CLUSTER OF EXCELLENCE
QUANTUM UNIVERSE

Small-Instanton induced Flavor Invariants and the Axion Potential

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DMLab meeting
17—18 Oct 2024, LPNHE, Paris

Preliminary & Outline of this talk

- Axion potential:

  How to make small(UV)-instanton becoming relevant?

  QCD contribution & UV contributions via Instanton effects

- Small-instantons & Axion potential:

  Topological susceptibilities and Flavour invariants

  UV completions of small-instanton

  Bounds from neutron EDM

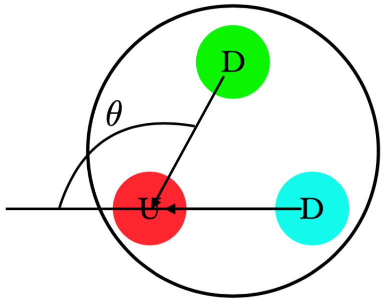
Preliminary

Strong CP problem & Axion solution

1.) QCD vacuum allows an effective(CP violating) term in the Lagrangian:

$$\mathcal{L} \supset \bar{\theta} \frac{g_s^2}{32\pi^2} G_{\mu\nu} \tilde{G}^{\mu\nu}$$

#Key feature: $\bar{\theta} = \theta_{\text{QCD}} - \arg\left(\det M_q\right)$ received contributions from both Strong & Electroweak sectors => theta-bar expected to be O(1)



2.) Bound from Neutron EDM: $\bar{\theta} < 10^{-10}$

Strong CP problem: Why is theta-bar so small?

Alternative questions: why no CP-violation in QCD? What make theta-bar so small? (any mechanism behind?)

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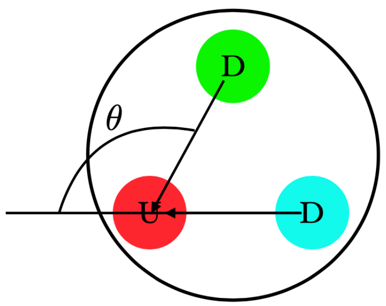

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Strong CP problem: Why is theta-bar so small?

Alternative questions: why no CP-violation in QCD? What make theta-bar so small? (any mechanism behind?)

3.) Axion solution: dynamically relaxes theta-bar to zero

QCD confinement gives axion potential:

$V_{\chi PT}(a) \sim 1 - cos(a/f_a)$

minimised at <a>=0

$$\mathcal{L} \supset \left(\bar{\theta} + \frac{a}{f_a}\right) \frac{g_s^2}{32\pi^2} G_{\mu\nu} \tilde{G}^{\mu\nu}$$

- Axion as Goldstone Boson of $U(1)_{PQ}$ anomalous symmetry
- Shift symmetry $\frac{a}{f_a} \to \frac{a}{f_a} + \epsilon$

=> absorb $\bar{\theta}$ effects

Preliminary & Outline of this talk

- Instanton #101:

QCD θ-vacuum = Superposition of n-vacua (energy degenerate but topologically distinct)

$$|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{-in\theta}|n\rangle = \cdots |0\rangle + e^{-i\theta}|1\rangle + \cdots$$

Instanton describes the tunnelling effect between degenerate n-vacua

Localized objects in Euclidean spacetime, satisfying the Euclidean equation of motion with non-trivial topologies and therefore minimize the Euclidean action.

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Explicit SU(2) BPST instanton solution with Q = 1:

$$\frac{g^2}{32\pi^2}\int d^4x\, G^A_{\mu\nu}\tilde{G}^{A,\mu\nu}(x)\Big|_{\text{inst.}} = Q\,, \text{ where } Q \in \mathbb{Z}\,.$$

(Background field configuration)

$$G^a_\mu(x)\Big|_{1-\text{inst.}} = 2\,\eta_{a\mu\nu}\frac{(x-x_0)_\nu}{(x-x_0)^2+\rho^2}$$

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Characterized by a set of collective coordinates => zero-modes (family of equivalent solutions)

x_0 — Location of instanton

ρ — Size of instanton

Preliminary & Outline of this talk

- Instanton #101: Path Integral with Instanton configurations

$$S_{\rm YM}^{\rm inst.} = \int d^4x \left(\frac{1}{4} G^A_{\mu\nu} G^{A,\,\mu\nu} + i\theta_{QCD} \frac{g^2}{32\pi^2} G^A_{\mu\nu} \tilde{G}^{A,\,\mu\nu} \right) \Bigg|_{\rm (a.-)inst.} = \frac{8\pi^2}{g^2} |Q| + i\theta_{QCD}\, Q$$

=> Within perturbative regime, $Q = \pm 1$ will dominate the Euclidean Path Integral

$$Z = \int \mathcal{D}A\, e^{-S_{\rm YM}^{\rm inst.}}$$

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Estimating instanton effects => vacuum-to-vacuum transition amplitude:

$$\langle 0|0\rangle \,\big|_{\rm 1\text{-}inst.} \sim \int \mathcal{D}\varphi_I^{(0)} \int \mathcal{D}\varphi_I^{(\neq 0)}\, e^{-\left[S^{\rm 1\text{-}inst.}_{\rm YM} + \int d^4x\, \varphi_I^\dagger \left(\frac{\delta^2 \mathcal{L}}{\delta \varphi_I^2}\right) \varphi_I\right]}$$

Zero modes measure Non-zero modes measure

't Hooft 76
Shifman, Vainshtein, Zakharov 79

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Integrating out non-zero modes

$$\langle 0|0\rangle \big|_{\text{1-inst.}} = e^{-i\theta_{\rm QCD}} \int d^4x_0 \int \frac{d\rho}{\rho^5}\, d_N(\rho) \int \prod_{f=1}^{N_f} \left(\rho\, d\xi_f^{(0)} d\bar{\xi}_f^{(0)}\right) e^{\int d^4x\, (-\bar{\psi} J \psi + \text{h.c.})}$$

Instanton density

't Hooft 76
Shifman, Vainshtein, Zakharov 79

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$$d_N(\rho) \sim e^{-8\pi^2/g^2(\Lambda = 1/\rho)}$$

Strongly suppressed at hight energy scale
(for asymptotic free theory)

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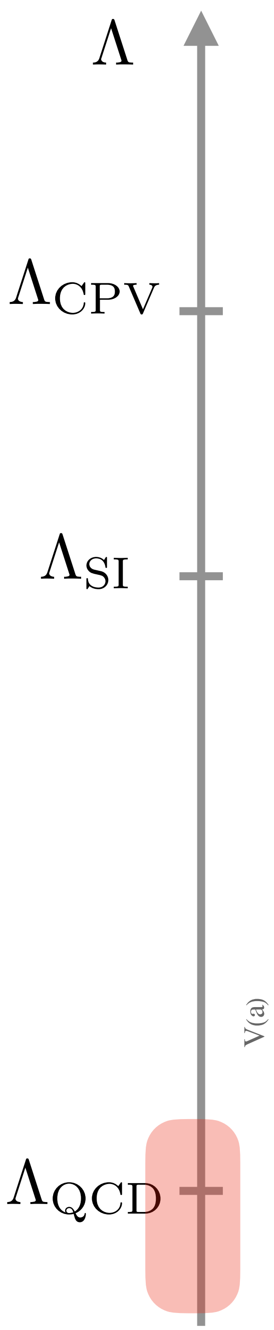
When small-instanton effects become relevant?
=> Boost the QCD coupling at high energy scale

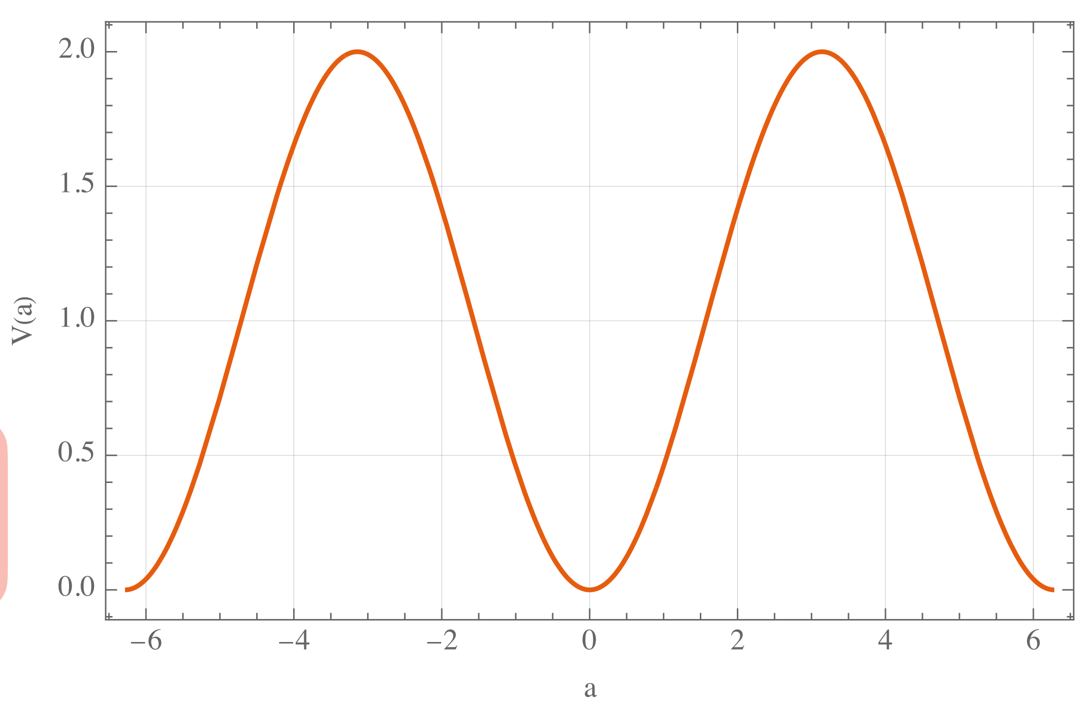
- Non-trivial embedding of QCD in UV theories: $SU(3)_1 \times SU(3)_2 \times \cdots \times SU(3)_k \to SU(3)_{\text{QCD}}$
- Extra-dimensions (5D instantons)

Agrawal and Howe (1710.04213)
C. Csáki, M. Ruhdorfer, Y. Shirman (1912.02197)
T. Gherghetta, V. V. Khoze, A. Pomarol, Y. Shirman (2001.05610)

