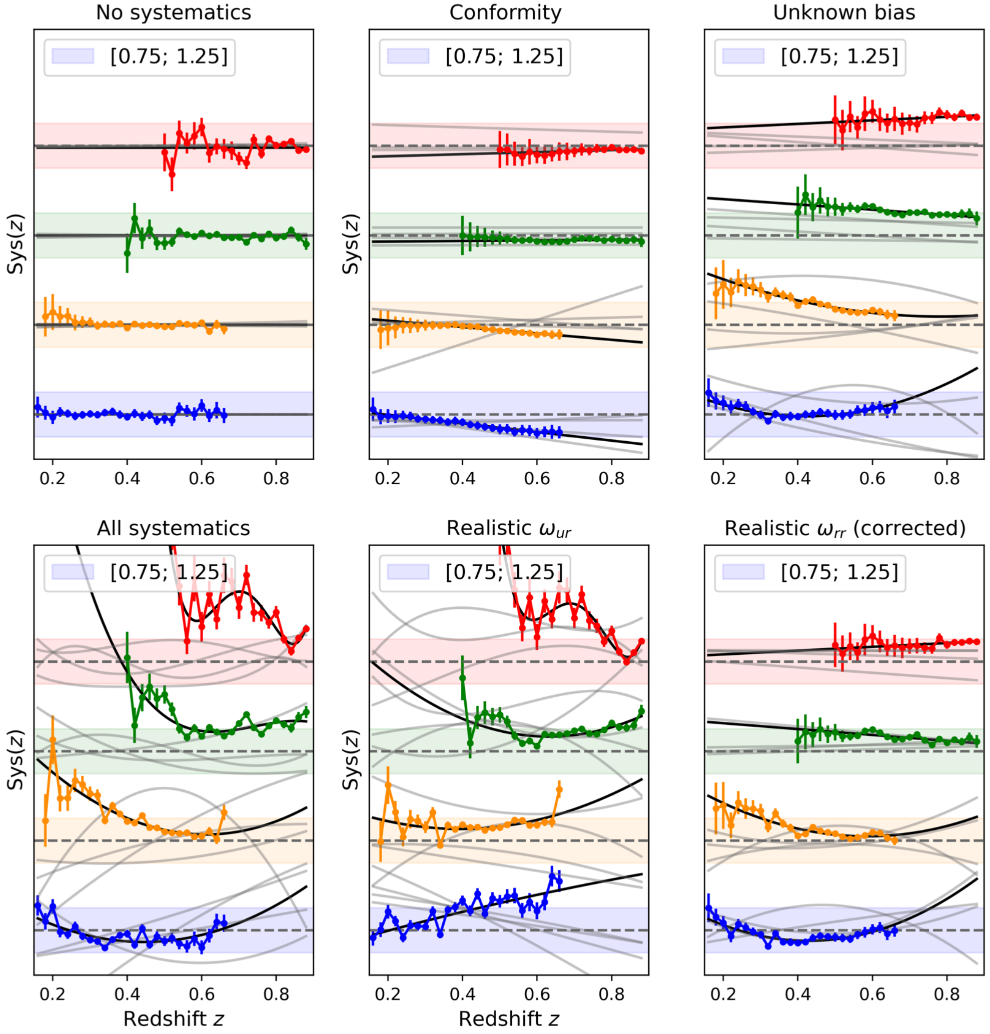
- Prior on the nuisance parameters **s** of the **Sys(z, s)** are estimated in simulations
- If independant systematics, add them linearly : **Sys(z, s) =** $\sum_i$ **Sys (z, s_i)**

$$\mathrm{Sys}(z,\, s) = \sum_k a_k\, \mathcal{P}_k(u(z))\, s_k$$

Maglim lenses

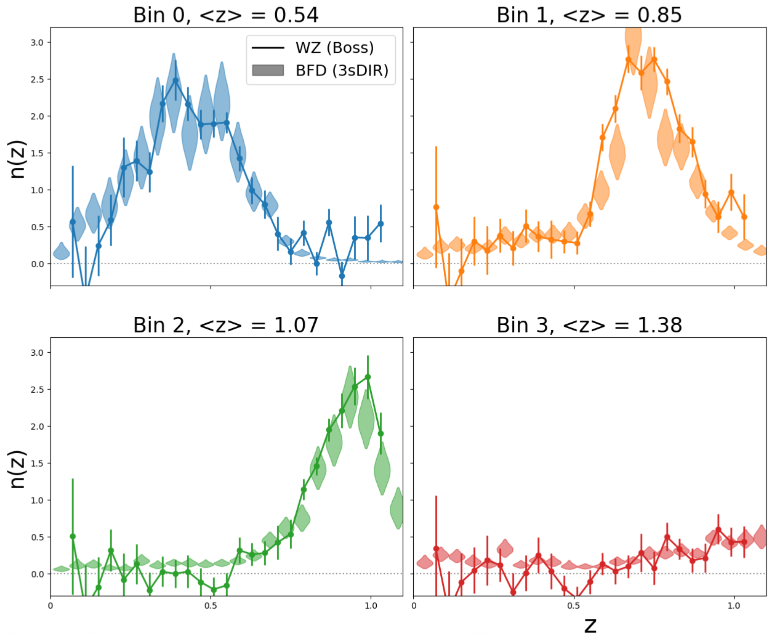


DES sources



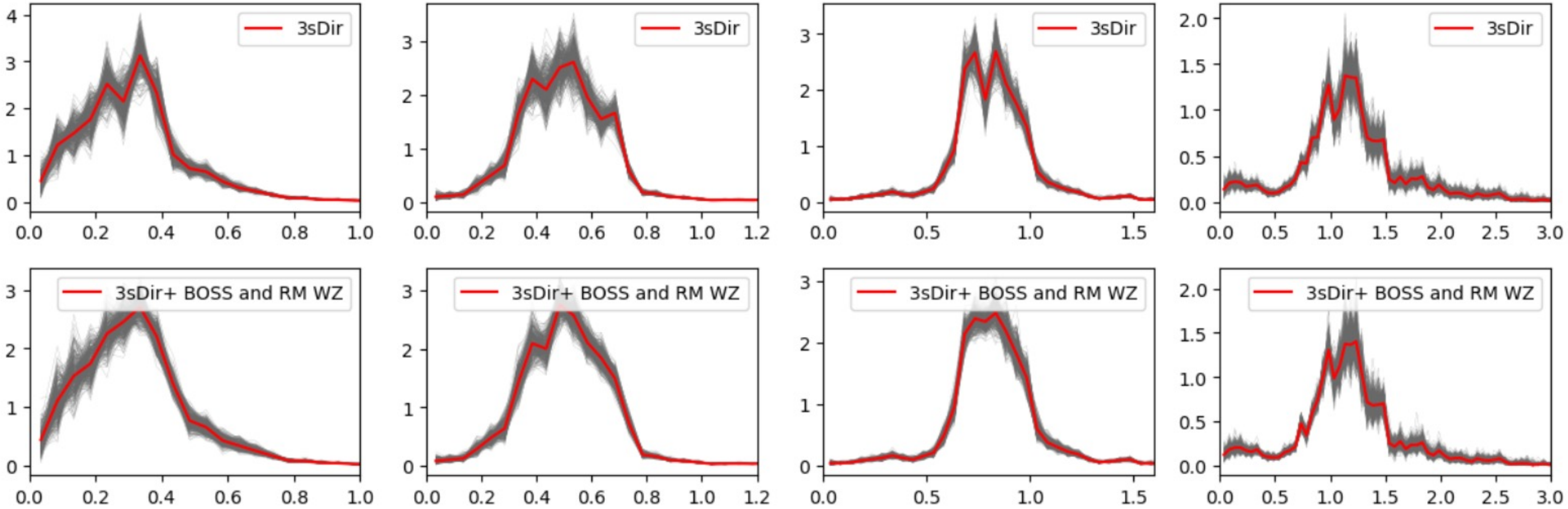
SOM and WZ joint constrains:

G. Bernstein, M. Troxel, A. Amon,
B. Yin, A. Alarcon, W. d'Assignies



- 4 DES source bin.
- Top panel, $n(z)$ generated with the SOM
- Bottom, $n(z)$ generated with the SOM, with high- Cz- Likelihood.
- Remove fluctuations and individual $n(z)$ are -smoother-

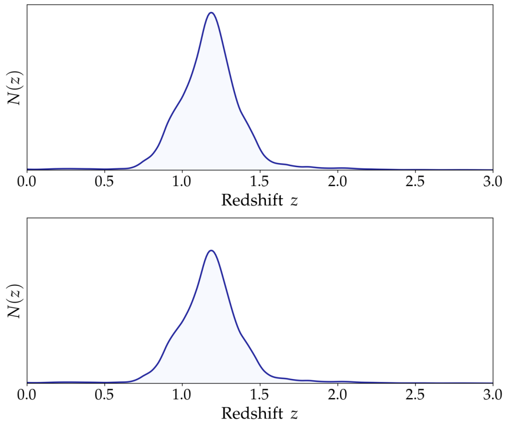
Preliminary



Redshift uncertainty marginalisation for cosmology

- The results of the SOM x WZ is a set of $n(z)$ realisations.
- Computationally we can't sample each realisation at each step of the inference process

- Need to explore the photo-z uncertainty thought a reduce small set of parameter.
- Usually it us a 'shift and stretch ' model. (simple but brutal)

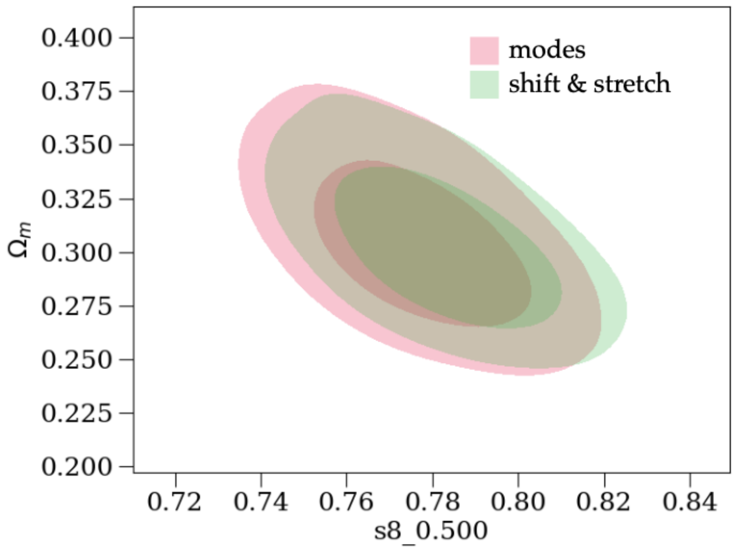


- Idea (DESY6): Use Principal Component Analysis to extract a set of orthogonal modes that capture (most) of the variations within the set of $n(z)$.

