Going beyond FLRW in an inhomogeneous Universe: the Szekeres exact solution of GR

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Outline

1. Motivations and assumptions
2. Szekeres solution
3. Constraints from cosmological observations
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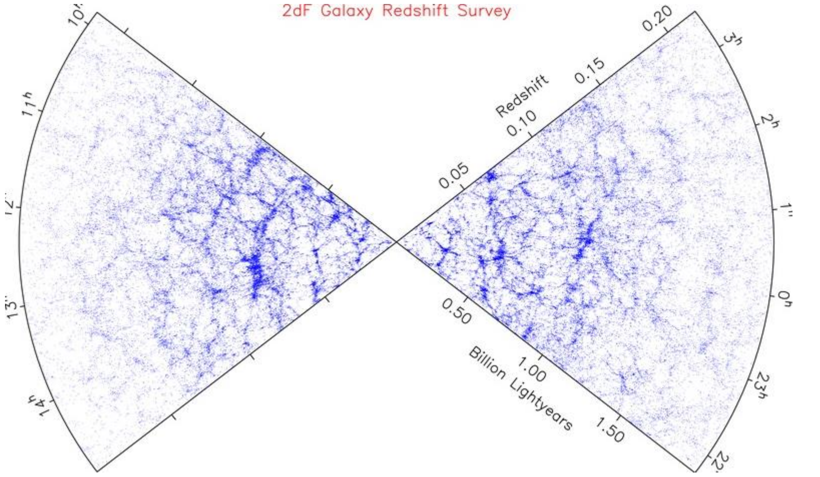
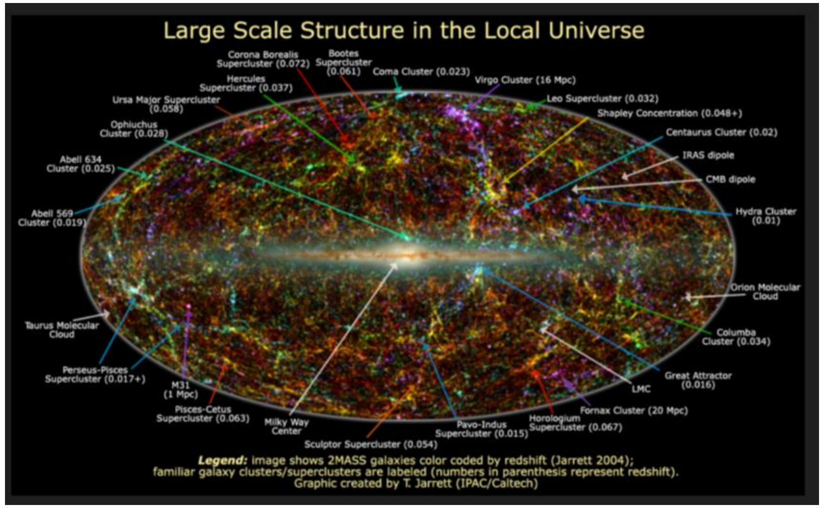
5. Take-home messages