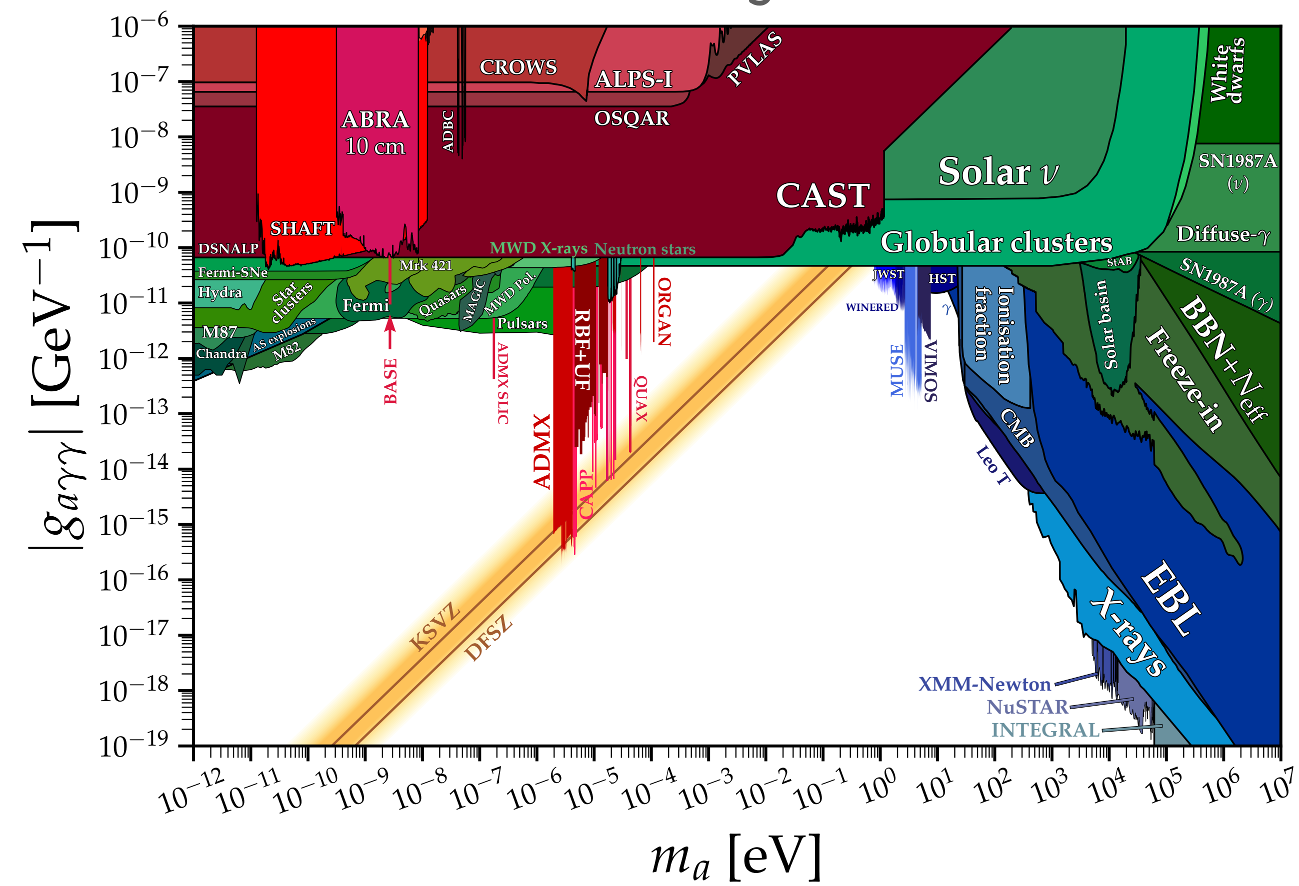
Preliminary & Outline of this talk

- Axion potential: QCD contribution





cajohare.github.io/AxionLimits/



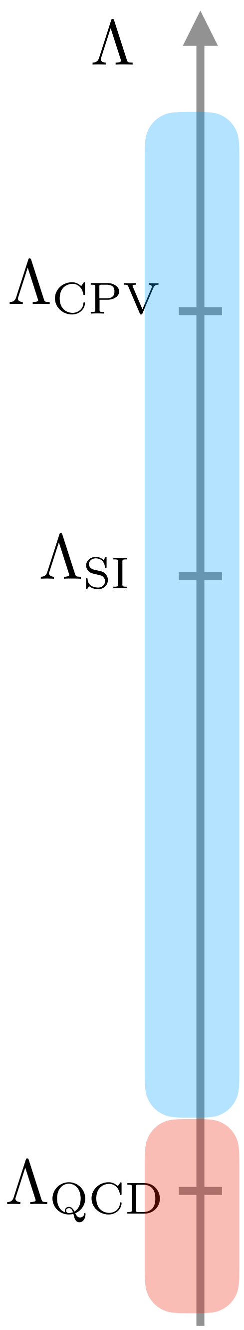
$$V(a) \sim m_\pi^2 f_\pi^2 \left[1 - \cos\frac{a}{f_a}\right] \rightarrow \langle \frac{a}{f_a} \rangle = 0$$

$$m_a^2 f_a^2 = \frac{m_u m_d}{(m_u + m_d)^2} m_\pi^2 f_\pi^2$$

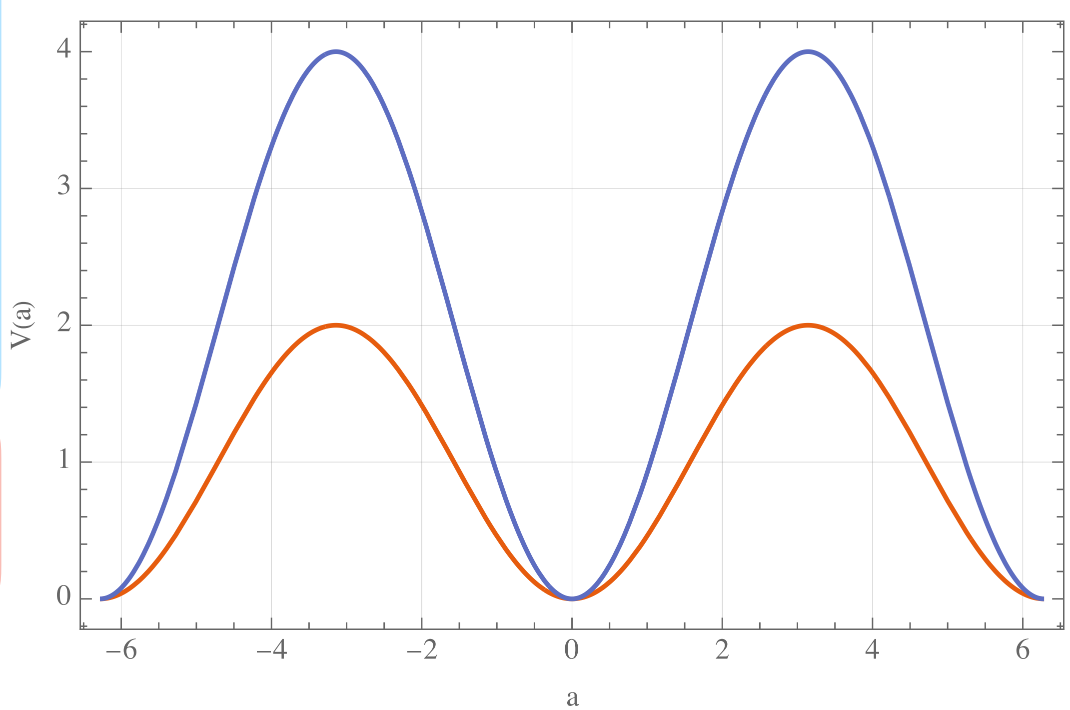
Preliminary & Outline of this talk

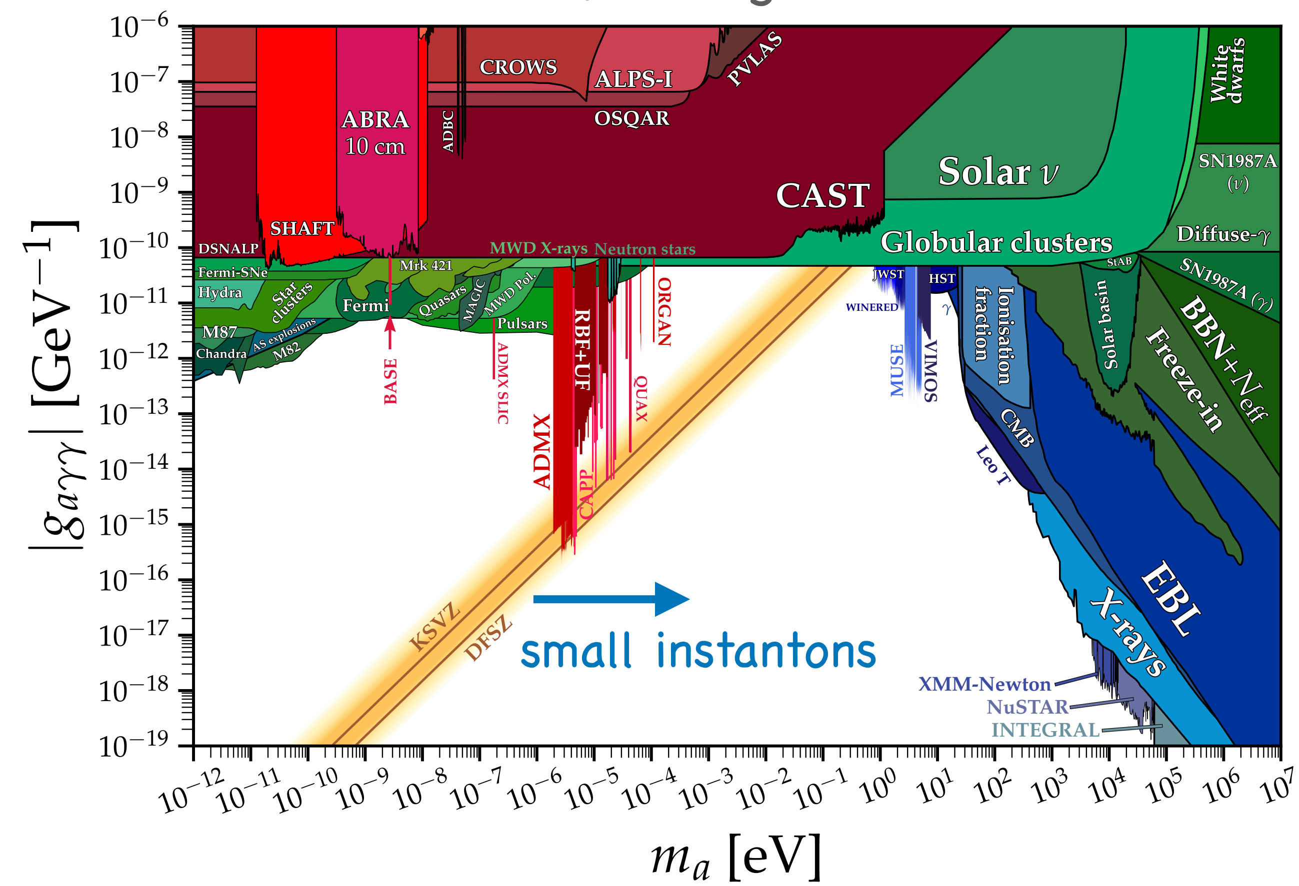
- Axion potential: UV aligned contribution

cajohare.github.io/AxionLimits/



Small instantons $\rho = 1/\Lambda$





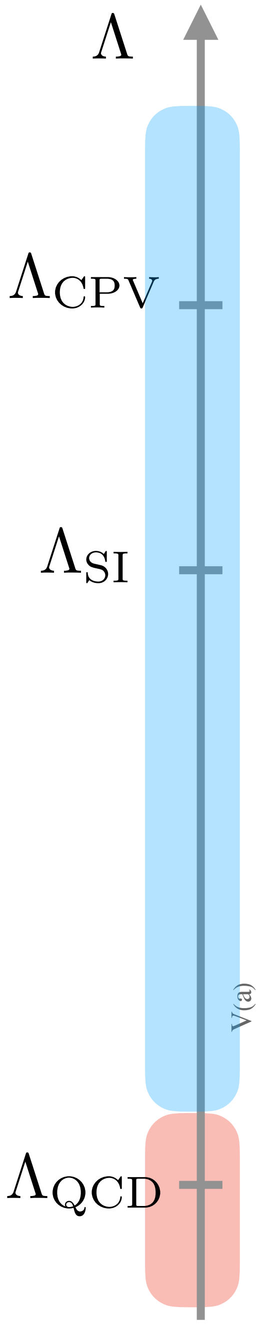
$$V(a) \sim (m_\pi^2 f_\pi^2 + \Lambda_{UV}^4)\left[1 - \cos\frac{a}{f_a}\right]$$