- MCMC: sample mode coefficients λ_i:

$$n(z) = n_0(z) + \sum_i \lambda_i \, . e_i(z)$$

Y3 2x2pt data



G. Bernstein, M. Troxel, A. Amon, B. Yin, A. Alarcon, W. d'Assignies

Two main methods:

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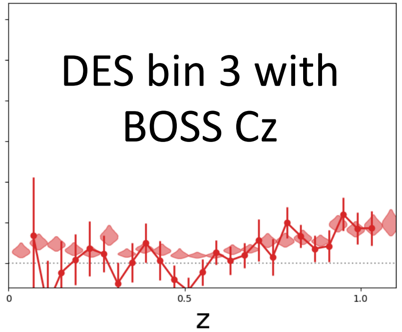
Combination of the two methods for DES

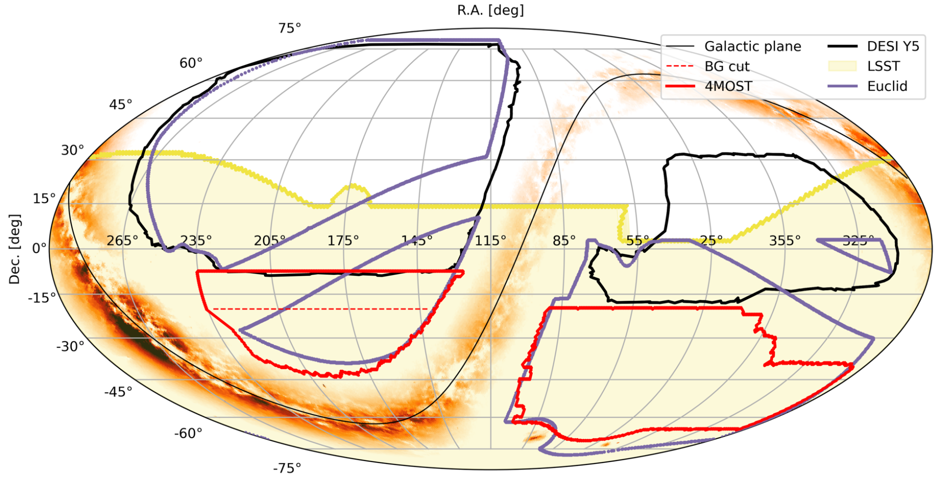
The two methods for Euclid

Clustering redshifts and future surveys

Past:
- BOSS / eBOSS.

Future:
- DESI (north),
- 4MOST(south),
- Euclid Hα (both)



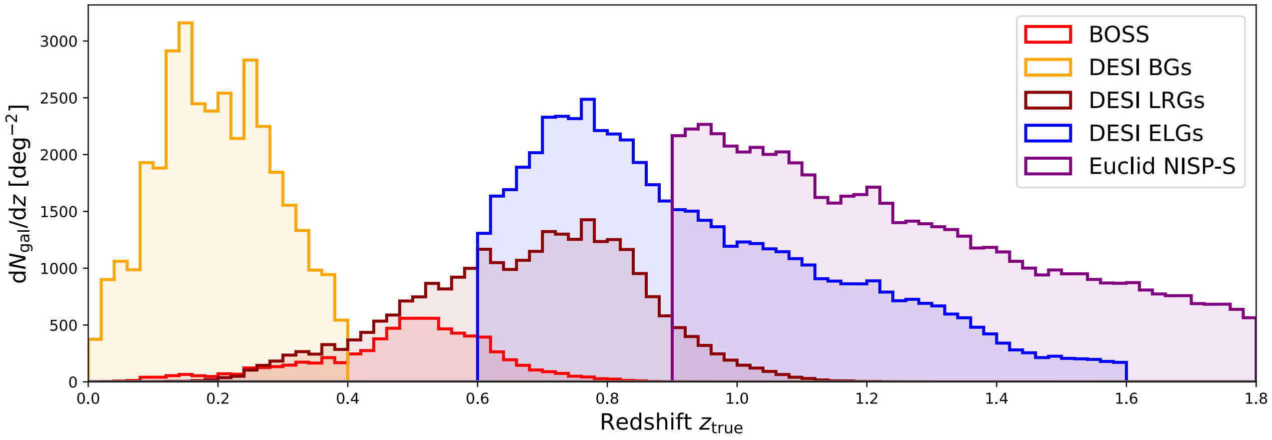


@Antoine Rocher

- Dense samples for $0 < z < 1.8$
- QSO covering $1 < z < 3.5$

- Photometric : Euclid and LSST:

→ Challenging requirements!



Clustering redshifts and scale choice

$n_p(z_i) \propto \omega_{sp}(\theta, z_i) / \xi_m(\theta, z_i)$

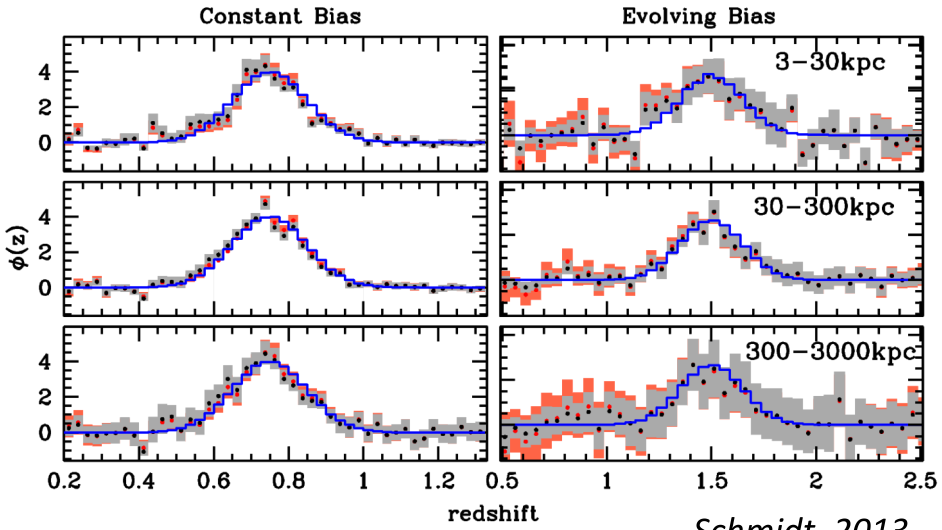
- Newman, 2008, 2-10 Mpc
- S. Schmidt, 2013, 0.003-3 Mpc (30-300 kpc)
- B. Menard, 2013, 0.003-3 Mpc

DES:
- M. Gatti, 2017, 0.5-1.5 Mpc
- C. Davis, 2018, 0.1-10 Mpc
- M. Gatti, 2022, 1.5-5 Mpc
- R. Cawthon, 2022, 0.5-1.5 Mpc

Kids:
- J.L. Van der Bush, 2021, 0.1-1 Mpc

Euclid:
- K. Naidoo, 2022, 0.1-1 Mpc

'Systematic errors introduced by violating the assumption of linear bias are out- weighed by an improved SNR for r < 1 Mpc'



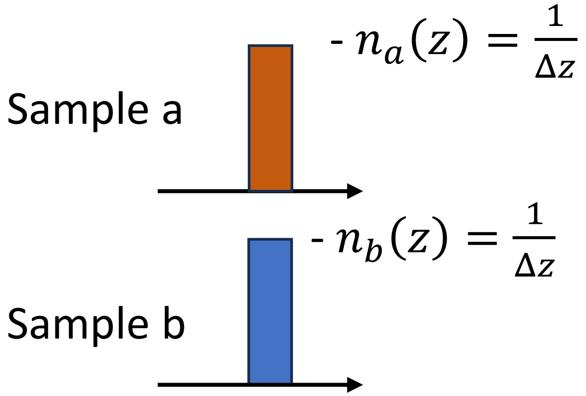
Schmidt, 2013

Mocks based on Millennium and Las- Damas simulations.
3 parameters HOD, different biases scenario.

William d'Assignies

Impact of scales and biases (1/2)

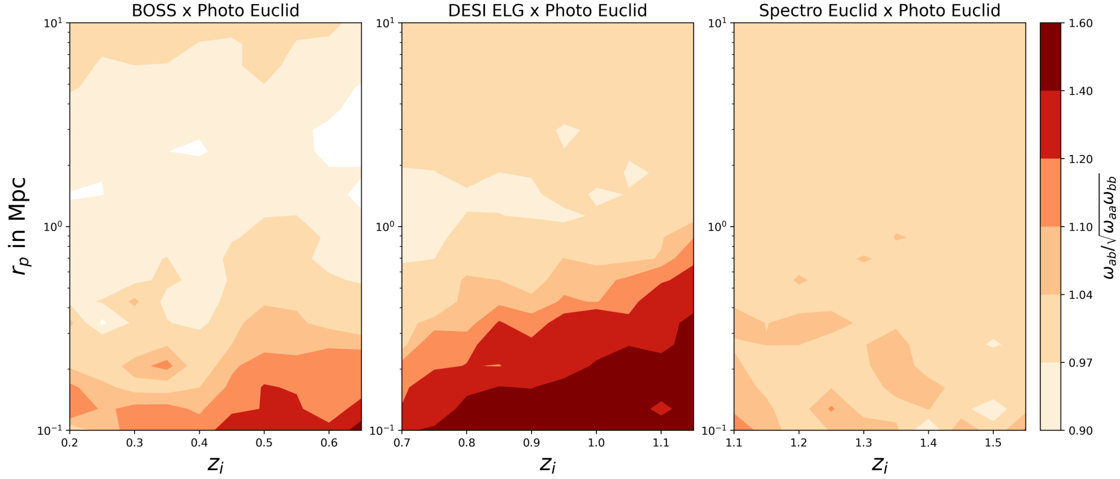
- The ansatz is the following: two samples in the same z-bin.



  - Cross-correlation: $\omega_{ab}(r_p, z_i) \approx b_a \times b_b \times \xi_m$
  - Auto-correlation : $\omega_{aa}(z_i, r_p) \approx b_a^2 \times \xi_m$
  - For clust-z, we don't want the biases : $\frac{\omega_{ab}}{\sqrt{\omega_{aa}\omega_{bb}}} \approx$ **1**

- For clust-z, very small scales ~ 1 Mpc or even smaller and we used linear biases…

- 1st idea: Use Flagship simulation, a and b into small redshift bins and measure the ratio for each bin, for different r_p



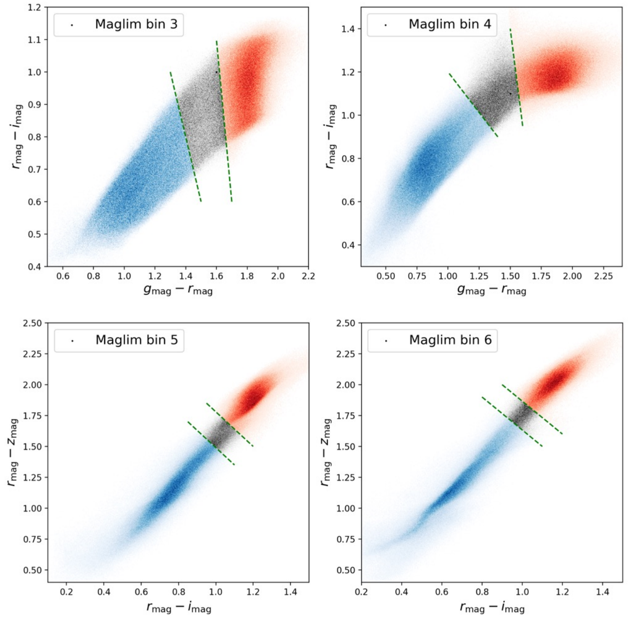
→ $\frac{\omega_{ab}(r_p, z_i)}{\sqrt{\omega_{aa}\omega_{bb}}} \approx 1$

Impact of scales and biases (2/2)

Q: is this 'real' or coming from the MOCKs

- We used the public DESY3 'lens' samples : Maglim (blue and red) and Redmagic. (red)