Motivations and assumptions

(In)homogeneity of the Universe

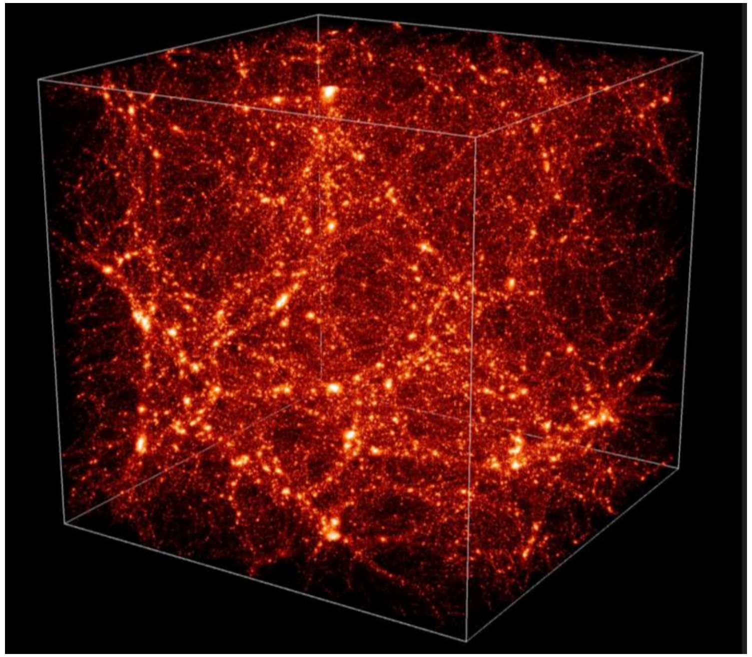


At large scales, the Universe is quasi-homogeneous.



At small scales it is not.

LCDM tree (IAP INC group)



Homogeneous standard model: perturbed ΛCDM

Theoretical drawback: perturbed FLRW is not a solution of the field equations of GR

Experimental issues: tensions, anomalies, etc. due to a mismatch between effects generated in the early universe and transported to the observer through a FLRW spacetime and more direct late universe observations

Assumptions

General Relativity as the gravitation theory

Geometric optics approximation:

Light wave-lengths are negligible wrt the space curvature scales:

- photons travel along null geodesics
- being test objects, the light rays have no effect on the geometry
- the light rays have no vorticity
- being geodesics, the light rays experience no acceleration

Reciprocity theorem:

Assumption: the photon number is conserved

Since we work in the framework of a metric theory of gravity (GR), the reciprocity theorem applies

The Szekeres solution

Szekeres metric

Exact solution of GR for a dust sourced spacetime with no symmetry, in comoving and synchronous coordinates:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{\left(\Phi_{,r} - \Phi E_{,r}/E\right)^2}{\epsilon - k}\mathrm{d}r^2 + \frac{\Phi^2}{E^2}(\mathrm{d}p^2 + \mathrm{d}q^2)$$

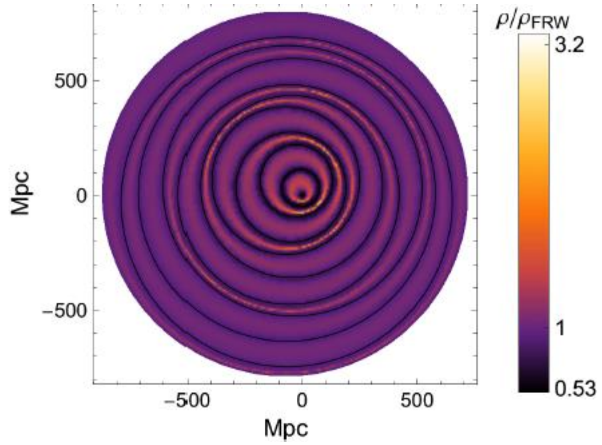
$\Phi(t, r)$, $k(r)$

$$E(r, p, q) = \frac{S}{2}\left[\left(\frac{p-P}{S}\right)^2 + \left(\frac{q-Q}{S}\right)^2 + \epsilon\right]$$

$S(r), \quad P(r), \quad Q(r)$

Szekeres geometry and mass dipoles

- Quasi-spherical solutions $(\epsilon = +1)$ = a set of non-concentric evolving spheres, with a dipole distribution of the energy density around each sphere
In QSS, $\Phi(t, r)$ is the areal radius of the spherical shell labelled by r at time t



Buckley and Schlegel 2020

- Quasi-hyperbolic solutions $(\epsilon = -1)$ = a set of evolving hyperboloids, with a pseudo-spherical dipole

The strenght and orientation on each comoving slice in both classes is determined by $S(r), P(r), Q(r)$

- Quasi-planar solutions $(\epsilon = 0)$ have no dipole-like structure

A given Szekeres spacetime can have simultaneously quasi-spherical, quasi-hyperbolic and quasi–planar regions