Enhance axion mass &

solve strong CP problem

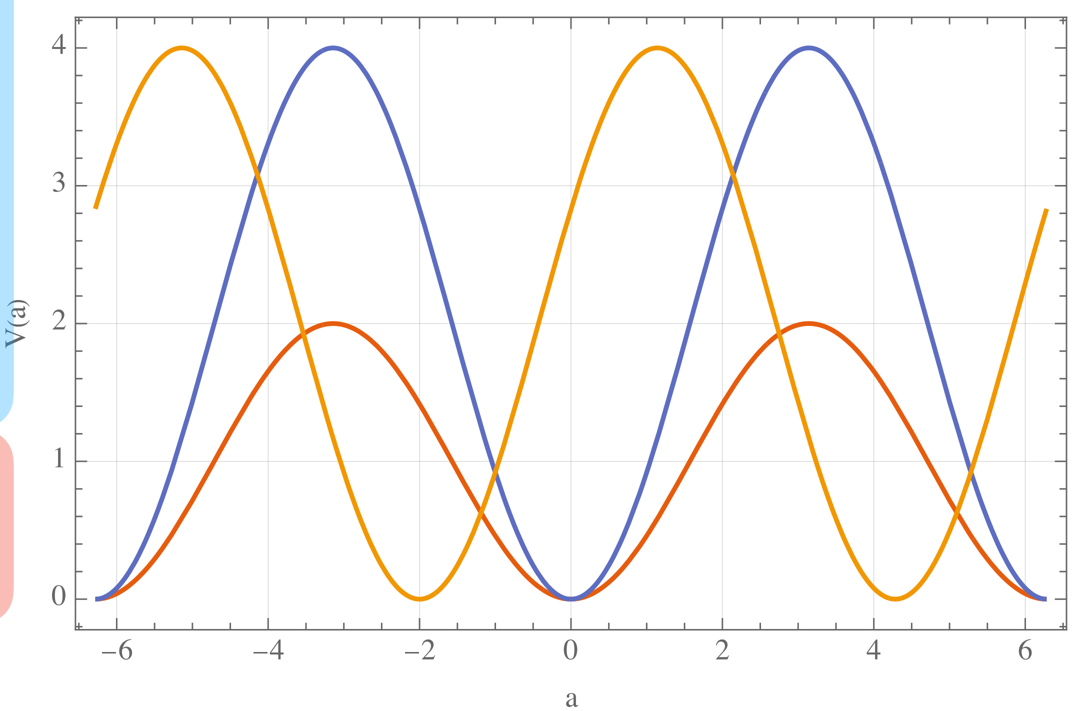
Preliminary & Outline of this talk

- Axion potential: UV misaligned contribution



New source of CPV, e.g. SMEFT CPV effective operators

Small instantons $\rho = 1/\Lambda$



$$V(a) \sim (m_\pi^2 f_\pi^2 + \Lambda_{UV}^4)\left[1 - \cos\left(\frac{a}{f_a} + \delta_{CPV}\right)\right]$$

Can potentially destroy the Axion solution

Preliminary & Outline of this talk

- CP-violation: The case of Standard Model(SM)

Jarlskog '85
Bernabeu, Branco, Gronau '86

CPV is parametrised by Jarlskog invariant:

$$J_4 = \mathrm{ImTr}\left([Y_u Y_u^\dagger, Y_d Y_d^\dagger]^3\right)$$

=> CP is conserved iff $J_4 = 0$ (neglecting $\bar{\theta}$)

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- CPV in the SM will not misalign the axion potential: Appear at 4-loop (from threshold corrections) and 7-loop (from radiative corrections) level

Ellis, Gaillard '79
Khriplovich '86
Georgi, Randall '86

$$\bar{\theta}_{\mathrm{ind}}^{(\mathrm{SM})} \sim 10^{-19}$$

Preliminary & Outline of this talk

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- BSM CP-violation: The case of SMEFT

$$\mathcal{L}_{\mathrm{SMEFT}} = \mathcal{L}_{\mathrm{SM}} + \sum_i \frac{C_i \mathcal{O}_i^{(6)}}{\Lambda^2}$$

Contain 1149 CP-odd couplings !!!

=> Generalise Jarlskog invariant to study CPV in the SMEFT systematically

Bonnefoy, Gendy, Grojean, Ruderman

2112.03889, 2302.07288

Preliminary & Outline of this talk

- CP-violation: The case of SMEFT

Considering non-perturbative effects => Use $\theta_{\rm QCD}$ as a spurion:

	$U(3)_{\rm Q}$	$U(3)_{\rm u}$	$U(3)_{\rm d}$	$U(3)_{\rm L}$	$U(3)_{\rm e}$
$e^{i\theta_{\rm QCD}}$	$\mathbf{1}_{+6}$	$\mathbf{1}_{-3}$	$\mathbf{1}_{-3}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$
$Y_{\rm u}$	$\mathbf{3}_{+1}$	$\bar{\mathbf{3}}_{-1}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$
$Y_{\rm d}$	$\mathbf{3}_{+1}$	$\mathbf{1}_{0}$	$\bar{\mathbf{3}}_{-1}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$
$Y_{\rm e}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$	$\mathbf{3}_{+1}$	$\bar{\mathbf{3}}_{-1}$

SM has one more CP-odd flavour invariant:

$$J_\theta = {\rm Im}[e^{-i\theta_{\rm QCD}} \det(Y_{\rm u} Y_{\rm d})]$$

Built flavour invariants featuring $\theta_{\rm QCD}$ for CP-violating SMEFT operators:

$$\mathcal{O}^{(1)}_{quqd} = \bar{Q}u\bar{Q}d$$

$$\mathcal{I}(C^{(1,8)}_{\rm quqd}) = {\rm Im}\left[e^{-i\theta_{\rm QCD}}\epsilon^{ABC}\epsilon^{abc}\epsilon^{DEF}\epsilon^{def}Y_{{\rm u},Aa}Y_{{\rm u},Bb}C^{(1,8)}_{{\rm quqd},CcDd}Y_{{\rm d},Ee}Y_{{\rm d},Ff}\right]$$

Note: $\bar{Q}u\bar{Q}d$ has 81 CP-odd phases

Bonnefoy, Gendy, Grojean, Ruderman
2112.03889, 2302.07288

Preliminary & Outline of this talk

- Small instanton & Axion potential: UV misaligned contributions

Small instantons generate axion potential of the form:

$$V(a) = \chi_{\mathcal{O}}(0)\frac{a}{f_a} + \frac{1}{2}\chi(0)\left(\frac{a}{f_a}\right)^2 \longrightarrow \langle\frac{a}{f_a}\rangle \equiv \theta_{\rm ind} = -\frac{\chi_{\mathcal{O}}(0)}{\chi(0)}$$

Induced by CP-violating operator

This talk

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Induced by CP-violating operator

This talk

Coefficients in the potential can be computed from following correlators: Witten '79

$$\chi(0) = -i\lim_{k\to 0}\int d^4x\, e^{ikx}\left\langle 0\middle| T\left\{\frac{g^2}{32\pi^2}G\tilde{G}(x)\,,\,\frac{g^2}{32\pi^2}G\tilde{G}(0)\right\}\middle|0\right\rangle\Bigg|_{\text{1-(a.-)inst.}}$$

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Evaluating these correlation functions within perturbative regime and one-(anti)instanton approximation. Making connection with SMEFT flavour invariants => Simplify the calculations

Small instanton & Axion potential: Evaluating the correlator $\chi_{\mathcal{O}}(0)$

- Core technique 1: Path Integral & Instanton background

$$\chi_{\mathcal{O}}(0) = -i \lim_{k\to 0} \int d^4x\, e^{ikx} \left\langle 0 \left| T \left\{ \frac{g^2}{32\pi^2} G\tilde{G}(x)\,,\ \frac{C_{\mathcal{O}}}{\Lambda^2_{\not{CP}}} \mathcal{O}[\varphi_{\rm I}, \varphi](0) \right\} \right| 0 \right\rangle ,$$

$$= e^{-i\theta_{\rm QCD}} \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \int \prod_{f=1}^{N_f} \left(\rho\, d\xi_f^{(0)}\, d\bar{\xi}_f^{(0)}\right)$$

$$\times \int \mathcal{D}\varphi\, e^{-S_0[\varphi]-S_{\rm int}[\varphi_{\rm I},\varphi]} \int d^4x \frac{g^2}{32\pi^2} G\tilde{G}(x) \times \frac{C_{\mathcal{O}}}{\Lambda^2_{\not{CP}}} \mathcal{O}[\varphi_{\rm I}, \varphi](0) \Big|_{\text{1-(a.-)inst.}}$$

Fields with instanton solutions (e.g. gluon, quark): φ_I

=> Expand the fields in their eigenmodes, replace zero mode wave function by instanton solutions, and integrate out non-zero modes:

Small instanton & Axion potential: Evaluating the correlator $\chi_{\mathcal{O}}(0)$

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Fields without instanton solutions: φ

=> Integrate over without performing the eigenmode expansion

Small instanton & Axion potential: Evaluating the correlator $\chi_{\mathcal{O}}(0)$

Core technique 2: Fermion zero mode & Grassmann integral

Fermion eigenmode expansion & fermion zero-mode solutions:

$$\psi_f(x) = \sum_k \xi_f^{(k)} \psi^{(k)}(x)\,; \quad \bar{\psi}_f(x) = \sum_k \bar{\xi}_f^{(k)} \bar{\psi}^{(k)}(x)$$

$$-i\not{D}\Big|_{\text{1-inst.}} \psi^{(0)}(x) = 0. \quad \longrightarrow \quad \psi^{(0)}(x)\Big|_{\text{1-inst.}} = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \frac{1}{\pi} \frac{\rho}{\left[(x-x_0)^2+\rho^2\right]^{3/2}} \begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \quad \varphi_{\alpha m} = \epsilon^{\alpha m}$$

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Fermion zero modes & Grassmann integral give rise to determinant-like structures:

- The well-know Grassmann integration identity:

$$\int d^3\xi_1 d^3\xi_2 \; e^{\xi_1 A \xi_2} = \det A$$

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$$\int d^3\xi_1 d^3\xi_2 \; e^{\xi_1 A \xi_2} = \det A$$

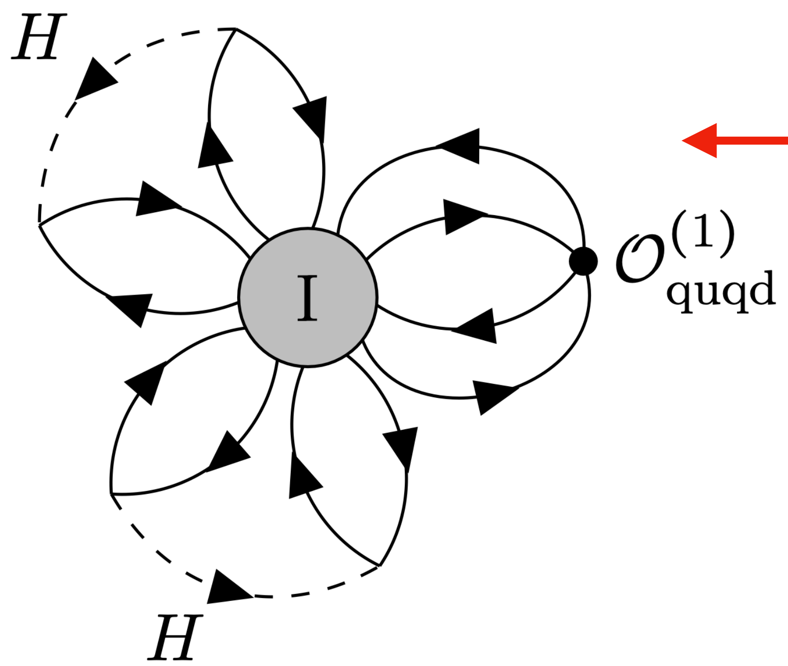
- Generalise the Grassmann integration identity for operator insertion:

Example: $$\int d^3\xi_1 d^3\xi_2 \; e^{\xi_1 A \xi_2} \xi_1 B \xi_2 = \frac{1}{2} \epsilon^{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} A_{i_1 j_1} A_{i_2 j_2} B_{i_3 j_3}$$

$$\int d^3\xi_1 d^3\xi_2 d^3\xi_3 d^3\xi_4 \; e^{\xi_1 A \xi_2 + \xi_3 B \xi_4} \xi_1 C \xi_2 = \frac{1}{2} \epsilon^{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} A_{i_1 j_1} A_{i_2 j_2} C_{i_3 j_3} \det B$$

