Locality and symmetry for $T\overline{T}$ and gravity

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(CERN)

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Based on work with Monica Guica, Per Kraus, Richard Myers, Stephen Ebert, Eliot Hijano, Konstantinos Roumpedakis, Ioannis Tsiaris, Seolhwa Kim

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Locality in gravity?

- Coordinates are unphysical
  $\rightsquigarrow$ no local observables

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- Event horizon thermodynamics:

$$S = \frac{k_B c^3}{\hbar G} \frac{A}{4}$$

Entropy not extensive!
E.g. black holes: [Bekenstein '73, Hawking '74]

$$T = \frac{\hbar c^3}{G k_B} \frac{1}{8\pi M}$$

  - First law: $c^2 \mathrm{d}M = T \mathrm{d}S$
  - Second law: $\Delta S_{\text{tot}} \geq 0$

Similar: cosmological horizon

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- String theory: asymptotic observables
  - S-matrix elements
  - AdS/CFT [Maldacena '97]

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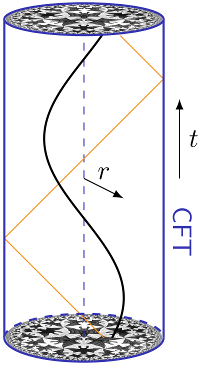
  - **AdS/CFT** [Maldacena '97]

[M. C. Escher, "Circle Limit IV"]



"AdS is a test tube"

$$\mathrm{d}s^2 = \frac{-\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sin^2\rho\, \mathrm{d}\Omega^2}{\cos^2\rho}$$

Introduction — holography

Anti-de Sitter space

- $d + 1$ (or more) dimensions

Conformal field theory

- d dimensions

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Dictionary [Gubser–Klebanov–Polyakov '98, Witten '98]

$$\int \mathcal{D}g_{\mu\nu}\,\mathcal{D}\Phi\, e^{-I[\gamma_{\alpha\beta},\alpha]} \quad \overset{\text{GKP/W}}{\longleftrightarrow} \quad e^{-W[J]} = \int \mathcal{D}\varphi\, e^{-I_{\text{CFT}} + J\cdot\mathcal{O}}$$

fixed boundary value α $\leftrightarrow$ source J

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$$\beta \propto \pi_0^{(r)} = \frac{\delta I_{\text{os}}}{\delta\alpha} \quad \leftrightarrow \quad \frac{\delta W}{\delta J} = \langle\mathcal{O}\rangle_J$$

Introduction — strategy

Beyond asymptotic observables?

- Simplify: **fewer dimensions**
  - gravitons need 3 spatial dimensions to propagate
    $\rightsquigarrow$ no bulk degrees of freedom
  - successful in 1+1d:
    JT gravity $\leftrightarrow$ matrix integrals

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  - *irrelevant* but *solvable* flow from QFT
  - unconventional UV but constrained by lots of symmetries
  - "somewhat gravitational"
  - move CFT into the bulk**?**

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$\rightsquigarrow$ **Boundary gravitons on finite surfaces**

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Outline

From $T\overline{T}$ to finite-volume holography

Symmetries

From finite-volume AdS to $T\overline{T}$

Flat space boundary modes: S-matrix & soft theorems

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The $T\overline{T}$ deformation

$\boldsymbol{T\overline{T}}$ **operator** in 2d field theory [Zamolodchikov '04]

- define $\mathcal{O}_{T\overline{T}}(x) \equiv \lim_{y\to x} \epsilon^{\alpha\beta}\epsilon^{\gamma\delta}T_{\alpha\gamma}(y)T_{\beta\delta}(x)$, up to derivatives
- expectation value factorizes: $\langle n|\mathcal{O}_{T\overline{T}}|n\rangle = \frac{E_n}{L}\frac{\partial E_n}{\partial L} + \frac{p_n^2}{L^2}$