Einstein's equations with a cosmological constant

$$\Phi_{,t}^2 = \frac{2M}{\Phi} - k + \frac{\Lambda}{3}\Phi^2$$

integrated as

$$t - t_B(r) = \int_0^{\Phi} \frac{\mathrm{d}\tilde{\Phi}}{\sqrt{\frac{2M}{\tilde{\Phi}} - k + \frac{\Lambda}{3}\tilde{\Phi}^2}}$$

$$4\pi\rho(t, r, p, q) = \frac{M_{,r} - 3ME_{,r}/E}{\Phi^2(\Phi_{,r} - \Phi E_{,r}/E)}$$

A given Szekeres model is a priori determined by 6 functions of r: $k(r),\ S(r),\ P(r),\ Q(r),\ M(r),\ t_B(r)$, one quasi-constant parameter ϵ, and a cosmological constant Λ. The number of independent functions can be reduced to 5 by rescaling r through, e. g., a choice of any arbitrary function.

Essential property of the Szekeres solutions: they possess the FLRW model as a homogeneous limit

$S = 2\epsilon,\ P = Q = 0,\ \Phi = ra(t),\ k = k_0 r^2,\ r$ freedom $\Longrightarrow\ t_B = const. = 0$

$\Downarrow$

A new parameter: the homogeneity-inhomogeneity transition scale < z_{eq} depending on the desired accuracy.

Null geodesic equations

Cosmological data obtained through electromagnetic signals (GW will come later)

Need of the Szekeres null geodesic equations $\Longrightarrow$

$$\frac{\mathrm{d}^2t}{\mathrm{d}s^2} + \left(\frac{\Phi_{,tr} - \Phi_{,t}E_{,r}/E}{\epsilon - k}\right)(\Phi_{,r} - \Phi E_{,r}/E)\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + \frac{\Phi\Phi_{,t}}{E^2}\left[\left(\frac{\mathrm{d}p}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}q}{\mathrm{d}s}\right)^2\right] = 0,$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2} + 2\left(\frac{\Phi_{,tr} - \Phi_{,t}E_{,r}/E}{\Phi_{,r} - \Phi E_{,r}/E}\right)\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}r}{\mathrm{d}s} + \left(\frac{\Phi_{,rr} - \Phi E_{,rr}/E}{\Phi_{,r} - \Phi E_{,r}/E} - \frac{E_{,r}}{E} + \frac{k_{,r}}{2(\epsilon - k)}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + 2\frac{\Phi}{E^2}\left(\frac{E_{,r}E_{,p} - EE_{,rp}}{\Phi_{,r} - \Phi E_{,r}/E}\right)\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}p}{\mathrm{d}s} + 2\frac{\Phi}{E^2}\left(\frac{E_{,r}E_{,q} - EE_{,rq}}{\Phi_{,r} - \Phi E_{,r}/E}\right)\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}q}{\mathrm{d}s}$$
$$- \frac{\Phi}{E^2}\left(\frac{\epsilon - k}{\Phi_{,r} - \Phi E_{,r}/E}\right)\left[\left(\frac{\mathrm{d}p}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}q}{\mathrm{d}s}\right)^2\right] = 0,$$