- We removed the Redmagic galaxies which are part of Maglim



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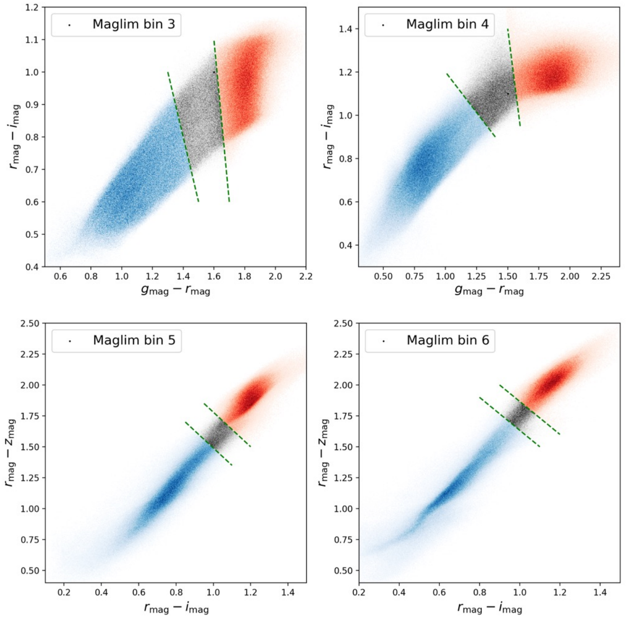
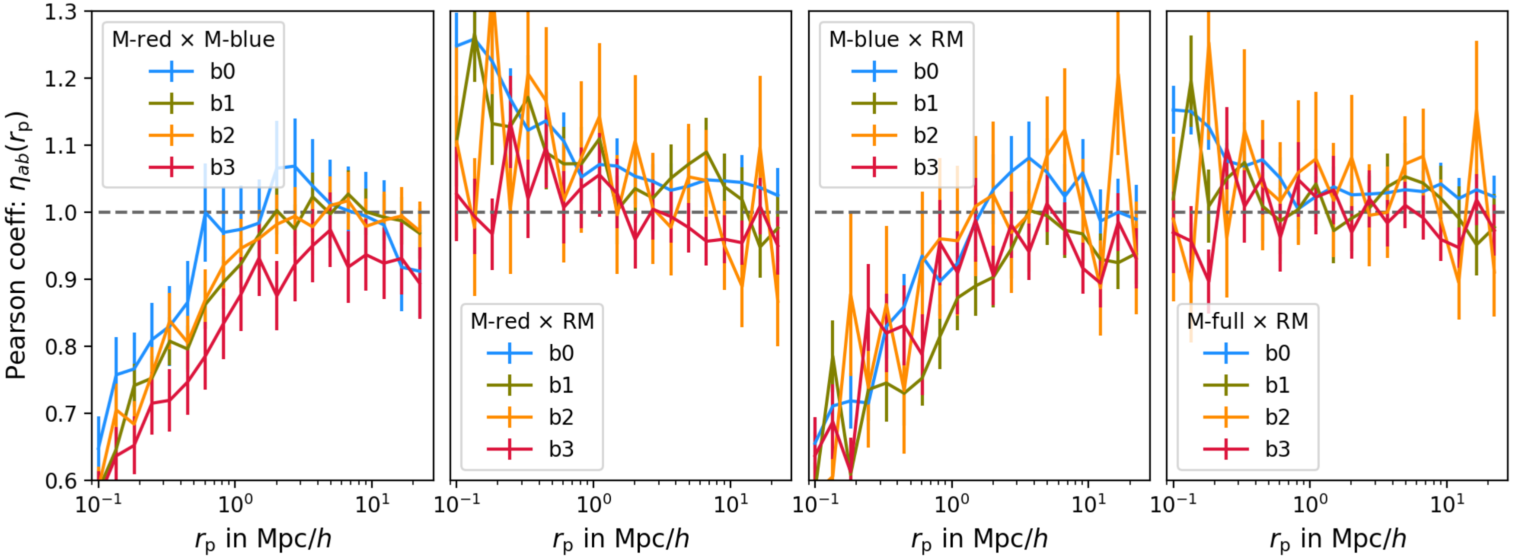


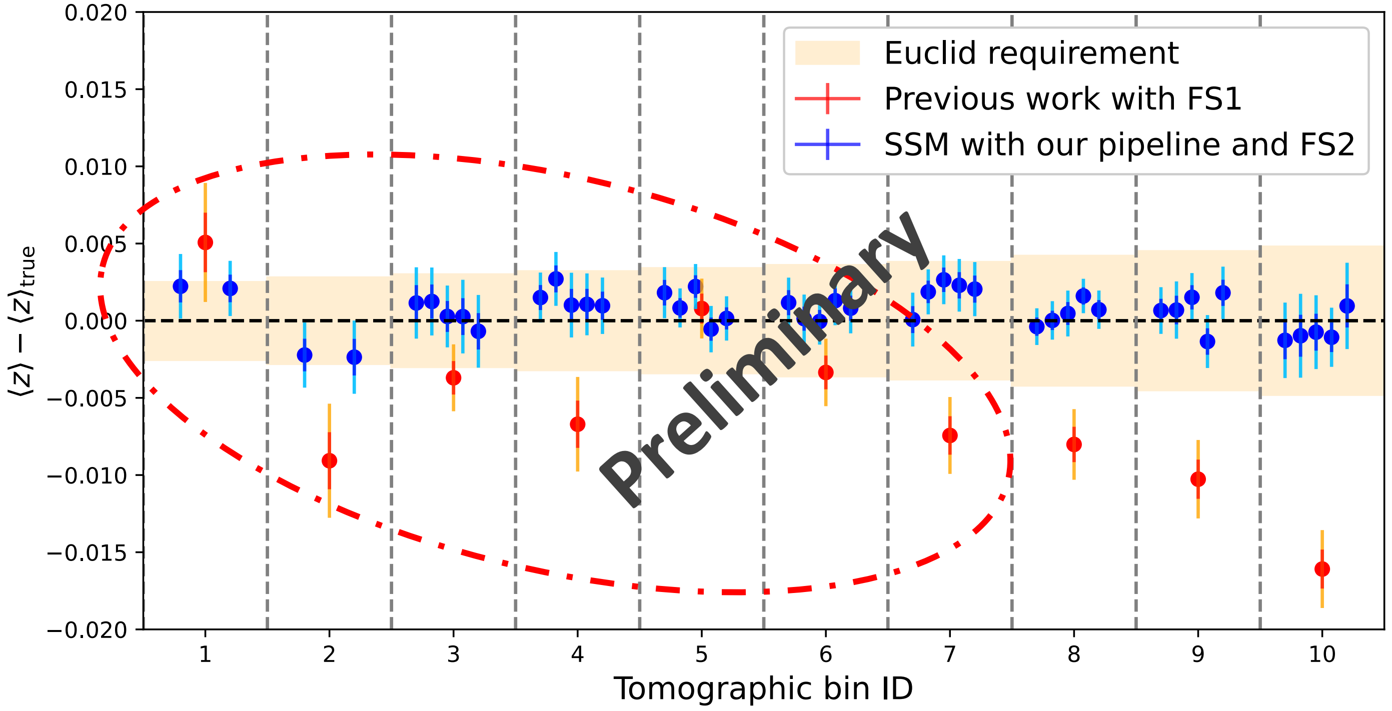
- We removed the Redmagic galaxies which are part of Maglim

- We report : $\eta_{ab}(r_p) = \frac{\omega_{ab}(r_p)}{\sqrt{\omega_{aa}(r_p)\omega_{bb}(r_p)}}$ ($\times C_n$ correction due to $n(z)$ mismatch)



- We detect under-correlation for red x blue for $r < 1$ Mpc/h
- We detect over-correlation for two red samples for $r < 1$ Mpc/h
- This indicates that the effect exists! We decided to discard $r < 1$ Mpc/h

Results with our new pipeline for Euclid



Main differences:

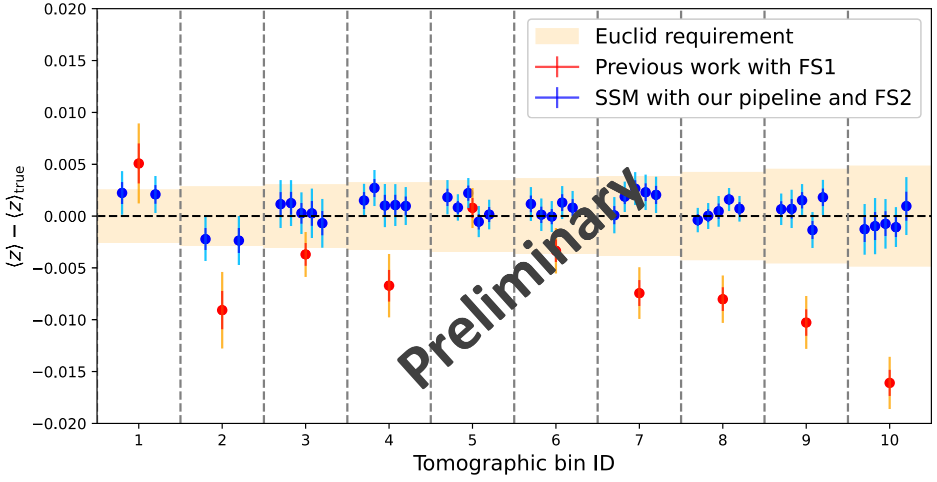
- FS1→ FS2 (400→ 1000 sq.deg)
- Estimator for pair count
- scale range and weighting
- $n(z)$ fitting
- do not combine samples of different spec. tracers (LRG,ELG…)
- photo bias correction
- How do extract the mean-z form the data vector

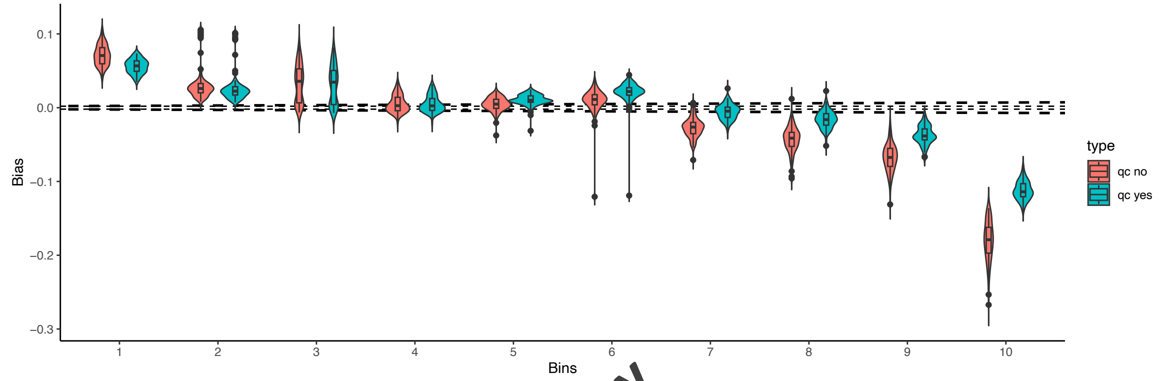
The new Cz pipeline fullfill the Euclid requirements !

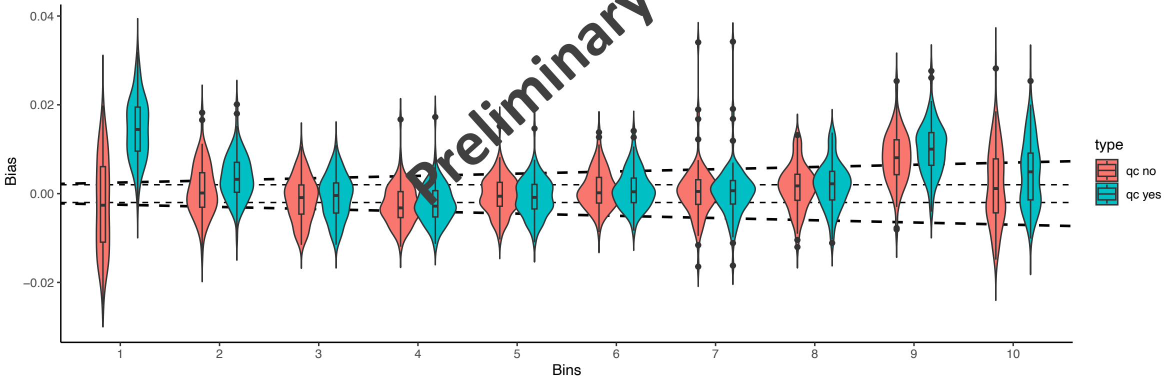
(new SOM pipeline for Euclid)

- William Roster, phd (Max Planck Institute) working on SOM for Euclid.
- He has a similar story:

- My results on the Cz pipeline for Euclid:





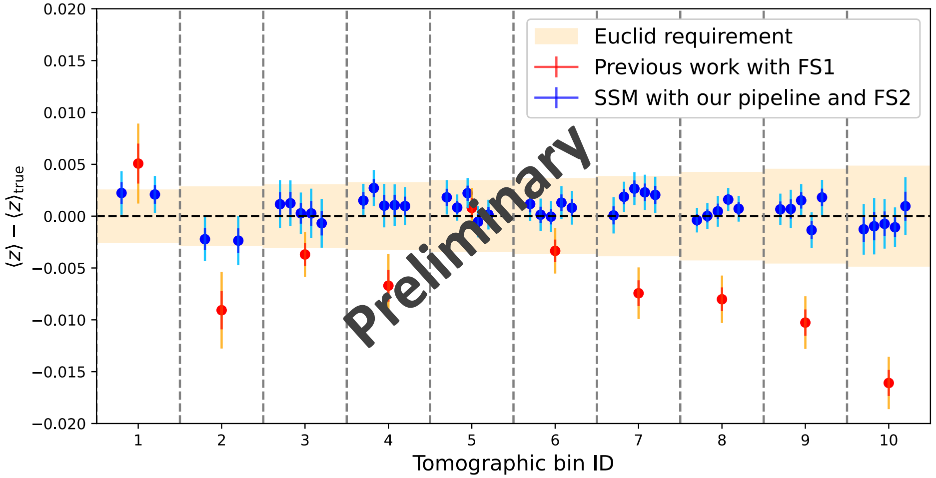


The new pipelines fullfill the Euclid requirements !
We can start thinking about the SOM x Cz!