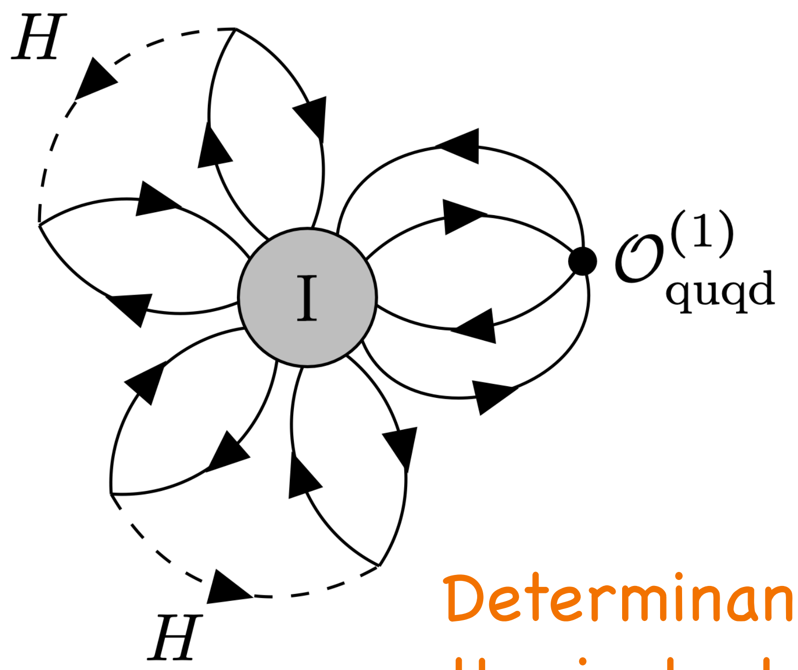
Determinant-like contraction

Topological Susceptibilities & Flavor invariants: Four-quark operator



$$\chi^{(1)}_{\rm ququd}(0)^{1-\rm inst.} = -i \lim_{k\to 0}\int d^4x\, e^{ikx}\left\langle 0\left|T\left\{\frac{1}{32\pi^2}G\widetilde{G}(x), \frac{C^{(1)}_{\rm ququd}}{\Lambda^2_{\not{CP}}}\mathcal{O}^{(1)}_{\rm ququd}(0)\right\}\right|0\right\rangle$$

Topological Susceptibilities & Flavor invariants: Four-quark operator



Determinant-like flavour invariants naturally arise in the instanton calculations

$$\chi^{(1)}_{\rm quqd}(0)^{1-\rm inst.} = \frac{1}{4\Lambda^2_{\not{CP}}}\Big[e^{-i\theta_{\rm QCD}}\epsilon^{i_1i_2m}\epsilon^{j_1j_2n}Y_{{\rm u},i_1j_1}Y_{{\rm u},i_2j_2}C^{(1)}_{{\rm quqd},mnop}\epsilon^{k_1k_2o}\epsilon^{l_1l_2p}Y_{{\rm d},k_1l_1}Y_{{\rm d},k_2l_2}$$

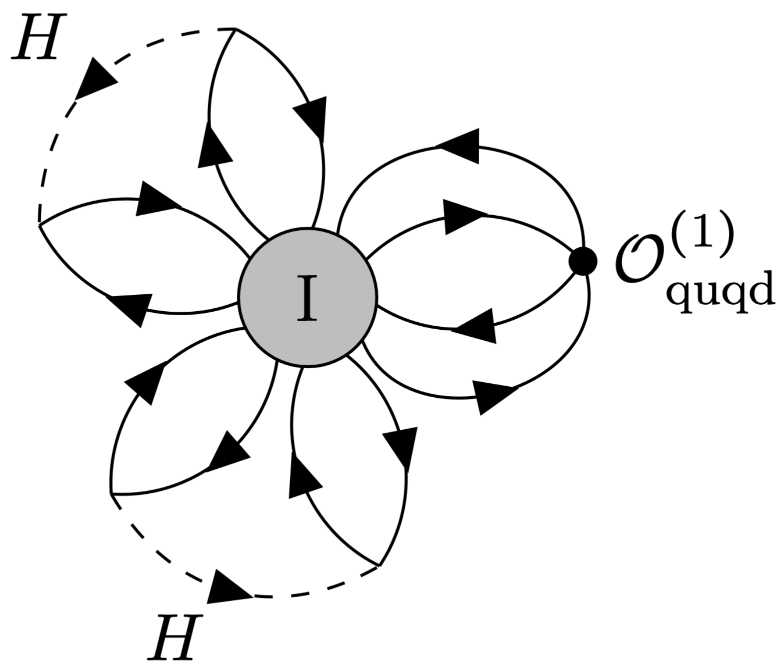
$$+e^{-i\theta_{\rm QCD}}\epsilon^{i_1i_2m}\epsilon^{j_1j_2n}Y_{{\rm u},i_1j_1}Y_{{\rm u},i_2j_2}C^{(1)}_{{\rm quqd},onmp}\epsilon^{k_1k_2o}\epsilon^{l_1l_2p}Y_{{\rm d},k_1l_1}Y_{{\rm d},k_2l_2}\Big]\int d^4x_0\int\frac{d\rho}{\rho^5}d_N(\rho)\rho^6$$

$$\times\underbrace{\int\mathcal{D}H\mathcal{D}H^\dagger\,e^{-S_0[H,H^\dagger]}\Big[\int d^4x_1d^4x_2(\bar\psi^{(0)}H_I^\dagger\epsilon^{IJ}P_R\psi^{(0)})(x_1)\,(\bar\psi^{(0)}\epsilon_{JK}H^KP_R\psi^{(0)})(x_2)\Big]^2}_{=2!\left[\int d^4x_1d^4x_2(\bar\psi^{(0)}P_R\psi^{(0)})(x_1)\Delta_H(x_1-x_2)\epsilon_{IJ}\epsilon^{JI}(\bar\psi^{(0)}P_R\psi^{(0)})(x_2)\right]^2\equiv 2!\,\mathcal{I}^2}$$

$$\times\left(\epsilon_{MN}\epsilon^{MN}\bar\psi^{(0)}P_R\psi^{(0)}\,\bar\psi^{(0)}P_R\psi^{(0)}\right)(0)\int d^4x\frac{G\widetilde{G}(x)}{32\pi^2}\,.$$

Fermion zero modes

Topological Susceptibilities & Flavor invariants: Four-quark operator

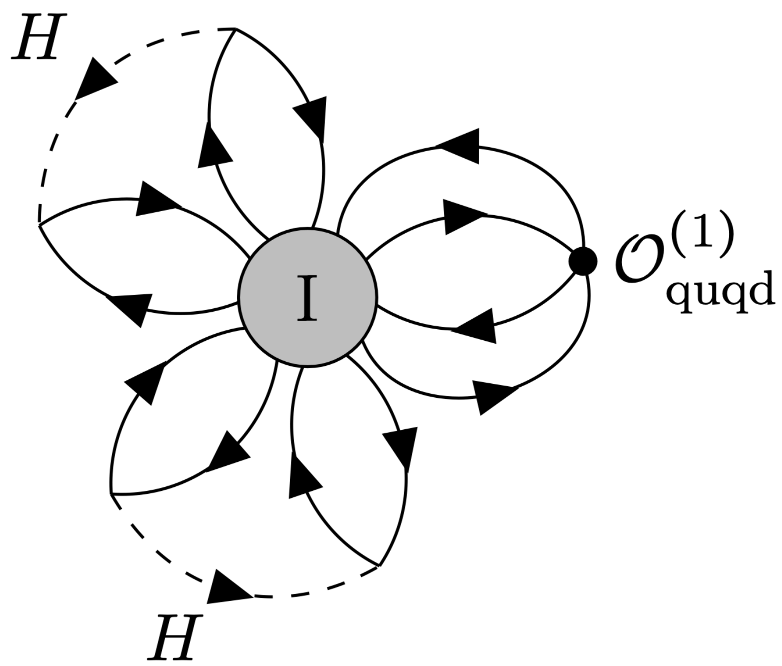


Plugging explicit form of fermion zero modes
Integrate over loop momenta,
collective coordinates

$$\chi^{(1)\,({\rm UV})}_{\rm ququd}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right)\right)\int \frac{d\rho}{\rho^5}\, d_N(\rho)\frac{2!}{(6\pi^2)^2}\frac{2}{5\pi^2\rho^2}$$

Contraction of Yukawa matrices
encapsulated in the Flavour invariants

Topological Susceptibilities & Flavor invariants: Four-quark operator



Can also use Instanton Naive Dimensional Analysis (NDA), result up to $\mathcal{O}(1)$

Csáki, D'Agnolo, Kuflik, Ruhdorfer (2311.09285)

$$\chi^{(1)\,(\mathrm{UV})}_{\rm ququd}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right)\right)\int \frac{d\rho}{\rho^5} d_N(\rho)\frac{1}{(256\pi^6)\rho^2}$$

Contraction of Yukawa matrices encapsulated in the Flavour invariants

Combining Flavour invariants & Instanton NDA
=> quickly estimate 't Hooft flower diagrams

Topological Susceptibilities & Flavor invariants: Semi-leptonic operator

$\mathcal{O}_{\rm lequ}^{(1)} = \bar{L}e\bar{Q}u + \text{h.c.}$

- Now start from the Flavour invariants:

$$\text{Im}\left(I_{\rm lequ}^{(1)}\right) = \text{Im}\left[e^{-i\theta_{\rm QCD}}\epsilon^{i_1 i_2 m}\epsilon^{j_1 j_2 n} Y_{{\rm u},i_1 j_1} Y_{{\rm u},i_2 j_2} C_{{\rm lequ},opmn}^{(1)} Y_{{\rm e},po}^{\dagger} \det Y_{\rm d}\right] = \mathcal{I}_{0000}^{0}\left(C_{\rm lequ}^{(1)}\right)$$

Trace-like contraction

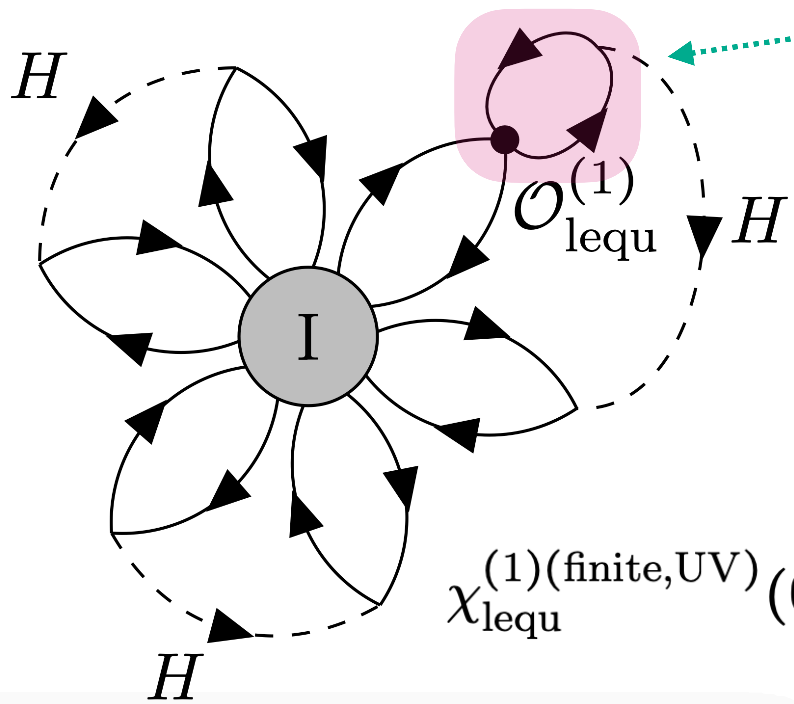
Topological Susceptibilities & Flavor invariants: Semi-leptonic operator