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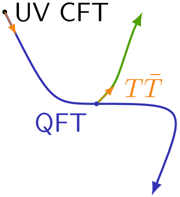
$T\overline{T}$ deformation [Smirnov–Zamolodchikov '16, Cavaglià et al. '16]

- Irrelevant deformation $\frac{\mathrm{d}I^{[\lambda]}}{\mathrm{d}\lambda} = \frac{1}{2}\int \mathrm{d}^2x \, \det T^{[\lambda]}$
- "Solvable": S-matrix, energy spectrum, partition function, ...



$$\text{e.g. } \partial_\lambda E_n = -\frac{1}{2}\int \mathrm{d}y \, \langle n | \mathcal{O}^{[\lambda]}_{T\overline{T}} | n \rangle = E_n \, \partial_L E_n + \frac{p_n^2}{L}$$

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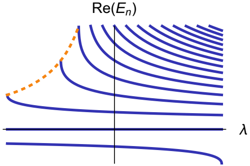
- For CFTs: KdV charges remain conserved



[McGough–Mezei–Verlinde '16]

$$E_n = \frac{1}{2\lambda}\left(1 - \sqrt{1 - 4\lambda E_n^{[0]} + 4\lambda^2 P_n^2}\right) , \qquad P_n = P_n^{[0]}$$

$T\overline{T}$ & gravity

$T\overline{T}$ ∼ 2d gravity:

$$Z_\lambda[f] = \int \mathcal{D}e\, e^{-\frac{1}{\lambda}\int \mathrm{d}^2x\, \epsilon^{\alpha\beta}\epsilon_{ab}(f^a_\alpha - e^a_\alpha)(f^b_\beta - e^b_\beta)} Z_0[e]$$

- ✓metric path integral; ✗kinetic term; ✗Σ topologies
- Classical solution: $\phi^{[\lambda]}(x) = \phi^{[0]}(y(x))$ [Conti, Negro, Tateo '18]
  with $\mathrm{d}y^\alpha = (\delta^\alpha_\beta - \lambda \hat{T}^\alpha_\beta)\mathrm{d}x^\beta$

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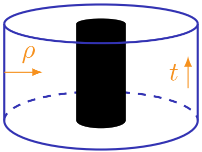
Holography: AdS_3 [McGough–Mezei–Verlinde '16]

BTZ geometries

$$\mathrm{d}s^2 = -f\mathrm{d}t^2 + \mathrm{d}r^2/f + r^2\mathrm{d}\varphi^2\ , \qquad f = r^2 - M + J^2/r^2$$



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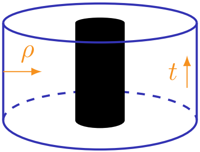

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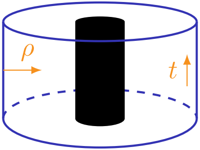

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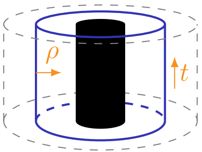
Energy within cylinder:

$$E = \frac{c}{6\rho_c}\left(1 - \sqrt{1 - \tfrac{12}{c}\rho_c M + \tfrac{36}{c^2}\rho_c^2 J^2}\right)$$

$\leftrightarrow$ $T\overline{T}$ flow with $\boxed{\lambda = 2G\rho_c \propto \rho_c/c}$



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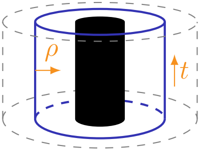

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Does $T\overline{T}$-deforming a holographic CFT "move it into the bulk"?