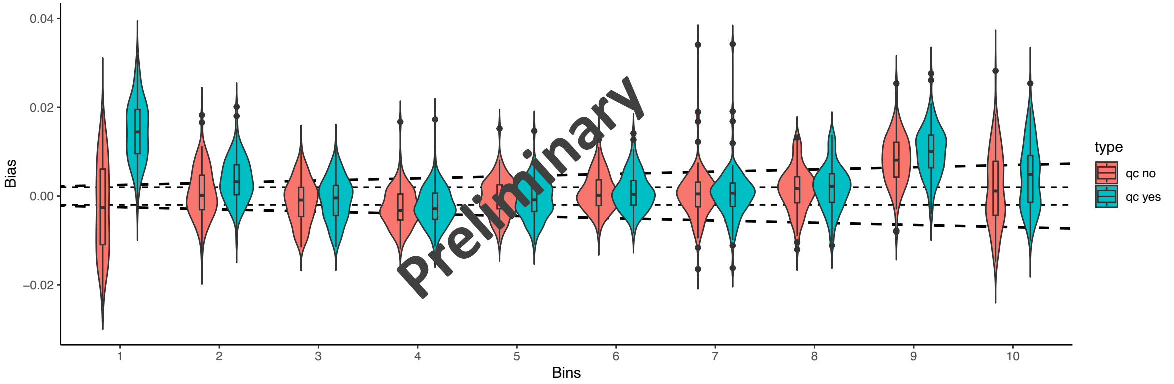
Conclusion !

- LSST, Euclid redshift calibration are challenging and crucial
- SOM methods: Direct use of photometry, high-z range, flexible systematics mitigation, but needs a representative redshift sample, limited by sample variance
- Cz methods: Insensitive to photometric systematics, sample variance, but limited by the z range and the sky coverage of spec-samples, sensitive to clustering systematics.
- Using both, through a direct combination or consistency checks.

William d'Assignies, Cz for Euclid



William Roster, SOM for Euclid



The redshift calibration challenge

- Precise measurements requires precise redshift calibration
- Unbias cosmology requires unbias redshift calibration
- Requirement on z-calibration:
  - Euclid: the mean-z of every tomographic bin: $\boldsymbol{\sigma_{\langle z \rangle_i} = 0.002 \times (1 + z)}$
  - Additional req. on the 2nd moment, but not 'official' req.

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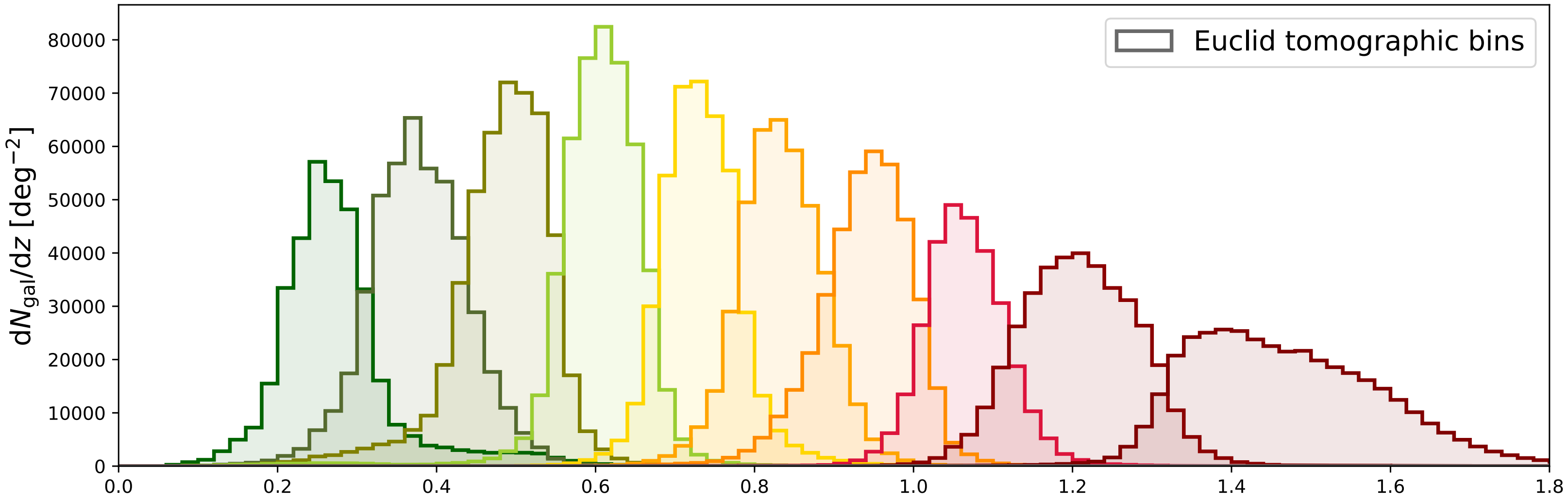
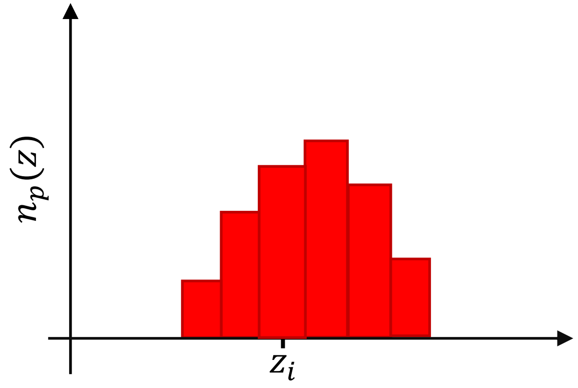


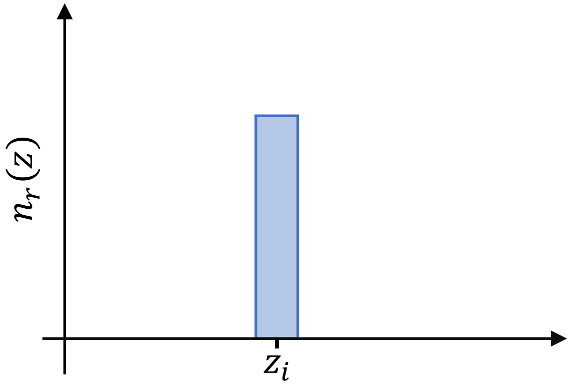
$$\langle z \rangle_i = \int dz\, n_i(z) \times z$$

Two main methods:

2- Clustering redshifts



A photometric sample with unknown $n(z)$



Spectroscopic samples in small z-bin

- The angular 2-pt cross-correlation is:

$$\omega_{sp}(\theta, z_i) = \int dz\, n_s(z)\, n_p(z)\, \langle \delta_s \delta_p \rangle_{\theta, z}$$
$$\approx n_p(z_i)\, \langle \delta_s \delta_p \rangle_{\theta, z_i}$$

- Assuming $\delta_x = b_x \delta_m$

$$\omega_{sp}(\theta, z_i) \approx n_p(z_i) \times b_s\, b_p\, \xi_m(\theta, z_i)$$

- Two cases:
  - Neglecting biases evolution:

$$n_p(z_i) \propto \omega_{sp}(\theta, z_i) / \xi_m(\theta, z_i)$$

  - Measuring biases with auto-correlations:

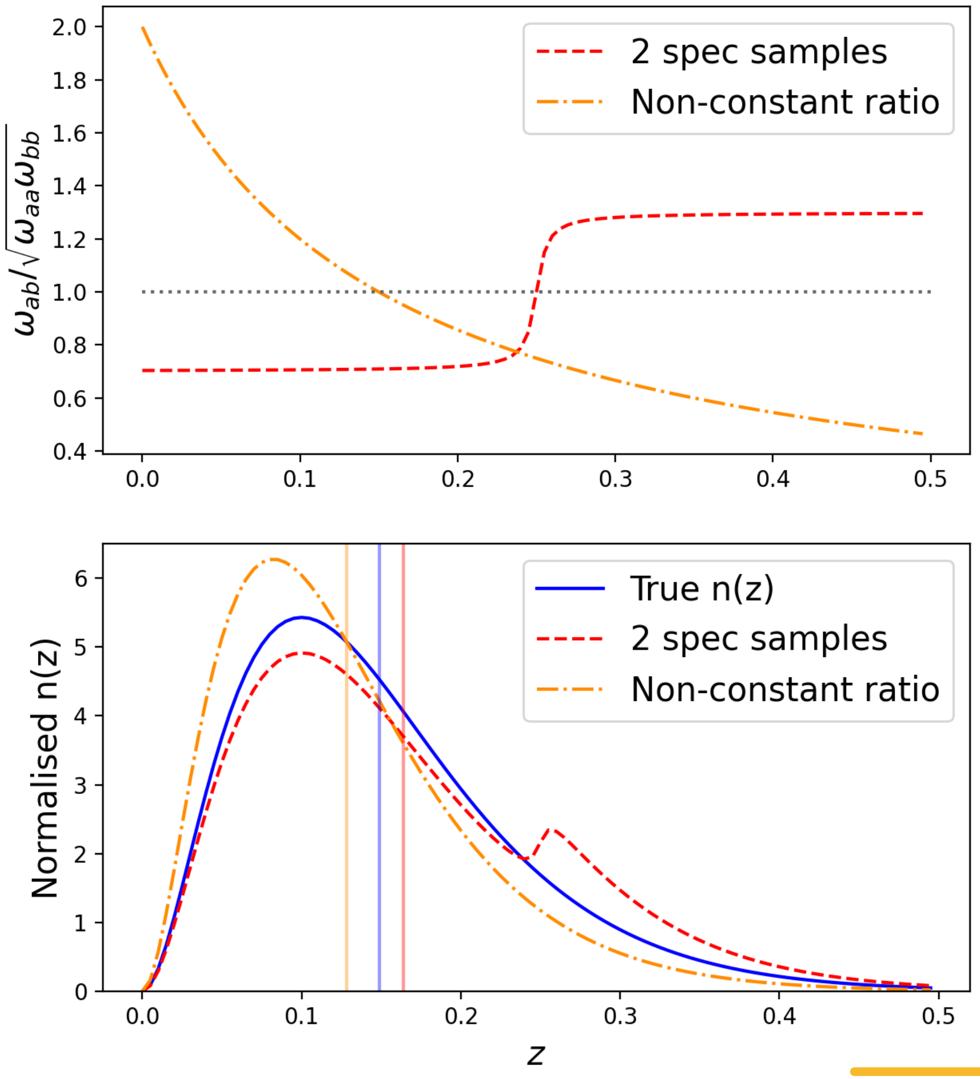
$$n_p(z_i) = \frac{\omega_{sp}(\theta, z_i)}{\Delta z \sqrt{\omega_{ss}(\theta, z_i) . \omega_{pp}(\theta, z_i)}}$$