- Now start from the Flavour invariants:

$$\mathcal{O}^{(1)}_{\rm lequ} = \bar{L}e\bar{Q}u + \text{h.c.}$$

$$\text{Im}\left(I^{(1)}_{\rm lequ}\right) = \text{Im}\left[e^{-i\theta_{\rm QCD}}\epsilon^{i_1 i_2 m}\epsilon^{j_1 j_2 n}Y_{{\rm u},i_1 j_1}Y_{{\rm u},i_2 j_2}C^{(1)}_{{\rm lequ},opmn}Y^{\dagger}_{{\rm e},po}\det Y_{\rm d}\right] = \mathcal{I}^{0}_{0000}\left(C^{(1)}_{\rm lequ}\right)$$

Trace-like contraction

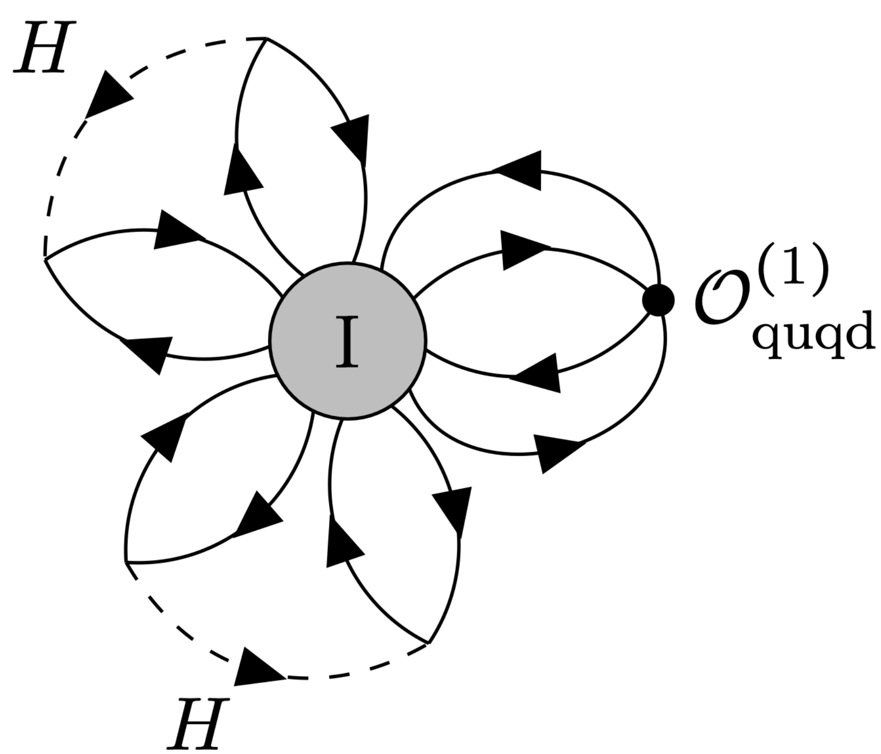


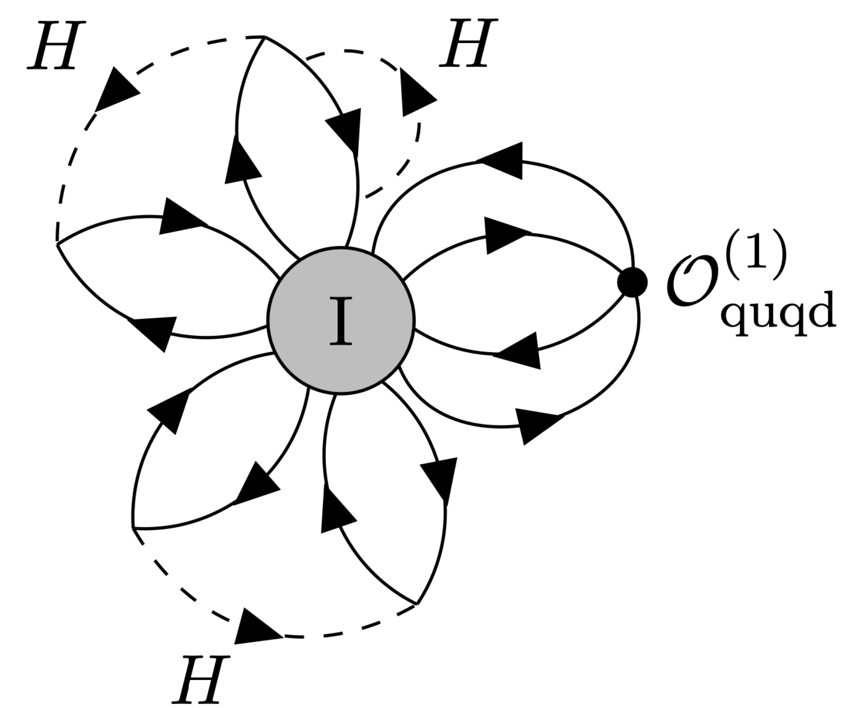
1-loop suppression induced by leptonic fields

$$\chi^{(1)(\text{finite,UV})}_{\rm lequ}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\mathcal{I}^{0}_{0000}\left(C^{(1)}_{\rm lequ}\right)\int\frac{d\rho}{\rho^5}d_N(\rho)\frac{3!}{(6\pi^2)^2}\frac{11+30\left(\log\left(\rho\Lambda_{\not{CP}}\right)+\gamma_{\rm E}-\log 2\right)}{600\pi^4\rho^2}$$

- Anticipating how CPV SMEFT operators participate in the instanton computations
- Classifying the leading effects from the Wilson coefficients

Topological Susceptibilities & Flavor invariants: Higher-order Invariants





$$\mathcal{A}^{a_1,b_1,c_1,d_1}_{a_2,b_2,c_2,d_2}(C^{(1,8)}_{\rm quqd}) = {\rm Im}\Big[e^{-i\theta_{\rm QCD}}\epsilon^{ABC}\epsilon^{abc}\epsilon^{DEF}\epsilon^{def}Y_{{\rm u},Aa}Y_{{\rm u},Bb}\left(X^{a_1}_{\rm u}X^{b_1}_{\rm d}X^{c_1}_{\rm u}X^{d_1}_{\rm d}\right)_C^{\ C'}$$
$$\times C^{(1,8)}_{{\rm quqd},C'cD'd}\left(X^{a_2}_{\rm u}X^{b_2}_{\rm d}X^{c_2}_{\rm u}X^{d_2}_{\rm d}\right)_D^{\ D'}Y_{{\rm d},Ee}Y_{{\rm d},Ff}\Big]\,,$$

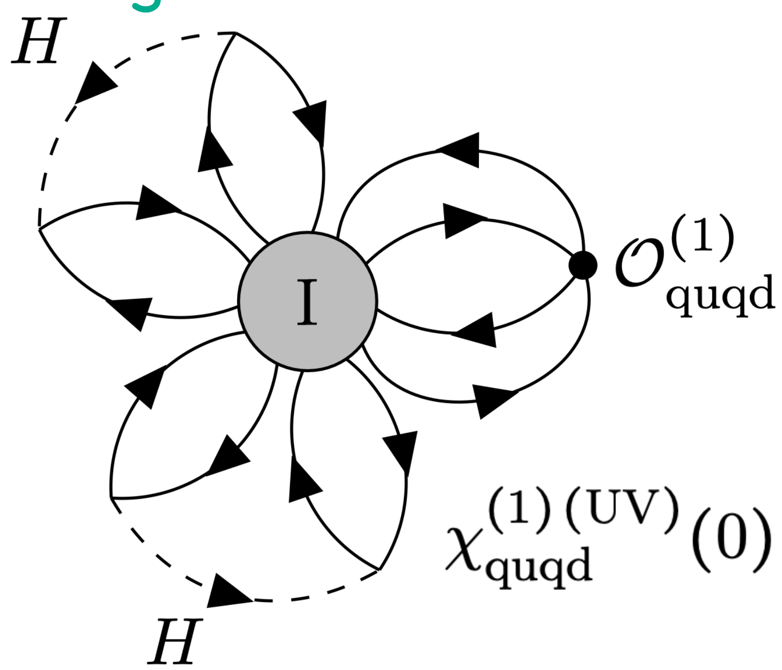
$$X_{\rm u,d} = Y_{\rm u,d}Y^\dagger_{\rm u,d}$$

=> Set X=1 for the lowest order flavour invariants

Topological Susceptibilities & Flavor invariants: Four-quark operator

#Integration over the size of instanton

$$\mathcal{O}^{(1)}_{\rm quqd} = \bar{Q}u\bar{Q}d + \text{h.c.}$$



$$\chi^{(1)\,({\rm UV})}_{\rm quqd}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\rm quqd}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\rm quqd}\right)\right)\int \frac{d\rho}{\rho^5}\, d_N(\rho)\frac{2!}{(6\pi^2)^2}\frac{2}{5\pi^2\rho^2}$$

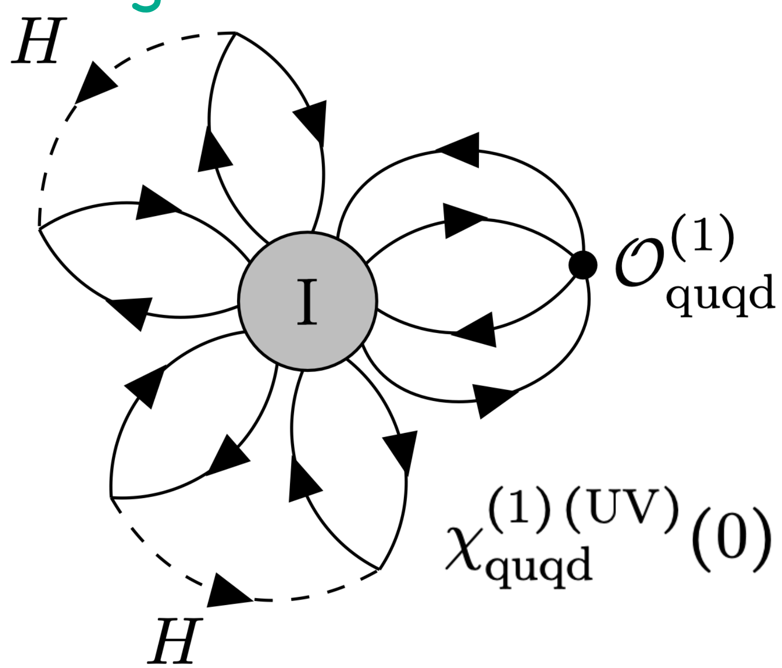
ρ-integral is IR divergent

=> Need a physical IR cut-off

Topological Susceptibilities & Flavor invariants: Four-quark operator

#Integration over the size of instanton

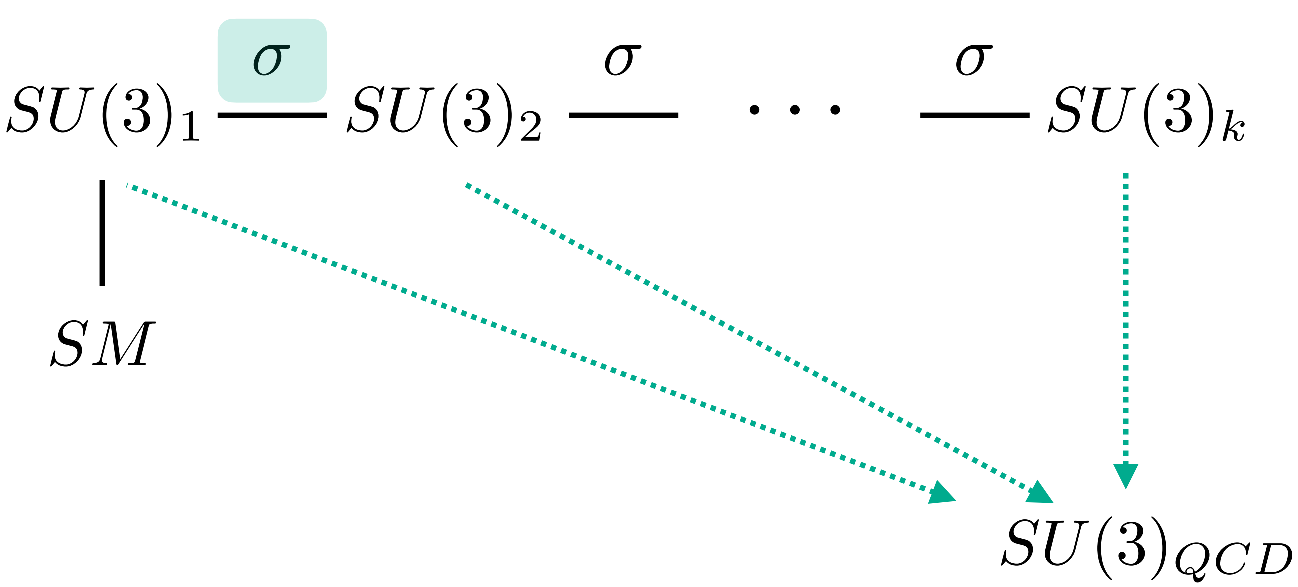
$$\mathcal{O}^{(1)}_{\rm quqd} = \bar{Q}u\bar{Q}d + \text{h.c.}$$



$$\chi^{(1)\,({\rm UV})}_{\rm quqd}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\rm quqd}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\rm quqd}\right)\right)\int \frac{d\rho}{\rho^5}\, d_N(\rho)\frac{2!}{(6\pi^2)^2}\frac{2}{5\pi^2\rho^2}$$