$T\overline{T}$: a double-trace deformation

Holographic CFT

[Gubser–Klebanov–Polyakov '98, Witten '98]

- Recall: holographic dictionary

$$e^{-I_{\mathsf{os}}[g_{\alpha\beta},\phi_0]} \xleftrightarrow{\text{GKP/W}} e^{-W[J]} = \int \mathcal{D}\phi\, e^{-I_{\mathsf{CFT}}+J\cdot\mathcal{O}}$$

$$\text{boundary value } \phi_0 \quad \leftrightarrow \quad \text{source } J$$

$$\pi_0 = \frac{\delta I_{\mathsf{os}}}{\delta\phi_0} \quad \leftrightarrow \quad \frac{\delta W}{\delta J} = \langle\mathcal{O}\rangle_J$$

- Add a (relevant) double-trace operator: [Witten '01; Sever et al. '01]

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Gravity side: now **fix** $\boldsymbol{\phi_0+\lambda\pi_0}$ and send $I\to I+\int_{\partial\mathcal{M}}\lambda\pi_0^2/2$

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Similar for $\boldsymbol{T\overline{T}}$:

now $\phi_0\to\gamma_{\alpha\beta}$ and $\pi_0\to\hat{T}_{\alpha\beta}\equiv T_{\alpha\beta}-T\,\gamma_{\alpha\beta}$

$$\partial_\lambda W[J] = \frac{1}{2}\int\sqrt{\gamma}\,\langle\mathcal{O}_{T\overline{T}}\rangle \qquad \delta W = \frac{1}{2}\int\sqrt{\gamma}\,\langle T_{\alpha\beta}\rangle\,\delta\gamma^{\alpha\beta}$$

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Similar for $\boldsymbol{T\overline{T}}$:

now $\phi_0\to\gamma_{\alpha\beta}$ and $\pi_0\to\hat{T}_{\alpha\beta}\equiv T_{\alpha\beta}-T\,\gamma_{\alpha\beta}$

$$\partial_\lambda W[J] = \frac{1}{2}\int\sqrt{\gamma}\,\langle\mathcal{O}_{T\overline{T}}\rangle \qquad \delta W = \frac{1}{2}\int\sqrt{\gamma}\,\langle T_{\alpha\beta}\rangle\,\delta\gamma^{\alpha\beta}$$

$$\Rightarrow \qquad \gamma^{[\lambda]}_{\alpha\beta} = \gamma^{[0]}_{\alpha\beta} - 2\lambda\hat{T}^{[0]}_{\alpha\beta} + \lambda^2\hat{T}^{[0]}_{\alpha\gamma}\,\hat{T}^{[0]\gamma}{}_\beta$$

$$\hat{T}^{[\lambda]}_{\alpha\beta} = \hat{T}^{[0]}_{\alpha\beta} - \lambda\hat{T}^{[0]}_{\alpha\gamma}\,\hat{T}^{[0]\gamma}{}_\beta$$

$T\overline{T}$: a double-trace deformation

$$\gamma^{[\lambda]}_{\alpha\beta} = \gamma^{[0]}_{\alpha\beta} - 2\lambda \hat{T}^{[0]}_{\alpha\beta} + \lambda^2 \hat{T}^{[0]}_{\alpha\gamma} \hat{T}^{[0]\gamma}{}_{\beta}$$
$$\hat{T}^{[\lambda]}_{\alpha\beta} = \hat{T}^{[0]}_{\alpha\beta} - \lambda \hat{T}^{[0]}_{\alpha\gamma} \hat{T}^{[0]\gamma}{}_{\beta}$$

Mixed boundary conditions

$$\mathrm{d}s^2_{\mathsf{CFT}} = \left(\gamma^{(0)}_{\alpha\beta} - \frac{\lambda}{2G}\gamma^{(2)}_{\alpha\beta} + \frac{\lambda^2}{4G^2}(\gamma^2_{(2)})_{\alpha\beta}\right) \mathrm{d}x^\alpha \, \mathrm{d}x^\beta$$

- Pure gravity: $\mathrm{d}s^2_{\mathsf{CFT}}$ is the induced metric at $\rho = \rho_c \equiv -\lambda/2G$
  (Recall $\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{\rho}(\gamma^{(0)}_{\alpha\beta} + \rho\,\gamma^{(2)}_{\alpha\beta} + \rho^2\gamma^{(4)}_{\alpha\beta})\mathrm{d}x^\alpha \, \mathrm{d}x^\beta$)
- With matter: Einstein equations change, mixed boundary conditions at infinity no longer coincide with induced metric at finite radius
- Imaginary energies: e.g. black hole larger than ρ_c
  More generally: coordinate transformation between undeformed and deformed metric becomes complex.