$$\frac{\mathrm{d}^2p}{\mathrm{d}s^2} + 2\frac{\Phi_{,t}}{\Phi}\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}p}{\mathrm{d}s} - \frac{\Phi_{,r} - \Phi E_{,r}/E}{\Phi(\epsilon - k)}(E_{,r}E_{,p} - EE_{,rp})\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + 2\frac{\Phi_{,r} - \Phi E_{,r}/E}{\Phi}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}p}{\mathrm{d}s} - 2\frac{E_{,q}}{E}\frac{\mathrm{d}p}{\mathrm{d}s}\frac{\mathrm{d}q}{\mathrm{d}s} + \frac{E_{,p}}{E}\left[-\left(\frac{\mathrm{d}p}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}q}{\mathrm{d}s}\right)^2\right] = 0,$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}s^2} + 2\frac{\Phi_{,t}}{\Phi}\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}q}{\mathrm{d}s} - \frac{\Phi_{,r} - \Phi E_{,r}/E}{\Phi(\epsilon - k)}(E_{,r}E_{,q} - EE_{,rq})\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + 2\frac{\Phi_{,r} - \Phi E_{,r}/E}{\Phi}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}q}{\mathrm{d}s} - 2\frac{E_{,p}}{E}\frac{\mathrm{d}p}{\mathrm{d}s}\frac{\mathrm{d}q}{\mathrm{d}s} + \frac{E_{,q}}{E}\left[\left(\frac{\mathrm{d}p}{\mathrm{d}s}\right)^2 - \left(\frac{\mathrm{d}q}{\mathrm{d}s}\right)^2\right] = 0.$$

and their first integral: $$\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 = \frac{(\Phi_{,r} - \Phi E_{,r}/E)^2}{\epsilon - k}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + \frac{\Phi^2}{E^2}\left[\left(\frac{\mathrm{d}p}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}q}{\mathrm{d}s}\right)^2\right].$$

Redshift

Definition $1 + z = -k^t_{em}$

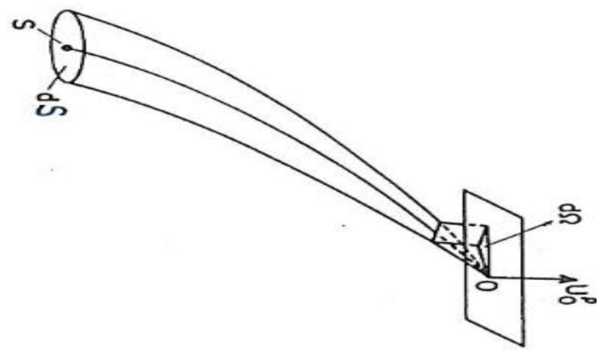
Adapt to Szekeres the method applied by Bondi (1947) to the Lemaître-Tolman-Bondi solution, and obtain

$$\frac{\mathrm{d}(\ln(1+z))}{\mathrm{d}s} = -\frac{1}{\frac{\mathrm{d}t}{\mathrm{d}s}}\left\{\left[\frac{\Phi_{,tr}\Phi_{,r} + \Phi\Phi_{,t}(E_{,r}/E)^2 - (\Phi_{,t}\Phi_{,r} + \Phi\Phi_{,tr})(E_{,r}/E)}{\epsilon - k}\right]\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + \frac{\Phi\Phi_{,t}}{E^2}\left[\left(\frac{\mathrm{d}p}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}q}{\mathrm{d}s}\right)^2\right]\right\}$$

Solve the null geodesic equations as 4 first order differential equations in $\mathrm{d}x^\mu/\mathrm{d}s$ with the two field equations satisfied, and calculate the redshift through the above.

Distance equations

Calculation of the area distance: $D_A^2 = \delta S / \delta\Omega$, two methods:



- The rate of change of the observer area distance depends on the expansion rate of the light bundle as

$$\frac{\mathrm{d}(\ln D_A)}{\mathrm{d}\lambda} = \theta = k^\alpha_{;\alpha}$$

which can be integrated as

$$D_A = \frac{\Phi^2}{E^2} \left(\frac{\Phi_{,r} - \Phi E_{,r}/E}{\sqrt{\epsilon - k}} \right) \exp \left[\int_{s_o}^{s} (k^t_{,t} + k^r_{,r} + k^p_{,p} + k^q_{,q}) \mathrm{d}\lambda \right]$$

with the k^α and derivatives emerging from the null geodesics.