- Possible UV completion of small-instantons:

  Product of Gauge groups

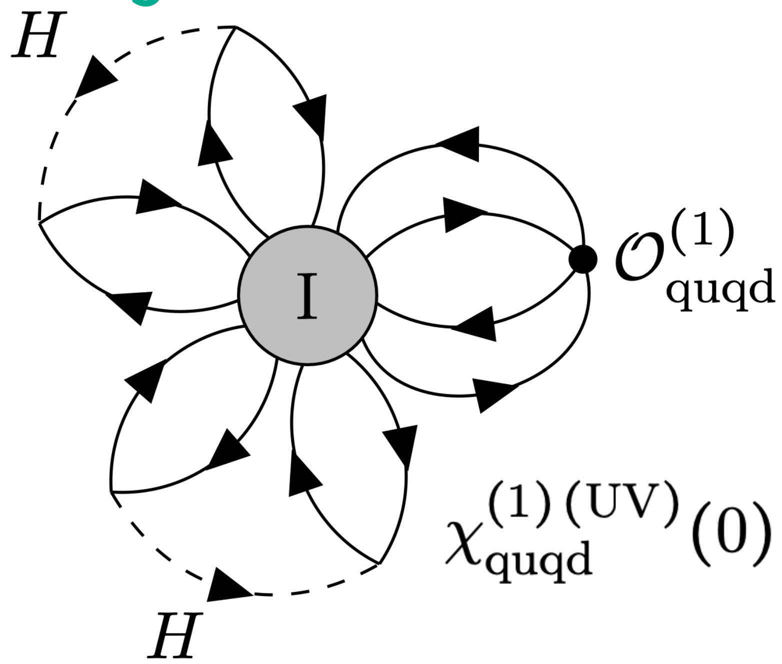


Agrawal and Howe (1710.04213)
C. Csáki, M. Ruhdorfer, Y. Shirman (1912.02197)

Topological Susceptibilities & Flavor invariants: Four-quark operator

#Integration over the size of instanton

$$\mathcal{O}^{(1)}_{\rm ququd} = \bar{Q}u\bar{Q}d + \text{h.c.}$$



$$\chi^{(1)\,({\rm UV})}_{\rm ququd}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right)\right)\int \frac{d\rho}{\rho^5}\, d_N(\rho)\frac{2!}{(6\pi^2)^2}\frac{2}{5\pi^2\rho^2}$$

- Possible UV completion of small-instantons:

Product of Gauge groups

$$SU(3)_1 \overset{\sigma}{\text{---}} SU(3)_2 \overset{\sigma}{\text{---}} \cdots \overset{\sigma}{\text{---}} SU(3)_k$$

$SU(3)_1$ — SM; $SU(3)_1, SU(3)_2, \ldots, SU(3)_k \rightarrow SU(3)_{QCD}$

Boost the coupling of each QCD subgroup:

$$\frac{1}{g^2_{\rm QCD}(\mu)} = \frac{1}{g_1^2(\mu)} + \frac{1}{g_2^2(\mu)} + \cdots + \frac{1}{g_k^2(\mu)}$$

Provide a physical cut-off:

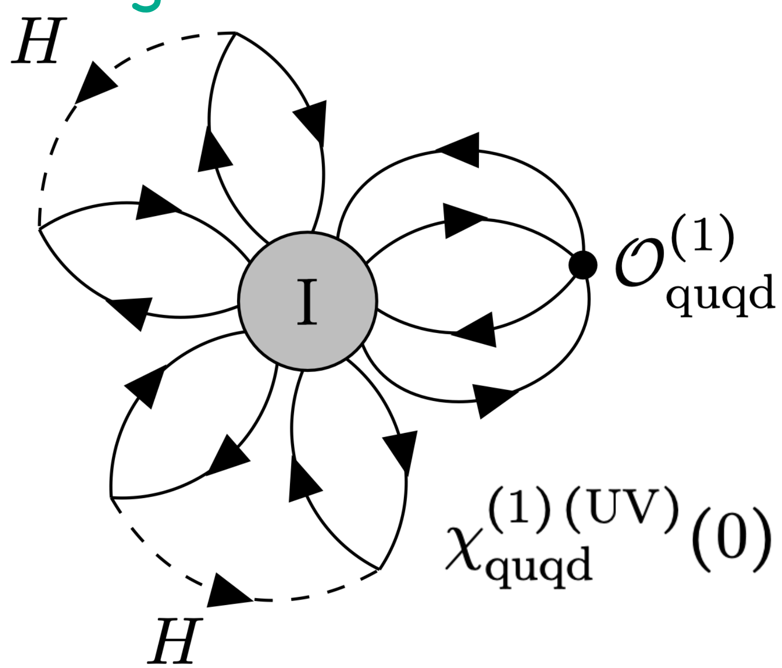
$$d_N(\rho) \rightarrow d_N(\rho)\, e^{-2\pi^2\rho^2 \sum |\langle\sigma\rangle|^2}$$

Agrawal and Howe (1710.04213)
C. Csáki, M. Ruhdorfer, Y. Shirman (1912.02197)

Topological Susceptibilities & Flavor invariants: Four-quark operator

#Integration over the size of instanton

$$\mathcal{O}^{(1)}_{\rm ququd} = \bar{Q}u\bar{Q}d + \text{h.c.}$$



$$\chi^{(1)\,(\rm UV)}_{\rm ququd}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right)\right) \int \frac{d\rho}{\rho^5}\, d_N(\rho) \frac{2!}{(6\pi^2)^2}\frac{2}{5\pi^2\rho^2}$$

- Possible UV completion of small-instantons:

  5D instantons

Uplift BPST instanton to a compact extra dimension of size R

$$d_N(\rho) \to d_N(\rho)\, e^{R/\rho}$$

T. Gherghetta, V. V. Khoze, A. Pomarol, Y. Shirman (2001.05610)

Bounds from non-measurement of theta-induced: Four-quark operator

$$\mathcal{O}^{(1)}_{\rm quqd} = \bar{Q}u\bar{Q}d + \text{h.c.}$$

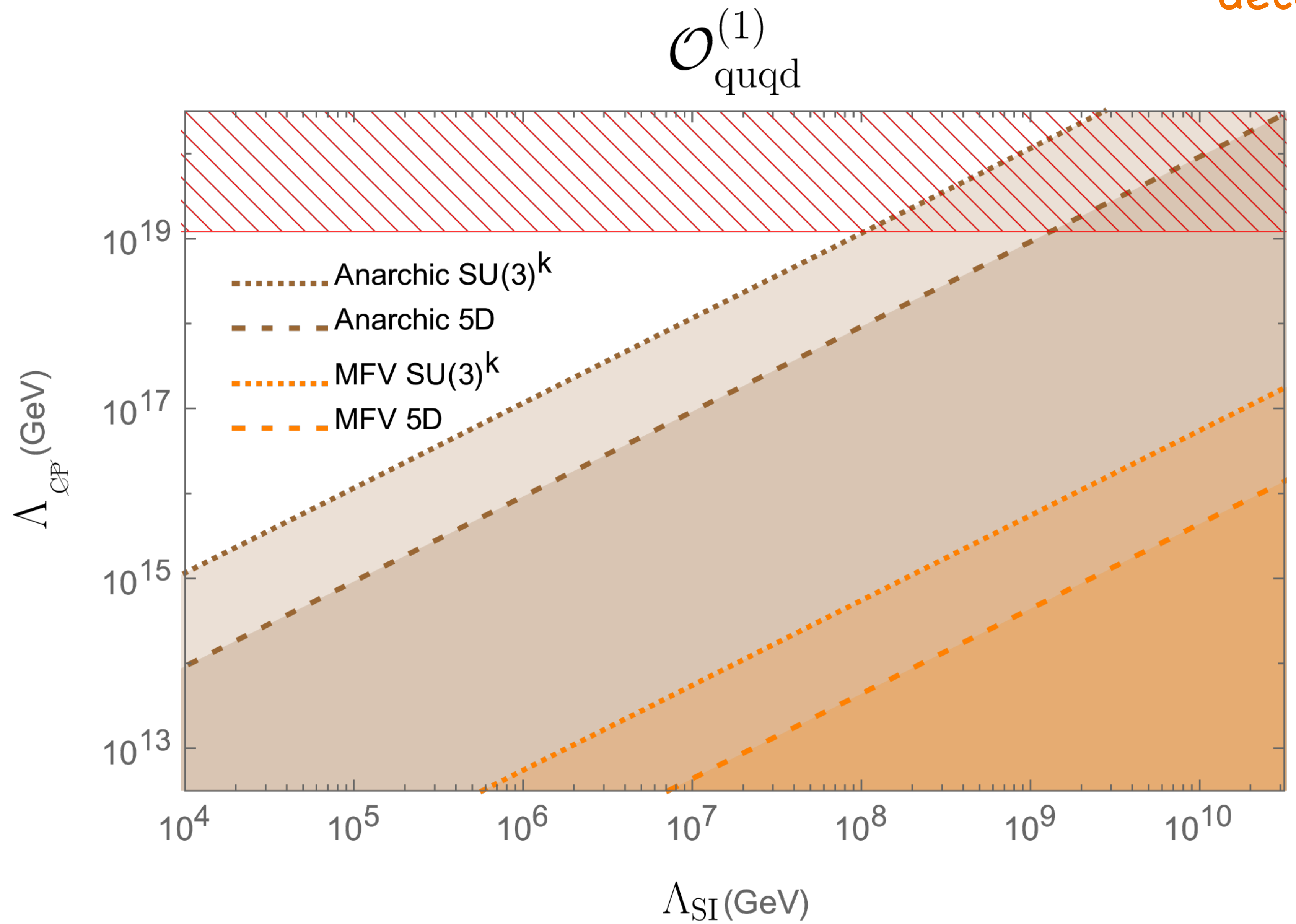
- Example: Product of gauge groups

$$\theta_{\rm ind} = \frac{16\pi^2}{5(b_0-6)K_\theta}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\rm quqd}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\rm quqd}\right)\right)\frac{\Lambda^2_{\rm SI}}{\Lambda^2_{\cancel{CP}}}$$

$$K_\theta = \text{Re}\left[e^{-i\theta_{\rm QCD}}\det\left(Y_{\rm u}Y_{\rm d}\right)\right]$$

Test different flavour scenarios

Finite ratio in the decoupling limit



Conclusions

- Enhancing the axion mass via small-instanton also (accidentally)enhances CPV effects that misalign potential
  => Dangerous effects which spoil Axion solution
  => The quality of Axion solution depends on UV scenarios