Impact of scales and biases (2/5)

What we measure (at best) is the product of the true $n(z)$ and the ratio $\frac{\omega_{ab}}{\sqrt{\omega_{aa}\omega_{bb}}}$

Some 'toy model' of 'catastrophic failures' :

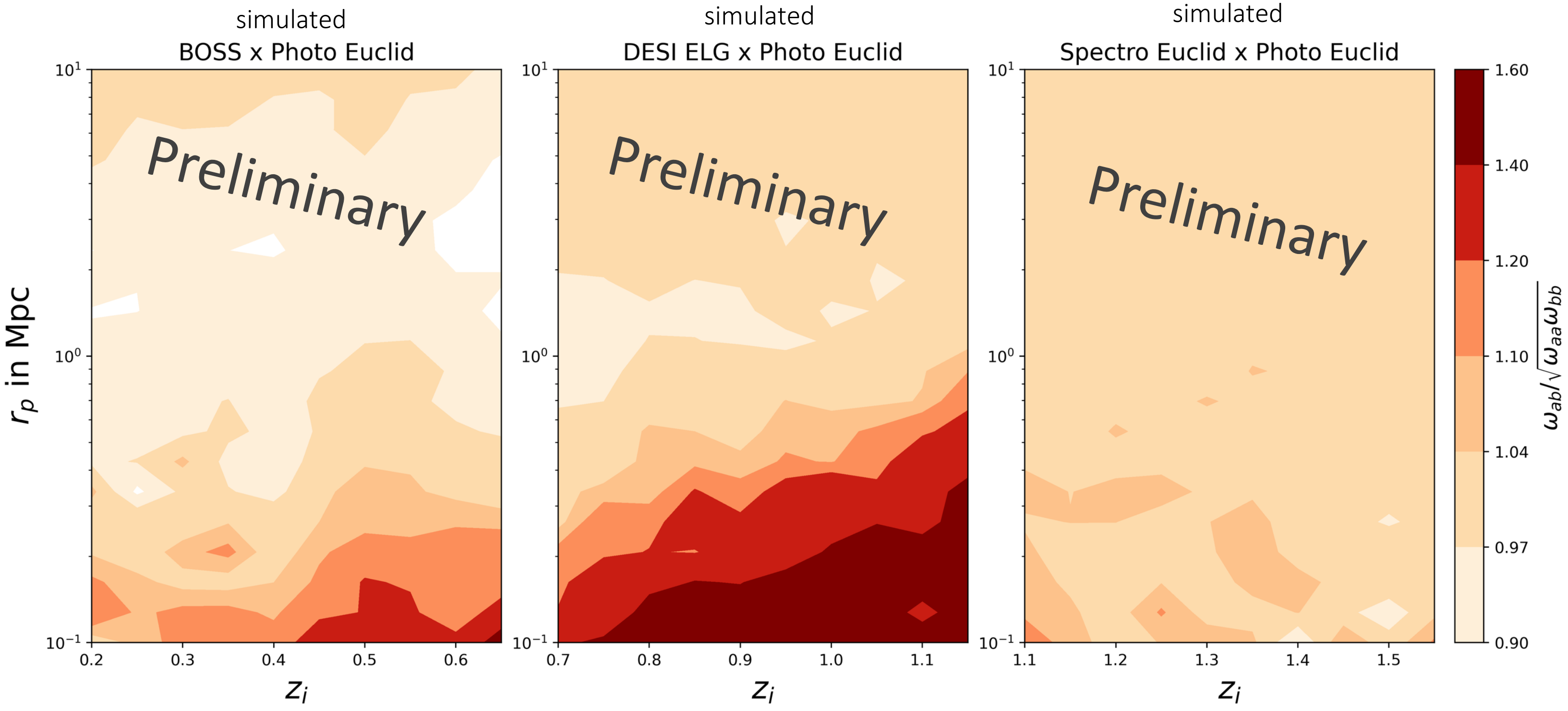
- If the ratio is z-dependant (decreases with z):
- If we combined two spec-samples into one spec sample, with two values for this ratio.
- From a true n(z), assuming this ratio to be 1, we would at best measure some biased n(z).



Impact of scales and biases (3/5)

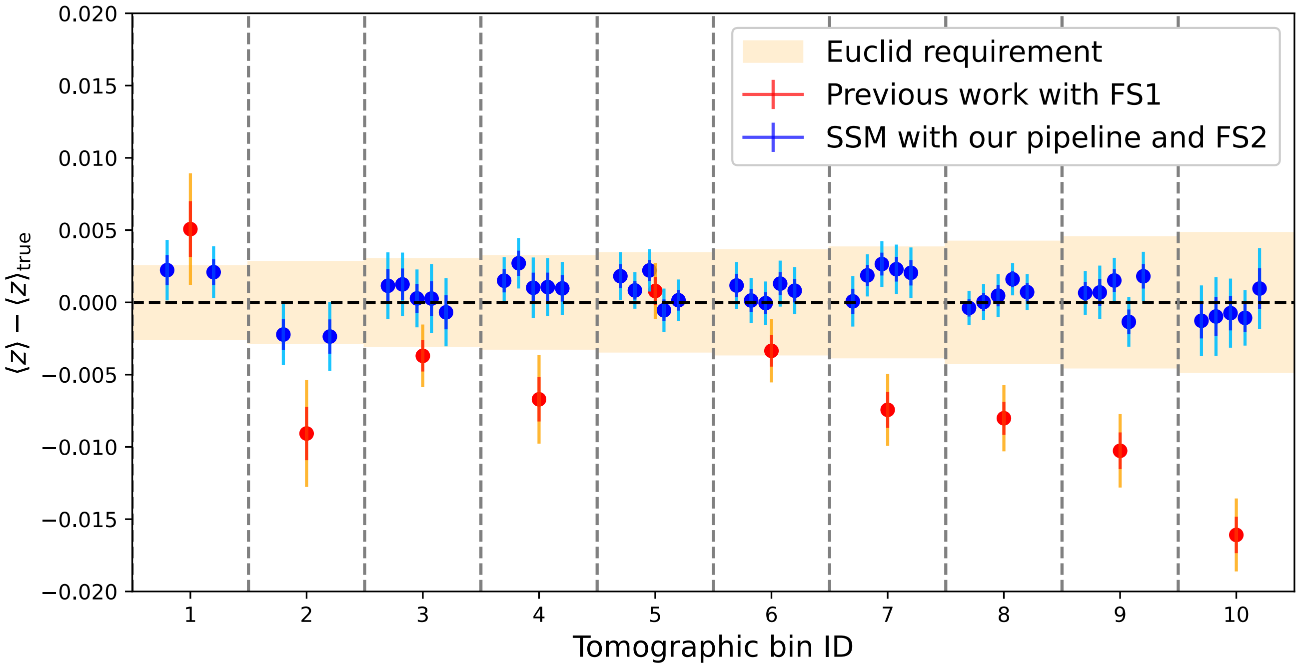
- Idea: Use Flagship 2, 'realistic mocks', split into redshift bins with $n_a(z_i) = n_b(z_i)$ and measure the ratio for each bin, for different r_p.

$$\rightarrow \frac{\omega_{ab}}{\sqrt{\omega_{aa}\omega_{bb}}}(r_p,\quad z_i) \approx 1. \quad \textit{(indeed what matters is to have a constant ratio)}$$



- high redshift very good
- ELG and BOSS stong over-correlation at (very) small scales
- BOSS under-correlation at scale 1-8 Mpc but probably ok since not a clear redshift evolution.
- Even if linear bias not true. 1< r_p < 6 Mpc, the ratio is ≈1 !

Future Euclid work:



- FS2, Cz for Euclid we will fulfil the requirements, so let's wait the data (?!)

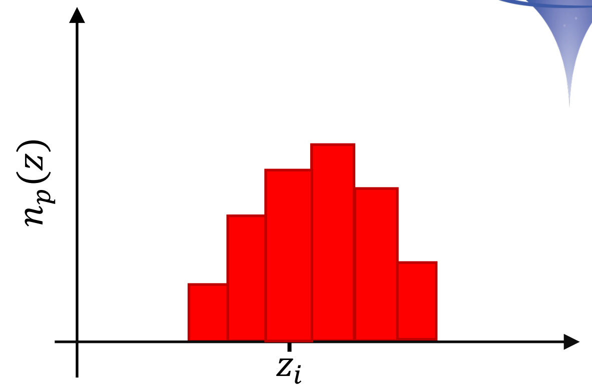
- Some caveats…
- TB, photo-z bias measurement (ω_{pp}) assuming LSST Y10 photometry…
- No interlopers in the Euclid spec-z (last two bins)
- Higher z TB ?
- DR1, what spectroscopic samples ?

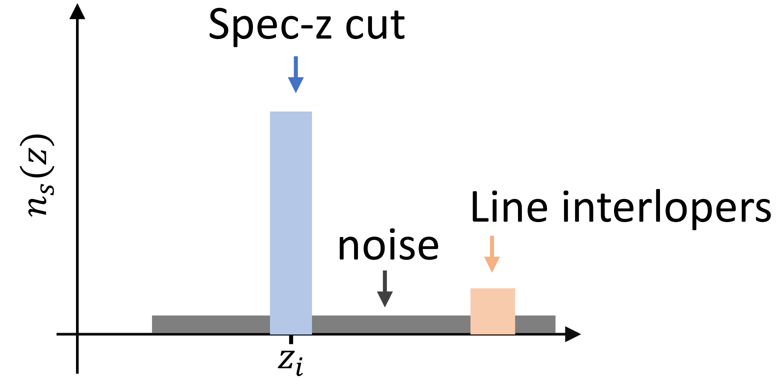
Future Euclid work:

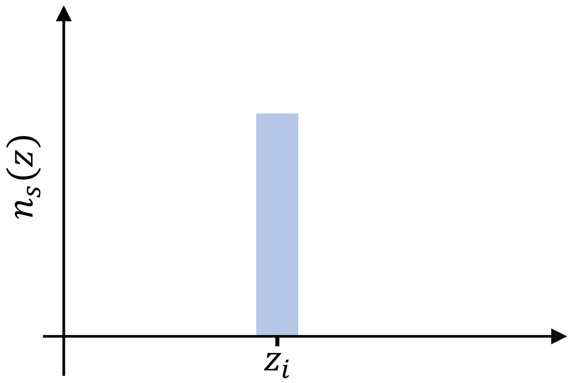
- Interlopers :
  - Clustering with the spec-z bin
  - Clustering with the line interloper
  - Clustering with noise interloper

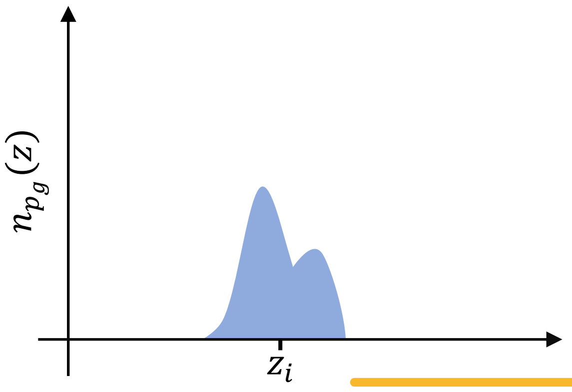
- DR1, not a lot of spectroscopic samples: need to find other methods.
- Typically in DES subsample with a qualitative photo-z. More complicated.

$$\omega_{p_g p}(z) \propto \int dz\, n_{p_g}(z)\, n_p(z)\, \xi_m(z)$$









Callibration at high-z : QSO

- From DESI, we will have 200 QSO per deg2, 1/3 at z>2.1
- Q and challenges since density is low. In theory what matters is the total number of spectras.