- Use the Sachs optical equation

$$\frac{\mathrm{d}^2 D_A}{\mathrm{d}\lambda^2} = - \left(|\sigma|^2 + \frac{1}{2} R_{\alpha\beta} k^\alpha k^\beta \right) D_A$$

which can be solved together with the Sachs equations for the expansion and shear

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} + \theta^2 + |\sigma|^2 = -\frac{1}{2} R_{\alpha\beta} k^\alpha k^\beta \qquad \frac{\mathrm{d}\sigma_{\alpha\beta}}{\mathrm{d}\lambda} + \theta\sigma_{\alpha\beta} = -C_{\alpha\beta\gamma\delta} k^\gamma k^\delta$$

to obtain the area distance

Then, once D_A is known, the reciprocity theorem gives the luminosity distance $D_L = (1+z)^2 D_A$

Constraints from cosmological observations

Supernova data

Type Ia supernovae are used as standard candles. Their apparent bolometric magnitude reads

$$m = \mathscr{M} + 5 \log D_L(z; p_a)$$

Calibrate the magnitude ``zero point'' $\mathscr{M}$ with low-redshift SNIa whose distance is evaluated through, e. g., Cepheid variables, TRGB or JAGB.

Another set of measurements at higher redshifts is used to constraint the parameters p_a of the Szekeres model, included in the expression for D_L previously obtained.

Galaxy number counts

The total Szekeres rest mass in a volume element defined as the proper volume on a constant time slice, evaluated on the null cone, is

$$\mathcal{M} = 4\pi\rho\frac{\Phi^2}{E^2}\left(\frac{\Phi_{,r} - \Phi E_{,r}/E}{\sqrt{\epsilon - k}}\right)\mathrm{d}r$$

Inserting the Szekeres expression for the energy density

=> $$4\pi m n\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{M_{,r} - 3ME_{,r}/E}{E^2\sqrt{\epsilon - k}}$$

=> the average mass density in redshift space, $m(z)n(z)$. Given the redshift calculated above, the mass density in real space follows and can be compared to the measured data => constraints on the Szekeres parameters.

Cosmic microwave background: CMB multipoles

In Szekeres, the dipole-like distribution of mass over each single (quasi)sphere is

$$\kappa\Delta\rho = \frac{\chi_{,r} - \chi E_{,r}/E}{\Phi^2(\Phi_{,r} - \Phi E_{,r}/} \frac{6M\Phi_{,r} - 2M_{,r}\Phi}{\Phi^2(\Phi_{,r}\chi - \Phi\chi_{,r})} \quad \text{with} \quad \chi = \frac{1 + P^2 + Q^2}{2S} + \frac{S}{2}$$

The Szekeres dipole = an intrinsic dipole

To compare it with the measured CMB dipole: subtract the kinetic dipole due to our motion wrt the CMB rest frame.

The Szekeres dipole strength and direction vary with redshift.

In (Secrest et al ApJ 2021) the dipoles are different in radio-galaxy and in quasar surveys. Due to redshift difference?

In an isotropic universe, there is no preferred direction => C_l independent of m

In Szekeres the dependence on m mut be taken into account.

Sunyaev-Zeldovich effects

Definition: added spectral distortion of the CMB through inverse Compton scattering of the CMB photons by high-energy electrons in galaxy clusters. Three types:

- Thermal SZ: the electrons owe their high-energy to their temperature. The spectral distortion of the CMB temperature is given by $$\frac{\Delta T_{TSZ}}{T_{CMB}} = f(x)y = f(x)\int n_e \frac{k_B T_e}{m_e c^2}\sigma_T \mathrm{d}l$$

which is independent of redshift => a powerful tool for analyzing the high-redshift universe.

More interestingly, the TSZE gives the cluster area distance in a cosmology-independent way through (Birkinshaw et al. ApJ 1991)

$$D_A = \frac{N_{RJ}^2}{N_X}\left(\frac{m_e c^2}{k_B T_{eo}}\right)^2 \frac{\Lambda_{eo}}{16\pi T_{CMB}^2 \sigma_T^2 (1+z)^3}$$

which can be used to compare the Szekeres D_A with its measured value given by the above rhs.

- Kinetic SZ: Doppler effect on the scattered photons due to the cluster bulk velocity. No proper way to use it in a Szekeres framework

- SZ polarization effect: too small to be measurable.