- The estimation of these effects can be made easier with the help of Determinant-like Flavour Invariants and Instanton Naive Dimensional Analysis
  => Allow to study the contributions of any SMEFT operators to the axion potential

Bonus slides

Preliminary & Outline of this talk

- Small instanton & Axion potential: UV misaligned contributions

Small instantons generate axion potential of the form:

$$V(a) = \chi_{\mathcal{O}}(0)\frac{a}{f_a} + \frac{1}{2}\chi(0)\left(\frac{a}{f_a}\right)^2 \longrightarrow \langle\frac{a}{f_a}\rangle \equiv \theta_{\text{ind}} = -\frac{\chi_{\mathcal{O}}(0)}{\chi(0)}$$

Induced by CP-violating operator — This talk

Coefficients in the potential can be computed from following correlators: Witten "79

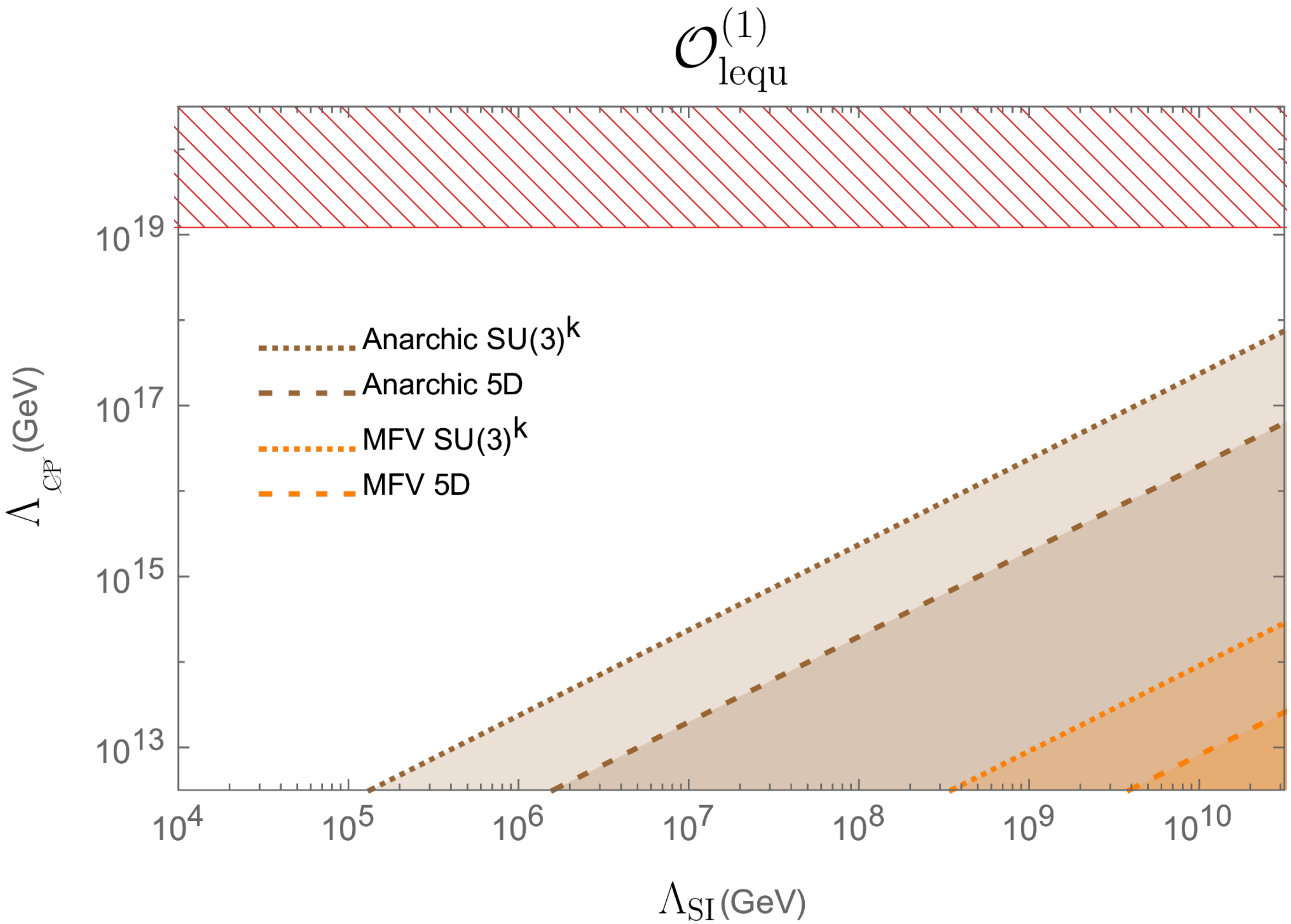
$$\chi(0) = -i\lim_{k\to 0}\int d^4x\, e^{ikx}\left\langle 0\left|T\left\{\frac{g^2}{32\pi^2}G\tilde{G}(x)\,,\,\frac{g^2}{32\pi^2}G\tilde{G}(0)\right\}\right|0\right\rangle$$

$$\chi_{\mathcal{O}}(0) = -i\lim_{k\to 0}\int d^4x\, e^{ikx}\left\langle 0\left|T\left\{\frac{g^2}{32\pi^2}G^a_{\mu\nu}\tilde{G}^{\mu\nu}_a(x)\,,\,\frac{C^{ij\cdots}_{\mathcal{O}}}{\Lambda^{D-4}_{\not{CP}}}\mathcal{O}^{D,ij\cdots}(0)\right\}\right|0\right\rangle$$

Note: In strongly coupled regime, one should use non-perturbative methods
=> Current algebra, QCD light cone sum rules

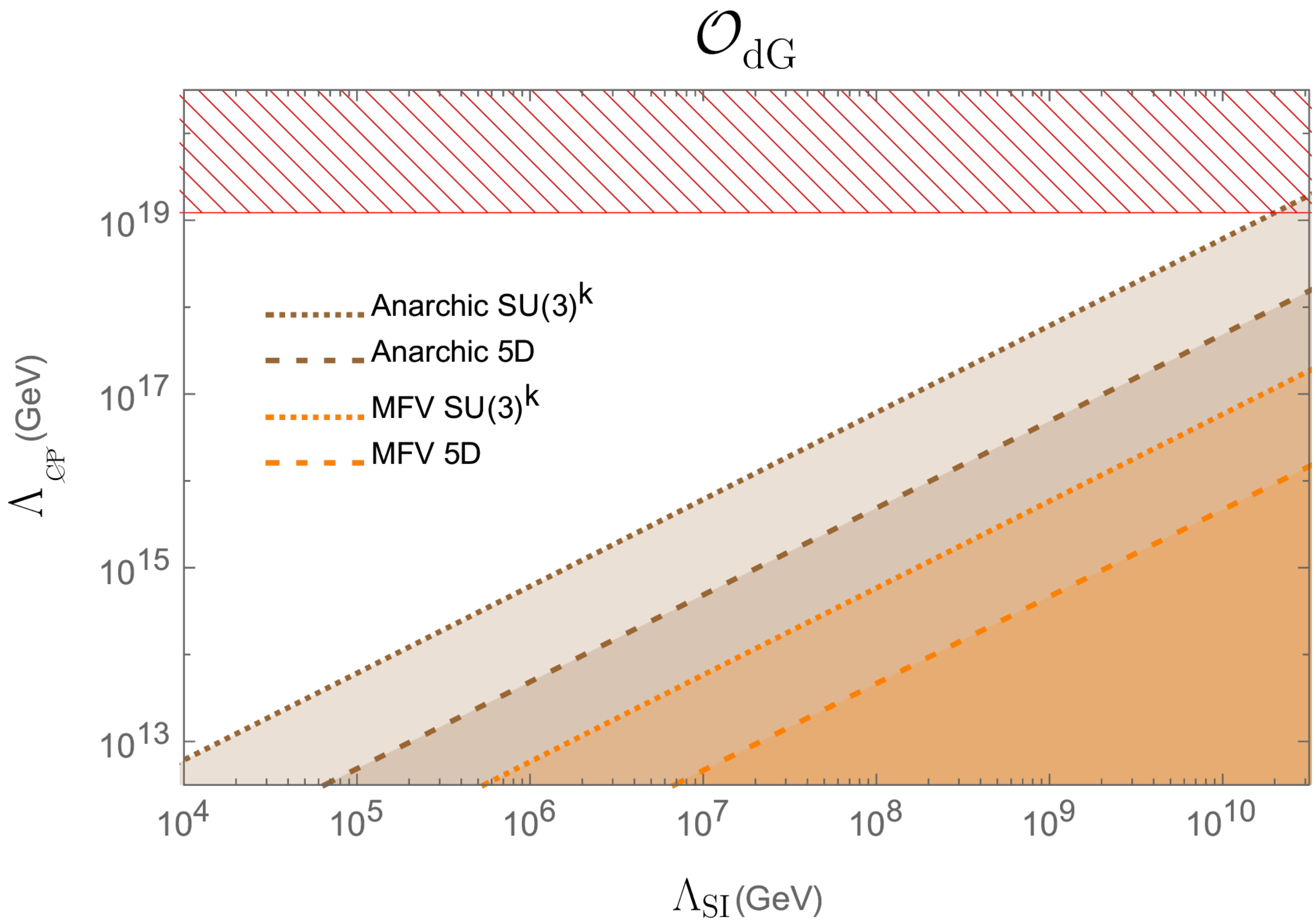
Bounds from non-measurement of theta-induced: Semi-leptonic operator

$$\mathcal{O}^{(1)}_{\rm lequ} = \bar{L}e\bar{Q}u + {\rm h.c.}$$

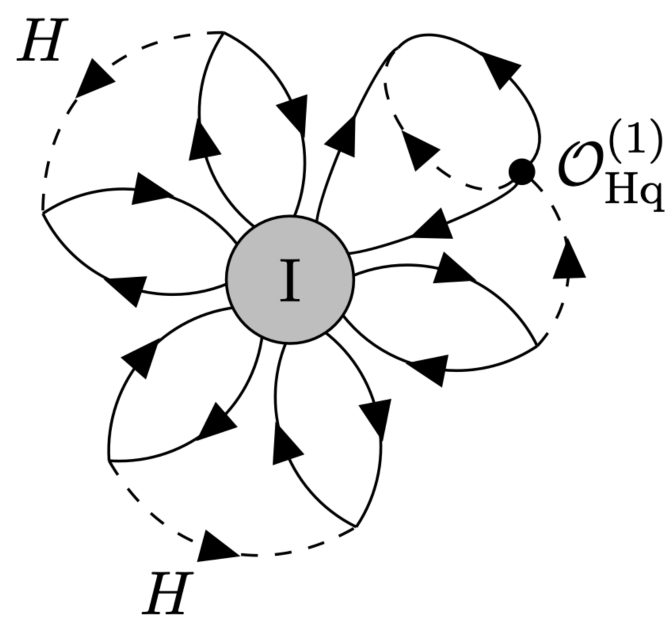


Bounds from non-measurement of theta-induced: Gluon dipole operator

$$\mathcal{O}_{\rm uG} = (\bar{Q}\sigma^{\mu\nu}T^A u)\tilde{H}G^A_{\mu\nu} + \text{h.c.}$$



Topological Susceptibilities & Flavor invariants: Higher-order Invariants



(c) Instanton diagram with an insertion of a non-chirality-flipping effective operator $\mathcal{O}^{(1)}_{\rm Hq}$.