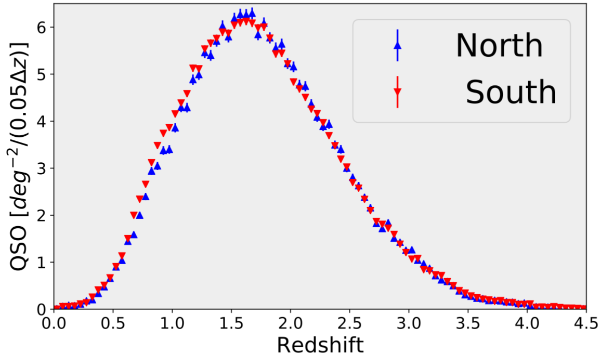
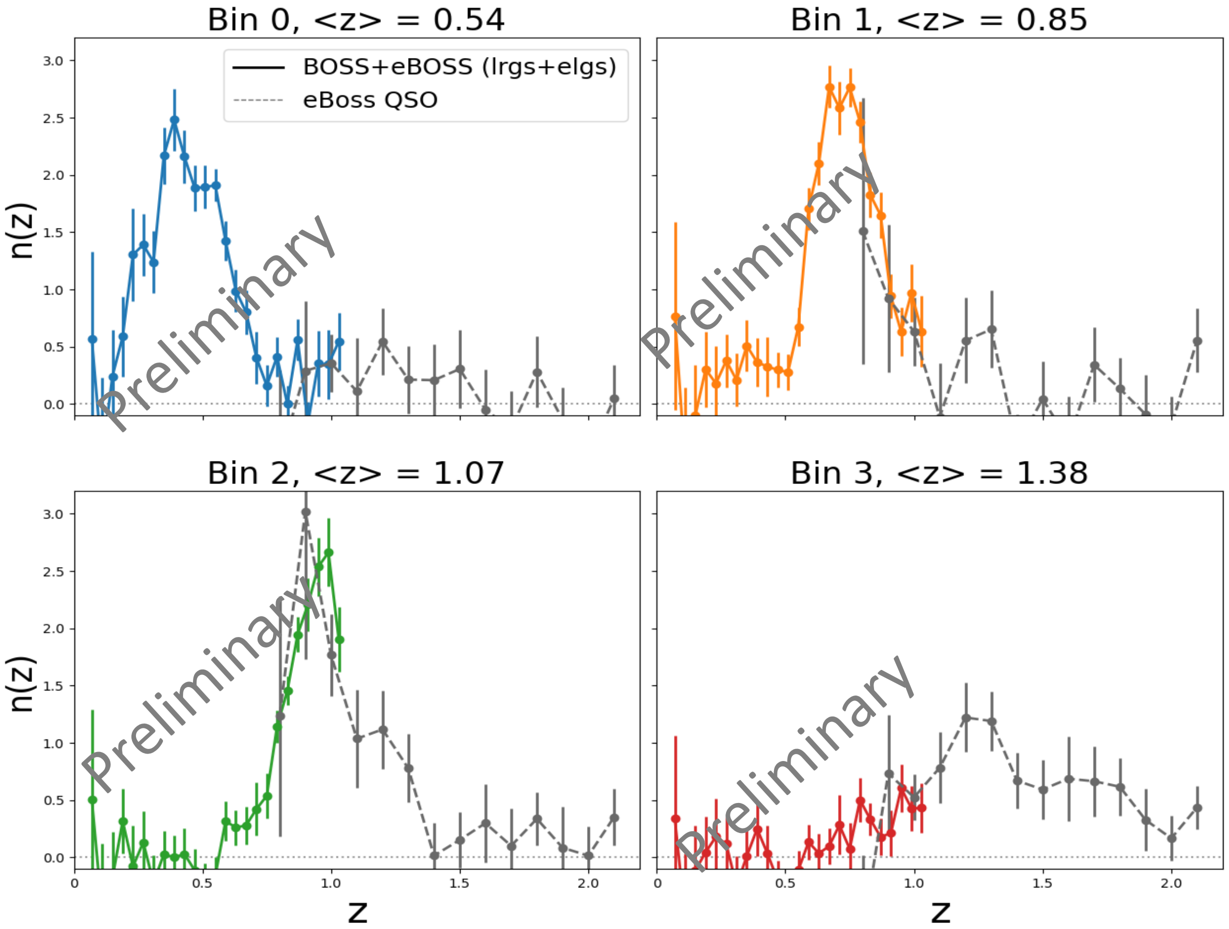


Figure 20. dN/dz for North and South regions.

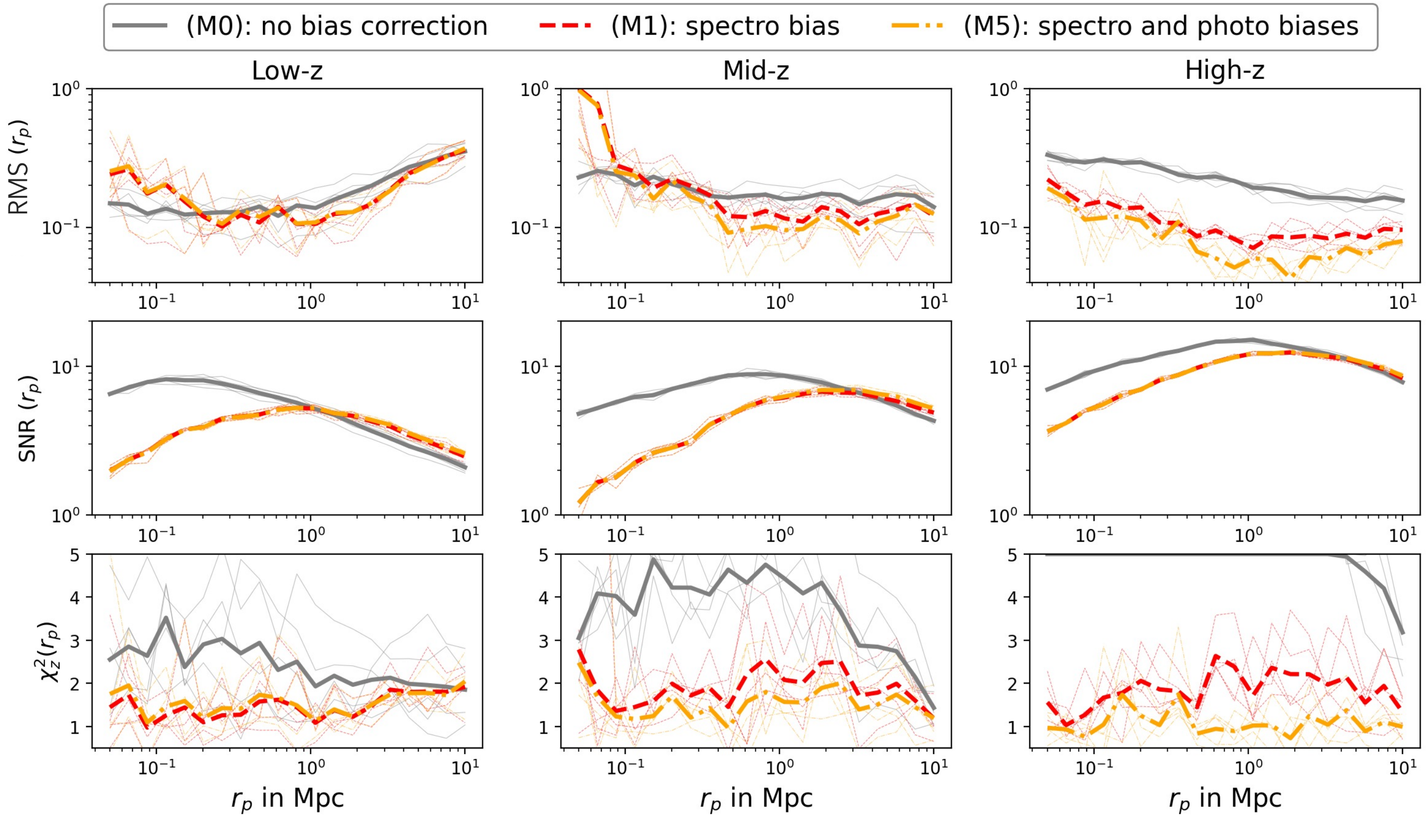
DESI : arXiv:2208.08511v2

- But still possible, eg. 700 deg2 of DES x eBOSS (scales 1-15 Mpc…) up to z=2.2
- With DESI, one might go up to z=3.



Finding the optimal scale range

- We consider the n(z) for 3 estimators (colors, bias correction), but with only one scale: $n_p(z_i|r_p) = \frac{\omega_{sp}(r_p, z_i)}{\Delta z \sqrt{\omega_{ss}(r_p, z_i)\omega_{pp}(r_p, z_i)}}$
- We plot the RMS of $\Delta n = n_{meas}(z) - n_{true}(z)$, SNR and χ^2(reduced so ~ 1)
- Thin lines: 5 sky realisations. Solid: their mean.
- Colors are different estimators, correcting the gal-bias(es)



- *The SNR peaks around 1 Mpc (except M0 low-z but bad χ^2 → systematic).*
- *It appears necessary to correct at least for spec-z (M1), and always better to correct both*
- *For M1 and M5 the $\chi^2 \sim 1$ between 0.1 and 10 Mpc.*
- *Potential additional systematics with data wrt simulation → conservative choice of scale >1 Mpc*

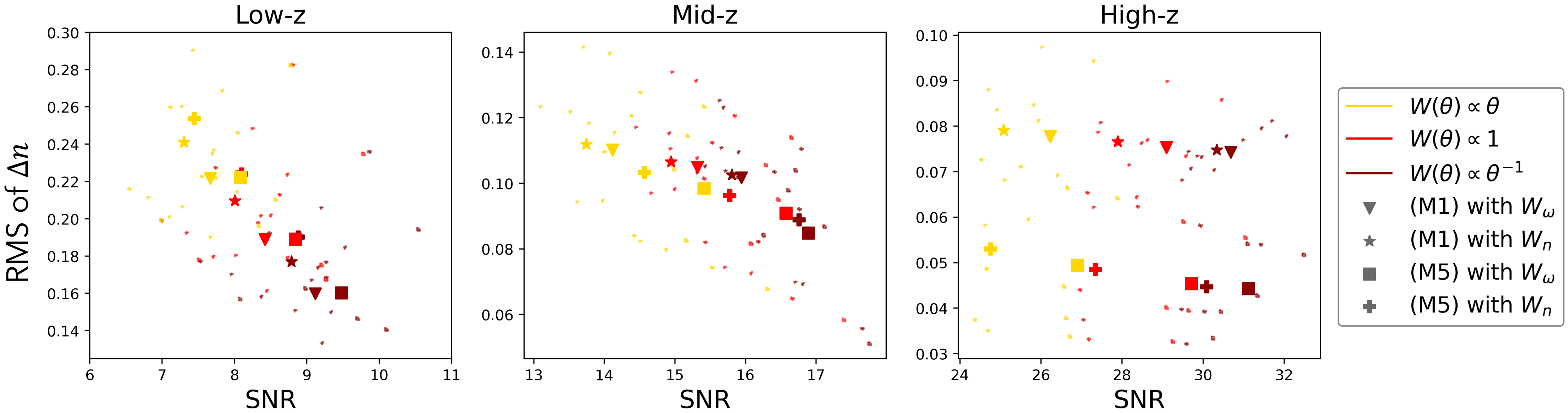
Optimal weighting

- Based on previous test we fix the scale range to be 1-5 Mpc

(why 5 and not 10: (i) SNR is decreasing with scale so in practice little impact

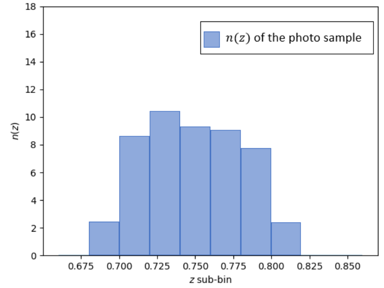
(ii) r>5 Mpc are used for clustering, lensing, so keep our study independant, avoid cov-matrix)