Baryon acoustic oscillations

The simplest model-independent BAO measurements involve the angular 2-point correlation function, using thin redshift bins => the angular BAO scale

$$\theta_{BAO}(z) = \frac{r_s}{(1+z)D_A(z)} \quad \text{where} \quad r_s = \int_{z_d}^{\infty} c_s(1+z)\mathrm{d}z \quad \text{and} \quad c_s(z) = \frac{1}{\sqrt{3\left(1+\frac{3\rho_b(z)}{4\rho_\gamma(z)}\right)}}$$

Assumption (standard): the acoustic scale is conserved along its travel through the late universe

$\Rightarrow$ considered as a standard ruler.

Interesting outcome of a typical galaxy survey: the comoving distances $D_M(z) = r_s/\Delta\theta$

Reciprocity theorem: $D_M = (1+z)D_A$ => constraints on Szekeres

Weak lensing

The magnification matrix of lens theory reads $\mathcal{A} = \begin{pmatrix} 1-\kappa-\gamma_1 & \gamma_2 \\ \gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}$

where the complex lensing shear $\gamma = \gamma_1 + \imath\gamma_2$ describes the streching of galaxy images and the convergence κ, a change in size and brightness.

The optical tidal matrix, which can be written in terms of the Szekeres Ricci and Weyl curvatures gives the evolution of $\mathcal{A}$.

Weak lensing surveys yield: source galaxy redshifts, galaxy positions on the sky, reduced shear estimates. The reconstruction of $\mathcal{A}$ can be done from these data and compared to the theoretical Szekeres lensing quantities.

Redshift-space distortion

The spatial distribution of galaxies appears distorted when plotted wrt redshift instead of distances, due to an additional (blue)redshift from their peculiar velocities.

Kaiser and finger-of-God effects, special relativistic in nature, are of no use for our purpose.

Gravitational redshift distortion arises from the net (blue)redshift gained by a photon climbing out the potential well of a galaxy before falling into that of the Milky Way. Galaxies at higher potential than Earth appear closer and those at lower potential appear farther away.

Integrated Sachs-Wolfe effect makes a photon passing through a low area of gravitational potential shielded from the cosmological expansion, and thus background galaxies appear closer.

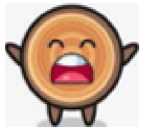
Accurate estimations of line-of-sight velocities is therefore strongly needed. A recent study (Chen et al. 2023) has done the work with a $\Lambda\mathrm{CDM}$ background. It will have to be adapted to a Szekeres background in the future.

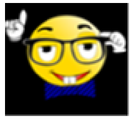
Redshift drift

Redshift drift = redshift increase (or decrease) measured by an observer looking at the same source at different instants. The source is measured on the observer's two different past light cones.

It occurs in any expanding universe.

But its magnitude depends on the geometry of the region travelled by the rays => constraints on the Szekeres parameters from its measurement.

Tiny amplitude: of the order $10^{-18}\ s^{-1}$ 

Solution (Piattella, Giani PRD 2017; Wucknitz et al. A&A 2021): use strong-gravitational-lens time delays to measure the drift; could be done within a much shorter time, the waiting for years between observations being replaced by the time delay. 

Differential cosmic expansion-Hubble flow anisotropies

Inhomogeneous models (LTB, Szekeres, etc.) exhibit differential cosmic expansion: expansion is a function of time and space: FLRW <-> Szekeres = $a(t) \leftrightarrow \Phi(t,r)$

The expansion rate can be split into a transverse and a longitudinal expansion. In Szekeres:

$$H_\perp = \frac{\Phi_{,t}}{\Phi} \qquad H_\| = \frac{\Phi_{,tr} - \Phi_{,t}E_{,r}/E}{\Phi_{,r} - \Phi E_{,r}/E}$$

The overall rate of change is given by the scalar expansion $\theta = 2H_\perp + H_\|$

Cosmography: Taylor expand the luminosity distance in powers of the redshift.