$$\mathcal{I}_{abcd}(C^{(1,3)}_{\rm Hq}) \equiv \mathrm{Im}\left[e^{-i\theta_{\rm QCD}}\epsilon^{IJK}\epsilon^{ijk}Y_{{\rm u},Ii}Y_{{\rm u},Jj}\left(X^a_{\rm u}X^b_{\rm d}X^c_{\rm u}X^d_{\rm d}\,C^{(1,3)}_{\rm Hq}\,Y_{\rm u}\right)_{Kk}\det Y_{\rm d}\right]$$

$$X_{\rm u,d} = Y_{\rm u,d}Y^{\dagger}_{\rm u,d}$$

=> Set X=1 for the lowest order flavour invariants

BPST instanton

$$G^a_\mu(x)\Big|_{1-\text{inst.}} = 2\,\eta_{a\mu\nu}\frac{(x-x_0)_\nu}{(x-x_0)^2+\rho^2}\,, \qquad \eta_{a\mu\nu} = \begin{cases} \epsilon_{a\mu\nu}, & \mu,\nu\in\{1,2,3\} \\ -\delta_{a\nu}, & \mu=0 \\ +\delta_{a\mu}, & \nu=0 \\ 0, & \mu=\nu=0 \end{cases}$$

An important property of the one-(anti-)instanton solution is that it satisfies the *(anti-) self-dual* equation

$$G^a_{\mu\nu} = \pm\tilde{G}^a_{\mu\nu}\,, \tag{C.6}$$

and thus, due to the Bianchi identity, automatically solves the gluon equation of motion $D^\mu G^a_{\mu\nu} = D^\mu \tilde{G}^a_{\mu\nu} = 0$. With all of these properties, the one-(anti)-instanton solution then yields the finite QCD classical action

$$S^{\text{1-inst.}}_{\text{YM}} = \int d^4x \left(\frac{1}{4}G^A_{\mu\nu}G^{A,\mu\nu} + i\theta_{\text{QCD}}\frac{g^2}{32\pi^2}G^A_{\mu\nu}\tilde{G}^{A,\mu\nu}\right)\Bigg|_{\text{1-(a.-)inst.}} = \frac{8\pi^2}{g^2} \pm i\theta_{\text{QCD}}\,. \tag{C.7}$$

Instanton density

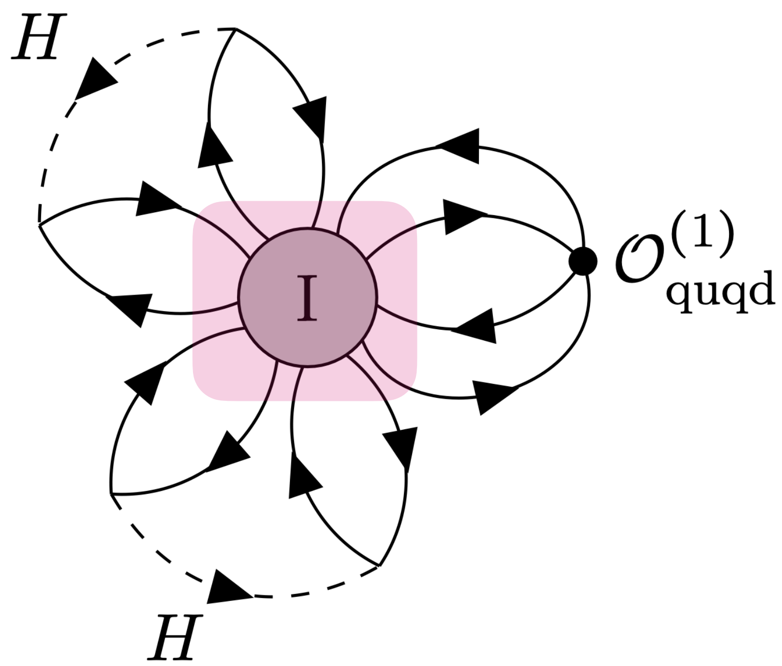
$$d_N(\rho) = C[N] \left(\frac{8\pi^2}{g^2} \right)^{2N} e^{-8\pi^2/g^2(1/\rho)}$$

$$C[N] = \frac{C_1\, e^{-C_2 N}}{(N-1)!(N-2)!}\, e^{0.292 N_f}$$

$$\frac{8\pi^2}{g^2(1/\rho)} = \frac{8\pi^2}{g_0^2(\Lambda_{\rm UV})} - b_0 \log \rho\Lambda_{\rm UV}\,, \qquad b_0 = \frac{11}{3}N - \frac{2}{3}N_f$$

't Hooft 76
Shifman, Vainshtein, Zakharov 79

Topological Susceptibilities & Flavor invariants: Four-quark operator



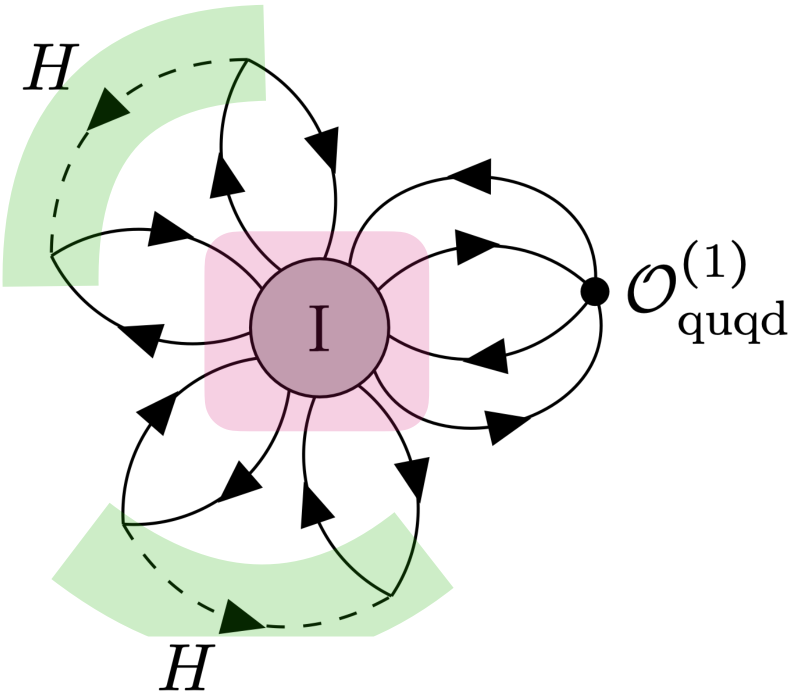
$$\chi^{(1)}_{\rm quqd}(0)^{\rm 1-inst.} = -i \lim_{k\to 0} \int d^4x\, e^{ikx} \left\langle 0 \left| T\left\{ \frac{1}{32\pi^2} G\widetilde{G}(x), \frac{C^{(1)}_{\rm quqd}}{\Lambda^2_{\not{CP}}} \mathcal{O}^{(1)}_{\rm quqd}(0) \right\} \right| 0 \right\rangle ,$$

$$= e^{-i\theta_{\rm QCD}} \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \int \mathcal{D}H\mathcal{D}H^\dagger\, e^{-S_0[H,H^\dagger]} \int \prod_{f=1}^{3} \left(\rho^2\, d\xi^{(0)}_{u_f} d\xi^{(0)}_{d_f} d^2\bar{\xi}^{(0)}_{Q_f} \right)$$

$$\times\, e^{\int d^4x (\bar{Q}Y_{\rm u}\widetilde{H}u + \bar{Q}Y_{\rm d}Hd + {\rm h.c.})(x)} \frac{1}{32\pi^2} \int d^4x\, G\widetilde{G}(x) \left(\frac{C^{(1)}_{\rm quqd}}{\Lambda^2_{\not{CP}}} \bar{Q}u\bar{Q}d(0) + {\rm h.c.} \right) ,$$

Q = 1

Topological Susceptibilities & Flavor invariants: Four-quark operator



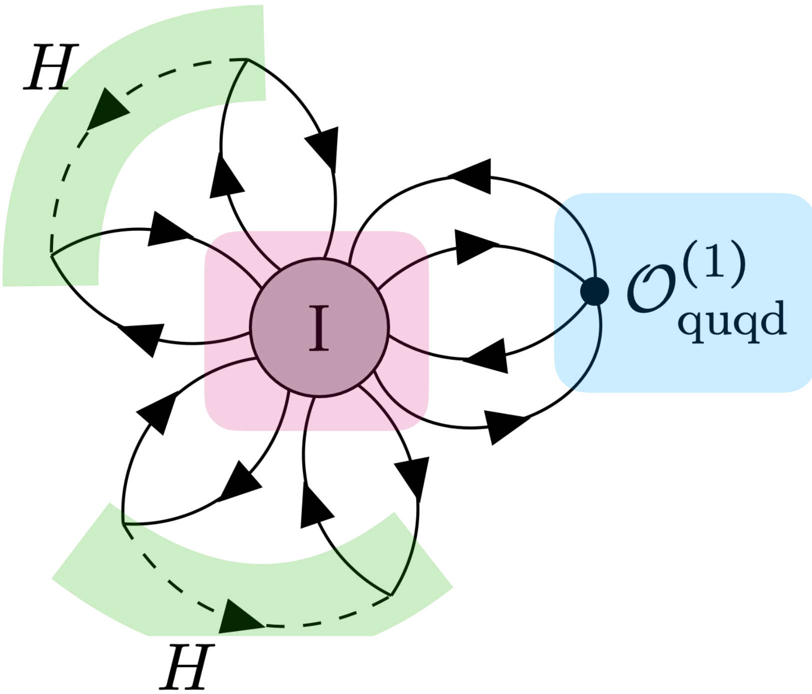
$$\chi^{(1)}_{\rm quqd}(0)^{1-\rm inst.} = -i \lim_{k\to 0} \int d^4x\, e^{ikx} \left\langle 0 \left| T\left\{ \frac{1}{32\pi^2} G\widetilde{G}(x), \frac{C^{(1)}_{\rm quqd}}{\Lambda^2_{\not{CP}}} \mathcal{O}^{(1)}_{\rm quqd}(0) \right\} \right| 0 \right\rangle ,$$

$$= e^{-i\theta_{\rm QCD}} \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \int \mathcal{D}H\mathcal{D}H^\dagger\, e^{-S_0[H,H^\dagger]} \int \prod_{f=1}^{3} \left(\rho^2\, d\xi^{(0)}_{u_f} d\xi^{(0)}_{d_f} d^2\bar{\xi}^{(0)}_{Q_f}\right)$$

$$\times\, e^{\int d^4x (\bar{Q} Y_{\rm u} \widetilde{H} u + \bar{Q} Y_{\rm d} H d + {\rm h.c.})(x)} \frac{1}{32\pi^2} \int d^4x\, G\widetilde{G}(x) \left(\frac{C^{(1)}_{\rm quqd}}{\Lambda^2_{\not{CP}}} \bar{Q}u\bar{Q}d(0) + {\rm h.c.} \right) ,$$

Q = 1

Topological Susceptibilities & Flavor invariants: Four-quark operator



$$\chi^{(1)}_{\rm quqd}(0)^{\rm 1-inst.} = -i \lim_{k\to 0} \int d^4x \, e^{ikx} \left\langle 0 \left| T \left\{ \frac{1}{32\pi^2} G\widetilde{G}(x), \frac{C^{(1)}_{\rm quqd}}{\Lambda^2_{\not{CP}}} \mathcal{O}^{(1)}_{\rm quqd}(0) \right\} \right| 0 \right\rangle ,$$

$$= e^{-i\theta_{\rm QCD}} \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \int \mathcal{D}H\mathcal{D}H^\dagger \, e^{-S_0[H,H^\dagger]} \int \prod_{f=1}^{3} \left(\rho^2 \, d\xi^{(0)}_{u_f} d\xi^{(0)}_{d_f} d^2\bar{\xi}^{(0)}_{Q_f} \right)$$

$$\times \, e^{\int d^4x (\bar{Q}Y_{\rm u}\widetilde{H}u + \bar{Q}Y_{\rm d}Hd + {\rm h.c.})(x)} \frac{1}{32\pi^2} \int d^4x \, G\widetilde{G}(x) \left(\frac{C^{(1)}_{\rm quqd}}{\Lambda^2_{\not{CP}}} \bar{Q}u\bar{Q}d(0) + {\rm h.c.} \right) ,$$

Q = 1