Outline

Symmetries from holography

CFT symmetries $\leftrightarrow$ AdS asymptotic (large gauge) symmetries

$$0 = \delta_\lambda I_{\mathsf{os}}[A_\alpha^{(0)}] = \int \mathrm{d}^d x \, \Pi^\alpha \underbrace{\delta_\lambda A_\alpha^{(0)}}_{\partial_\alpha \lambda} = -\int \mathrm{d}^d x \, \lambda \, \partial_\alpha \Pi^\alpha$$

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Undeformed AdS$_3$: [Brown, Henneaux '86]

- Asymptotic Killing vectors $\mathcal{L}_\xi \gamma_{\alpha\beta}^{[\lambda]} = 0$: $\xi^w = f(w)$ and $\xi^{\bar{w}} = \bar{f}(\bar{w})$
- Associated charges $Q_f \propto \int_{\partial\Sigma} T_{tw} f(w) = \sum_n L_n e^{inw}$

$$i\{L_m, L_n\} = (m-n)L_{m+n} + \frac{c}{12} m^3 \delta_{m+n}$$

Virasoro^2 with $c = 3\ell/2G$

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- Algebra

$$i\{L_m, L_n\} = \frac{m-n}{R_u}(L_{m+n} + \# L_m L_n) + \frac{c}{12 R_u^2} m^3 \delta_{m+n}$$

Expected to be valid up to $\hbar^1$ in field theory

Symmetries in the field theory

Classically

Semi-classically

Quantum (work in progress)

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- Symmetries: $0 \stackrel{?}{=} \delta I = \int \mathrm{d}^2 x\, f'(u)(T_{ww}\partial_{\bar{w}} u + T_{w\bar{w}}\partial_w u)$
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- Operator ordering & renormalization ambiguities **?**

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- Expect the following structure

$$\begin{array}{ccc}
\text{undeformed: } I_s^{(0)} & \xrightarrow{e^{i\hat{\mathcal{X}}}\bullet e^{-i\hat{\mathcal{X}}}} & \text{flowed charge } \tilde{I}_s\ (|n\rangle) \\
\frac{\bullet}{(1+2\lambda I_1)^s}\downarrow & & \downarrow \frac{\bullet}{(1+2\lambda I_1)^s} \\
\text{"fake" charge } \hat{I}_s\ (\langle I_s\rangle) & \xrightarrow{e^{i\hat{\mathcal{X}}}\bullet e^{-i\hat{\mathcal{X}}}} & \text{true charge } I_s\ (\langle I_s\rangle\ \&\ |n\rangle)
\end{array}$$

Outline

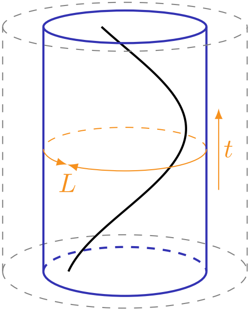
From $T\overline{T}$ to finite-volume holography

Symmetries

From finite-volume AdS to $T\overline{T}$

Flat space boundary modes: S-matrix & soft theorems

Setup



~~AdS$_3$~~

- pure 3d gravity,
  fixed topology $\mathbb{R} \times$ Disk
- fixe boundary geometry
  at finite red-shift,
  "fluctuating" bulk metric
- goal:
  - sensible**?**
  - observables & algebra
  - energy spectrum
  - correlation functions
- how is $T\overline{T}$ deformation realized?