- Small points: sky realisations. Big points: their mean



- *We want to be in the bottom right region (low-std-dev and high-SNR)*
- *Inverse weigting is the best (darkred).*
- *Two possibility for weighting: W_ω is better than W_n*
- *We also see at high-z the increase of std dev when neglecting the photo bias (M1 vs M5)*

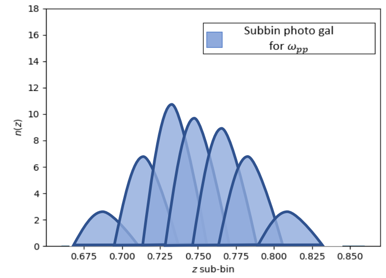
How to measure the redshift evo of $b_p(z)$



- For a discrete and 'perfect' redshift bin (top plot),

$$\omega_{pp}(z) = b_p(z)^2 \omega_{DM}(z) \frac{1}{\Delta z}$$

- Photo-z: not possible to split the sample into discrete redshift bins with a certain width (bottom plot)
- Idea: Measure the n(z) of the subbins with clustering redshift again and correct it



- Bin width correction:

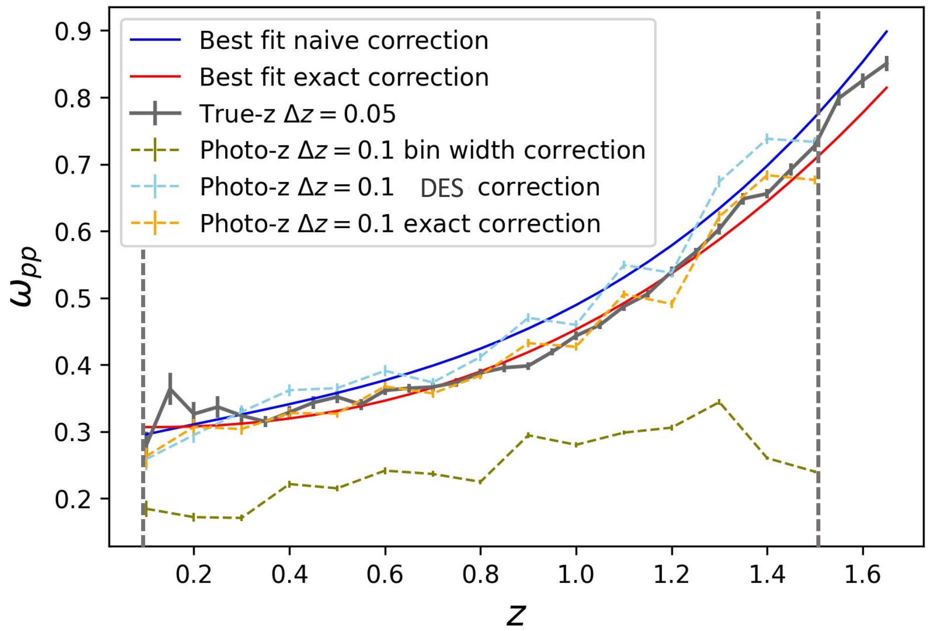
$$\omega_{pp}(z) = b_p(z)^2 \boldsymbol{\omega_{DM}(z)} \frac{\mathbf{1}}{\boldsymbol{\Delta z}}$$

- 'DES' Y3 correction:

$$\omega_{pp}(z) = b_p(z)^2 \boldsymbol{\omega_{DM}(z)} \int \mathbf{dz\, n(z)^2}$$

- Mine ('exact' correction):

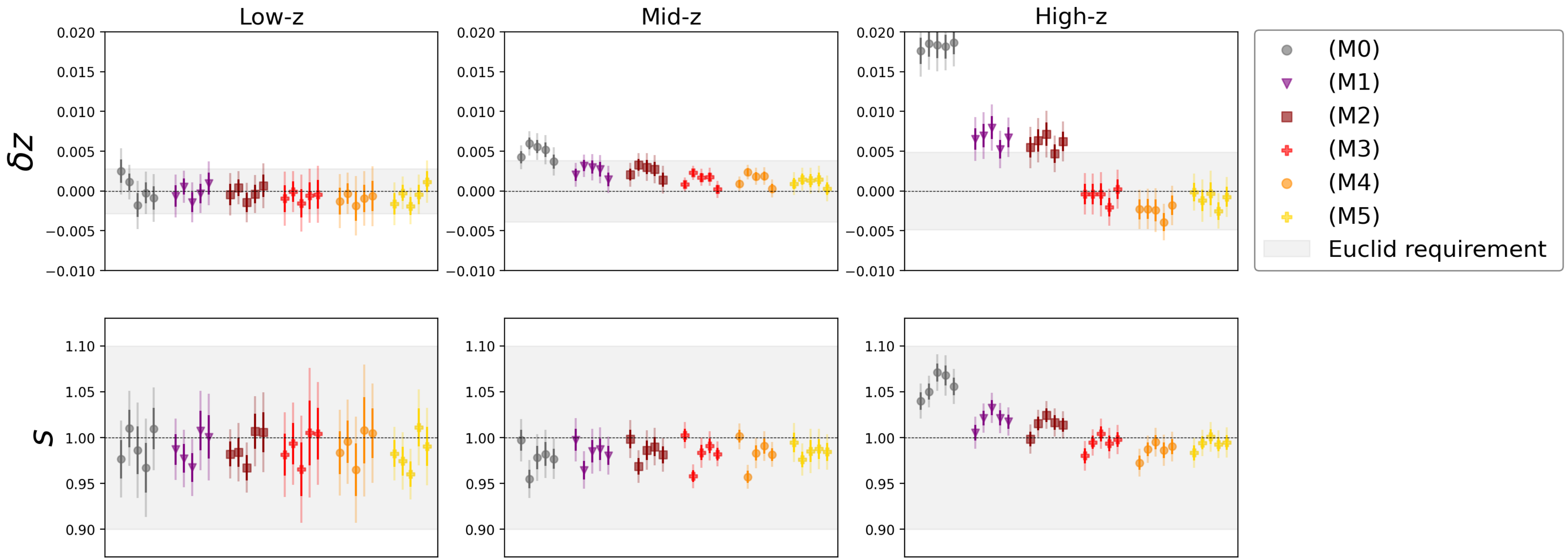
$$\omega_{pp}(z) = b_p(z)^2 \int \mathbf{dz_1 dz_2 n(z_1) n(z_2)\, \omega_{DM}(z_1, z_2)}$$



Caveats: the photo-z needs to be good enought such that the redshift variation accross the subbins is negligeable.

Bias correction scheme

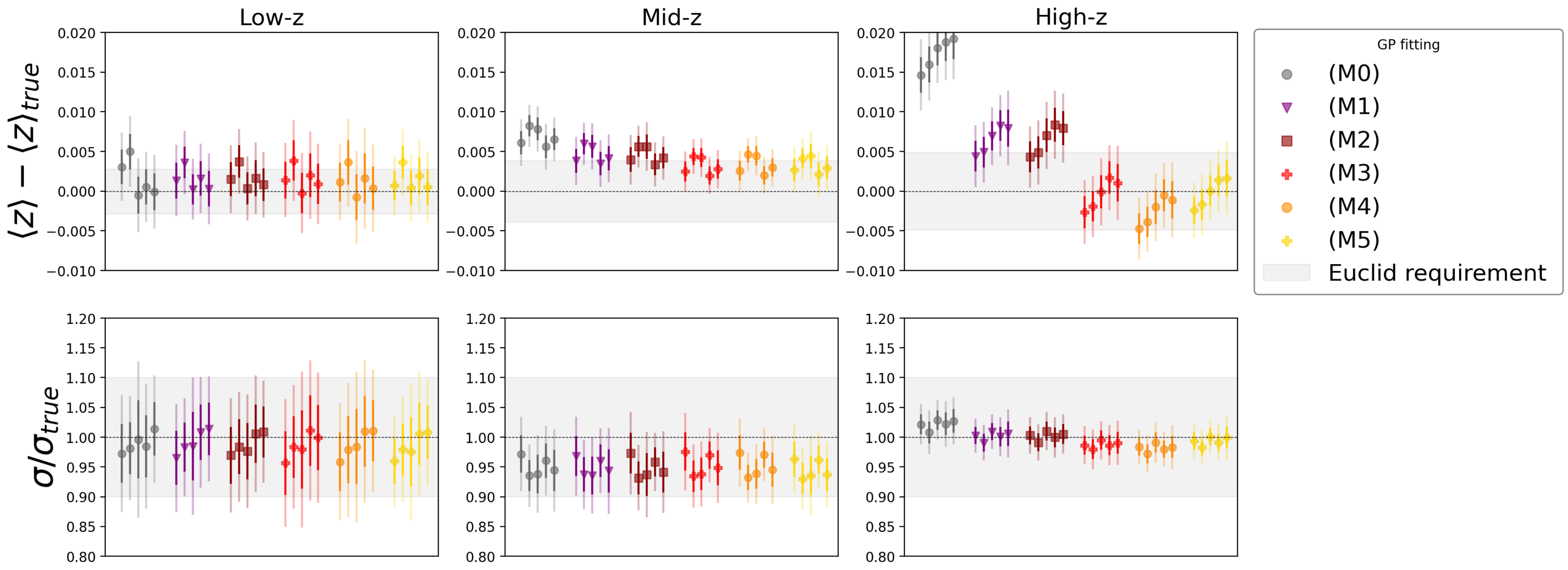
- Here we used the shifted-stretched fitting, conclusion are similar for GP
- We plot the mean z (δz) and shape (s) biases for different bias corrections



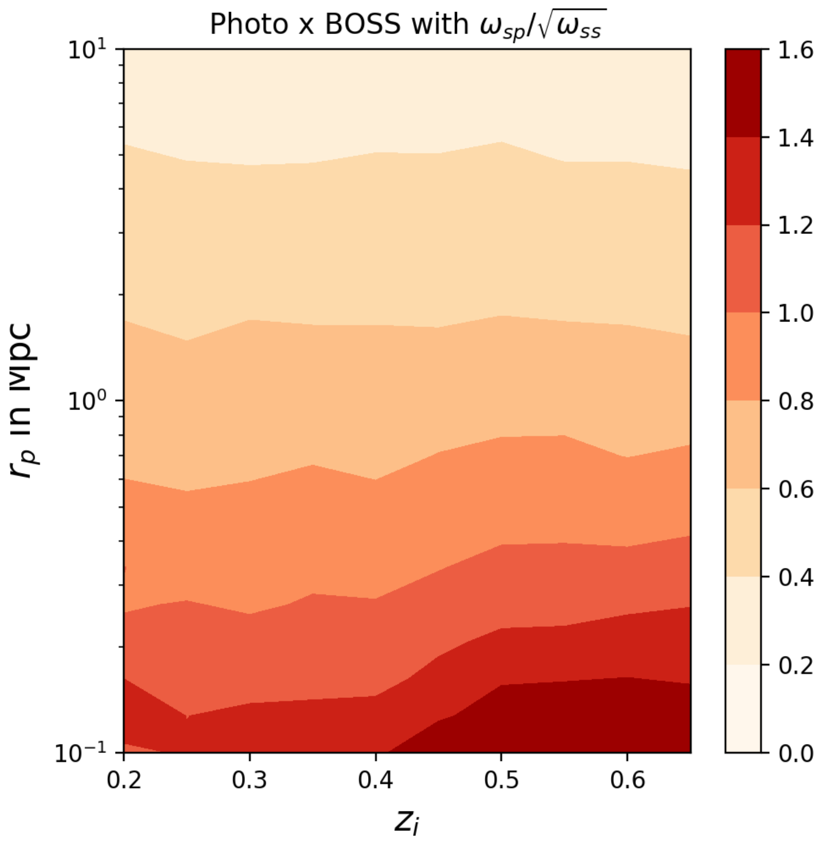
- M0: without bias correction: excluded for mid-z and high-z
- M1: spec-z bias correction only is ok for low-z and mid-z, not high-z
- M5 using the true-z to measure the photo-bias—> sim only, best case possible
- M3 and M4: spec-bias and photo- bias with the mehodology presented (2 choices, story of binning)
- M3 and M4 similar performance than M5 at high-z (M3 slightly better)