Express the coefficients as functions of the Szekeres expansion scalar and shear tensor (acceleration and vorticity are zero).

Result: in Szekeres, the quadrupole in the effective Hubble parameter and the dipole in the effective deceleration parameter are dominated by the shear.

Compare to observations (Dhawan et al. 2023, Cowell et al. 2023): these quadrupole and dipole are dominant in the corresponding effective parameters => hint in favor of Sz with high shear.

Position drift

A Szekeres observer who measures the same light source at different instants observes a drift in the redshift, but also in the position of the source.

Both can be calculated by a method inspired from Krasinski and Bolejko (PRD 2011).

The only spacetimes of the Szekeres family where the position drift vanishes are FLRW => an observation of this drift for any remote source = the Universe is inhomogeneous and FLRW ruled out at the corresponding scales

Effect too tiny ($\sim 10^{-6} - 10^{-7} \mathrm{arcsec/year}$) to be measured soon

But technical progress can be hoped

Data fitting with neural networks

Physical constraints

Metric signature preservation: $\epsilon - k > 0$ (save where $\Phi_{,r} - \Phi E_{,r}/E = 0$) => different evolution type depending on the value of ϵ

Shell-crossing avoidance: SCs are loci where the energy density diverges => must be avoided. Conditions for SC avoidance have been worked out by Szekeres (PRD 1975) and Hellaby and Krasinski (PRD 2002,2008)

Regular origin: implies $M \sim \Phi^3, \quad k \sim \Phi^2, \quad (S, P, Q) \sim \Phi^n, n \geq 0$

Positive areal radius: $\Phi(t, r) > 0, \quad \forall t \quad \text{and} \quad r$

Metric non degenerate and nonsingular: => non-vanishing E and S

Weak energy condition: either $M_{,r} - 3ME_{,r}/E \geq 0 \quad \text{and} \quad \Phi_{,r} - \Phi E_{,r}/E \geq 0$ or $M_{,r} - 3ME_{,r}/E \leq 0 \quad \text{and} \quad \Phi_{,r} - \Phi E_{,r}/E \leq 0.$

Asymptotic recovery of an FLRW universe at large scales: => $S = 2\epsilon, \quad P = Q = 0, \quad \Phi(t, r) = rR(t), \quad k = k_0 r^2$ for $t \geq t_{trans}$ and $r \geq r_{trans}$

Fixation of the r coordinate freedom: multiple possible choices depending on the considered effects MNC (2024)

All the free functions must be differentiable

Fitting method

Problem: determine the 5 functions and 7 constant parameters ($\Lambda,\ \epsilon,\ t_{trans},\ r_{trans},\ r_{obs},\ p_{obs},\ q_{obs}$) defining the cosmological Szekeres model from huge data sets at some given precision

Need: very powerful calculating device and method adapted to the fitting of functions

Device: machines designed to implement deep learning

Method: symbolic regression

Ultimate goal: use the whole set of available cosmological data to constrain the model = the same goal as the one pursued up to now with FLRW, but using Szekeres -> FLRW instead

Warning: implies adapting to the new model all the survey data reduction processes

Take-home messages

- Averaged homogeneous cosmological models such as FLRW can no more satisfy the requirements of precision cosmology (tensions, anomalies, etc)
- Szekeres inhomogeneous models, including FLRW models as homogeneous limits, appear as a great available GR alternative
- The main equations determining the behaviour of Szekeres models have been established and can be used to reproduce the physical effects at work in the Universe
- They exhibit an intrinsic mass dipole which has been observed in recent survey analyses
- Theoretical cosmography results are consistent with other recent observation analyses
- The five functions of r and the seven constants defining the Szekeres universe model can be constrained through current and in-coming data surveys at a given precision
- This will imply the use of symbolic regression implemented by machine learning
- An adaptation to the new model Szekeres -> FLRW of the data reduction processes in all cosmological surveys is essential to avoid systematic bias

THANK YOU FOR YOUR ATTENTION