Phase space & observables

[Peierls '52, Bergmann–Schiller '53, Soriau '70, Crnković–Witten '87, Marolf '92, Iyer–Lee–Wald–Zoupas '90s, ...]

Covariant phase space:

1. Phase space $\mathcal{P} = \{$solutions to equations of motion & boundary conditions$\}$
   - no need to choose equal-time slice & momenta
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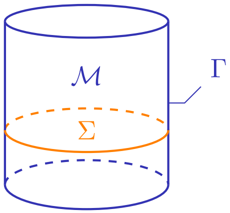
   from $\delta S = \int_\mathcal{M} E_a \, \delta\phi^a + \int_{\partial\mathcal{M}} (\Theta + \delta\ell)$



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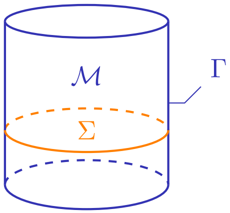
   from $\delta S = \int_{\mathcal{M}} E_a \, \delta\phi^a + \int_{\partial\mathcal{M}} (\Theta + \delta\ell)$
   - require $\delta S|_\Gamma = 0$
   - independent of Cauchy surface Σ
   - zero modes of $\int_\Sigma \delta\Theta$: gauge symmetries $\rightsquigarrow$ mod out



Phase space & observables

Asymptotically AdS$_3$ solutions [Bañados '92]

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{\rho}(\mathrm{d}w + \rho\bar{\mathcal{L}}(\bar{w})\,\mathrm{d}\bar{w})(\mathrm{d}\bar{w} + \rho\mathcal{L}(w)\,\mathrm{d}w)$$

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✂**AdS$_3$ solutions** with finite cylinder boundary

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{[(1-\rho\rho_c\mathcal{L}\bar{\mathcal{L}})\mathrm{d}w + (\rho-\rho_c)\bar{\mathcal{L}}\,\mathrm{d}\bar{w}][(1-\rho\rho_c\mathcal{L}\bar{\mathcal{L}})\mathrm{d}\bar{w} + (\rho-\rho_c)\mathcal{L}\,\mathrm{d}w]}{\rho(1-\rho_c^2\mathcal{L}\bar{\mathcal{L}})^2}$$

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Phase space & observables

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Degrees of freedom: $\mathcal{L}(w,\bar{w})$ and $\bar{\mathcal{L}}(w,\bar{w})$ $\sim$ boundary modes

Phase space & observables

(Pre)-**symplectic form** for GR [Crnković–Witten '87]

$$\Omega = \frac{1}{8\pi G} \int \mathrm{d}\Sigma_\alpha \sqrt{g}\, \delta^{\alpha\mu}_{\rho\sigma} \left[\delta\Gamma^\rho_{\mu\nu} \wedge \left(\delta g^{\sigma\nu} + \tfrac{1}{2} g^{\sigma\nu} \delta \ln g\right)\right]$$

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Infinitesimal diffeomorphisms

- spacetime vectors ξ
  $\rightsquigarrow$ phase space vectors $V_\xi \equiv \int_{\mathcal{M}} \mathcal{L}_\xi g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}}$

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  - bulk diffs are gauge transformations ($\xi|_{\partial\Sigma} = 0$)

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  - bulk diffs are gauge transformations $(\xi|_{\partial\Sigma}=0)$
- preserve boundary metric at $\rho=\rho_c$:

$$\partial_t\xi^w = \partial_\phi\xi^w + \frac{2\rho_c\bar{\mathcal{L}}\,\partial_\phi(\xi^w+\xi^{\bar w})}{(1-\rho_c\mathcal{L})(1-\rho_c\bar{\mathcal{L}})}\;,\; -\partial_t\xi^{\bar w} = \partial_\phi\xi^{\bar w} + \frac{2\rho_c\mathcal{L}\,\partial_\phi(\xi^w+\xi^{\bar w})}{(1-\rho_c\mathcal{L})(1-\rho_c\bar{\mathcal{L}})}$$