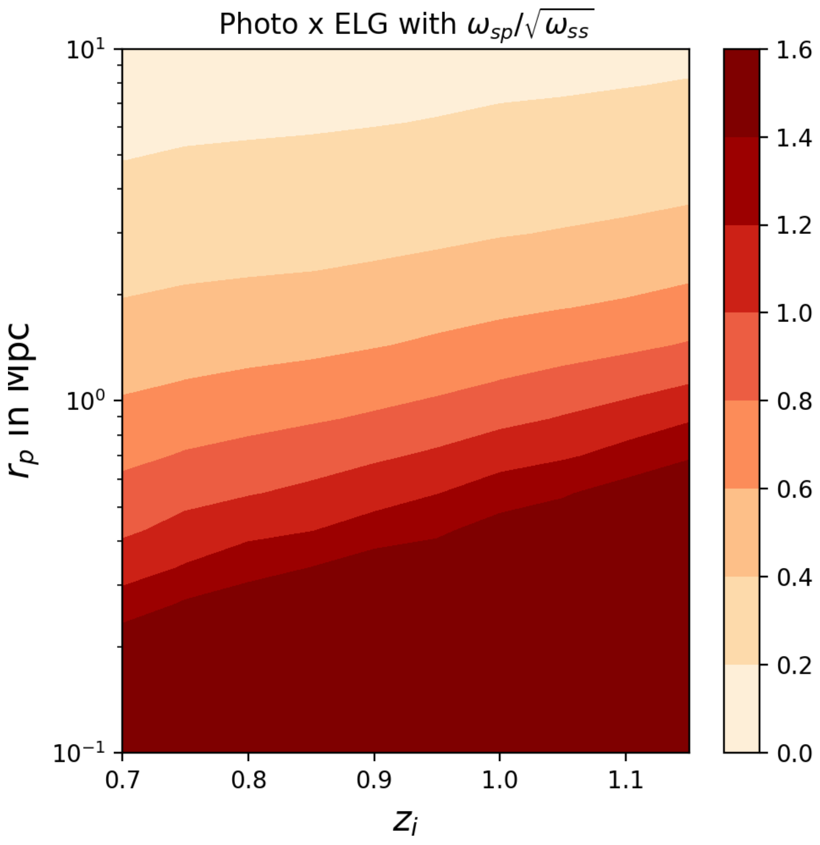
Bias correction scheme

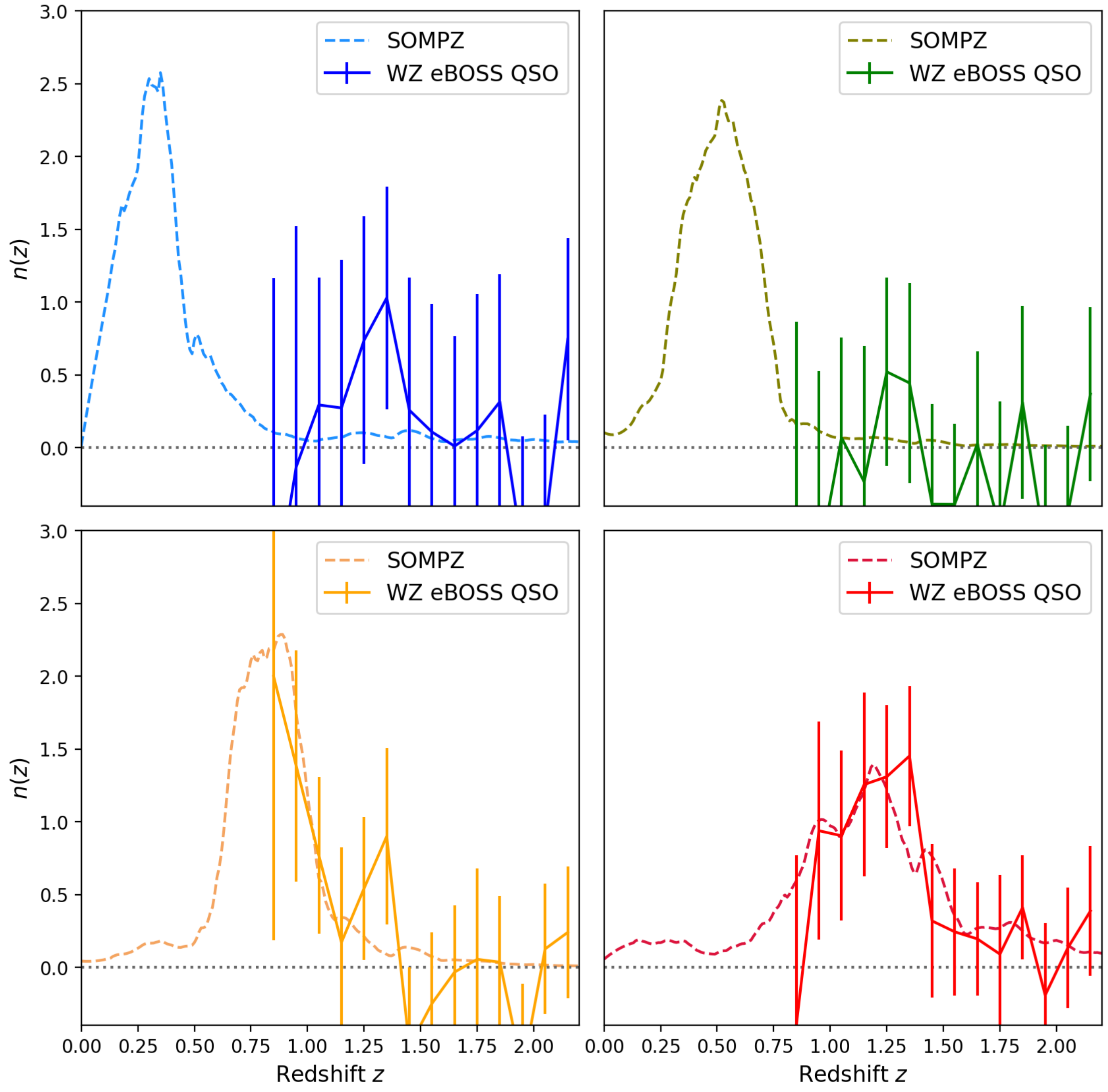
- Same plot but with suppressed GP
- Same conclusion, with bigger errorbars



Another way of visualising the photo-bias impact



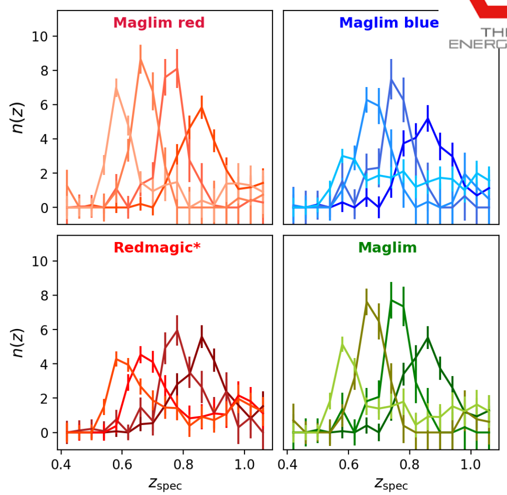




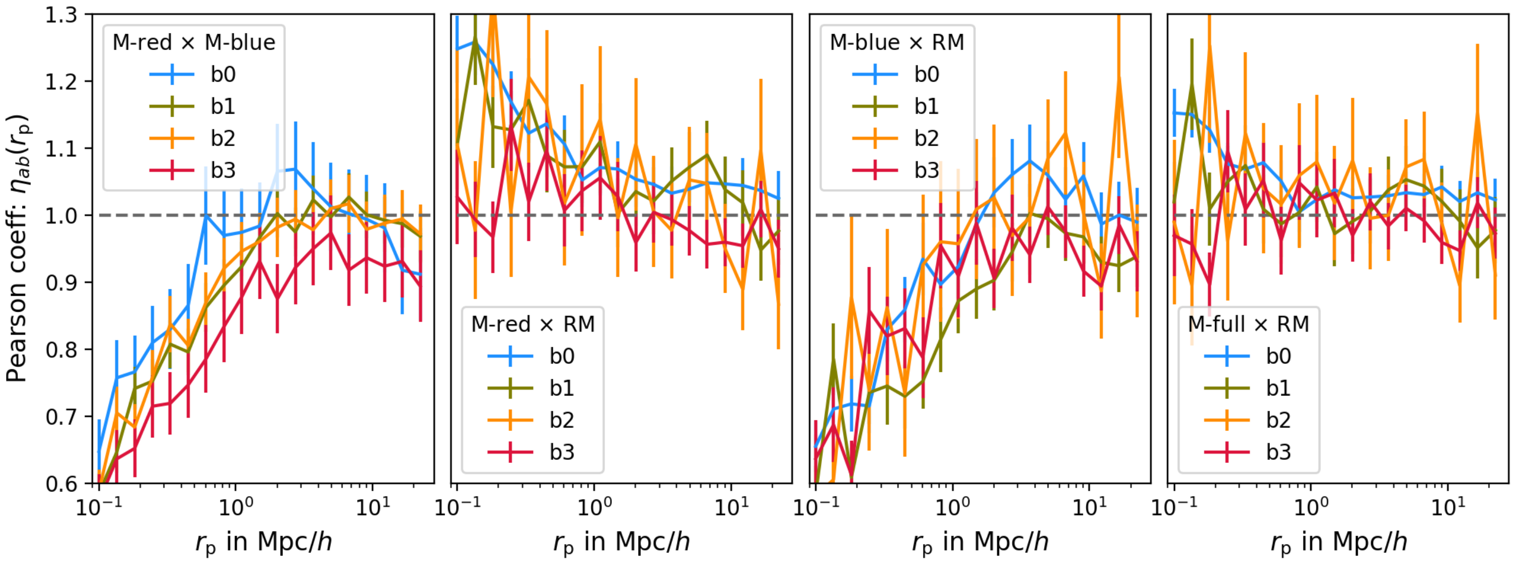
SOMPZ
WZ eBOSS QSO
SOMPZ
WZ eBOSS QSO
SOMPZ
WZ eBOSS QSO
SOMPZ
WZ eBOSS QSO
n(z)
n(z)
Redshift z
Redshift z
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

Impact of scales and biases (3/3)

$n(z)$ measured with clustering-z with eBOSS (large scale)



- We report : $\eta_{ab}(r_p) = \frac{\omega_{ab}(r_p)}{\sqrt{\omega_{aa}(r_p)\omega_{bb}(r_p)}}$ ($\times\ C_n$ correction due to $n(z)$ mismatch)



- With $C_n = \frac{\left(\int dz\, n_a^2\, \xi_m\right)^{0.5}\left(\int dz\, n_b^2\, \xi_m\right)^{0.5}}{\int dz\, n_a n_b \xi_m}$

- We detect under-correlation for red x blue for $r <$ 1 Mpc/h
- We detect over-correlation for two red samples for $r <$ 1 Mpc/h
- This indicates that the effect exists! We decided to discard $r < 1$ Mpc/h

- $$C_n = \frac{\left(\int dz\, n_a^2\, \xi_m\right)^{0.5}\left(\int dz\, n_b^2\, \xi_m\right)^{0.5}}{\int dz\, n_a n_b \xi_m}$$