Again state-dependent through $(\mathcal{L},\bar{\mathcal{L}})$

Canonical quantization

Asymptotically $\mathbf{AdS_3}$: [Brown–Henneaux '86]

boundary gravitons $\leftrightarrow$ boundary reparameterizations

- Global AdS: $T_{ww} = -\frac{c}{24} = -\frac{\ell}{16G}$
- Finite boundary diffeomorphism: $\phi \to F(\phi) + \bar{F}(\phi)$:

$$T_{ww} = -\frac{c}{12}\left(\frac{F'^2}{2} + \frac{F'''}{F'} - \frac{3}{2}\frac{F''^2}{F'^2}\right)$$

- Using $\{Q[\epsilon_1], Q[\epsilon_2]\} = \delta_{\epsilon_1} Q[\epsilon_2]$, extract [Alekseev–Shatashvili '89]

$$\Omega = \frac{c}{48\pi}\int \mathrm{d}\phi \left(\frac{\delta F' \wedge \delta F''}{F'^2} - \delta F \wedge \delta F'\right) + \mathrm{c.c.}$$

- Action $S = \frac{1}{2\pi}\int \mathrm{d}^2 x \left(\frac{c}{24}\left[\frac{\dot{F}''}{F'} + F\dot{F}'\right] - T_{ww}\right) + \mathrm{c.c.}$
- Quantize the "coadjoint orbit" [Kirillov–Kostant '72]

Canonical quantization

Asymptotically AdS$_3$: [Brown–Henneaux '86]

boundary gravitons $\leftrightarrow$ boundary reparameterizations

- Global AdS: $T_{ww} = -\frac{c}{24} = -\frac{\ell}{16G}$
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✂**~~AdS$_3$~~**: finite boundary preserving diffs (BPDs) **?**

$\rightsquigarrow$ perturbation theory

& Chern–Simons formulation of 3d gravity

details

details

Canonical quantization

Result:

$$S = \frac{1}{32\pi G}\int \mathrm{d}t\,\mathrm{d}y\,\Big[f'\partial_{\bar{w}}f + \bar{f}'\partial_w\bar{f} \\ + \tfrac{\rho_c}{2}f'^2\bar{f}'^2\Big(1 + \tfrac{\rho_c}{4}(f'^2+\bar{f}'^2) \\ + \tfrac{\rho_c^2}{16}(f'^4 + 3f'^2\bar{f}'^2 + \bar{f}'^4) + \dots\Big)\Big]$$

Canonical quantization

Result:

$$S \overset{?}{=} \frac{1}{8\pi G} \int \mathrm{d}t\,\mathrm{d}y \left[\frac{1}{4}(f'\dot{f} + \bar{f}'\dot{\bar{f}}) + \frac{1}{\rho_c}\left(1 - \sqrt{1 - \frac{\rho_c}{2}(f'^2 + \bar{f}'^2) + \frac{\rho_c^2}{16}(f'^2 - \bar{f}'^2)^2}\right)\right]$$

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Define $\phi \propto f+\bar{f}$ and $\Pi \propto f'-\bar{f}'$:

Canonical quantization

Result:

$$S \stackrel{?}{=} \int \mathrm{d}^2x \left[i\Pi\dot{\phi} + \frac{1}{\lambda}\left(1 - \sqrt{1 - \lambda(\phi'^2 + \Pi^2) + \lambda^2\phi'^2\Pi^2}\right)\right]$$
$$= \frac{1}{\lambda}\int \mathrm{d}^2x \left(1 - \sqrt{-\det(\partial_a X^\mu\, \partial_b X_\mu)}\right)$$

Define $\phi \propto f + \bar{f}$ and $\Pi \propto f' - \bar{f}'$:

- 3d Nambu–Goto in static gauge $X^\mu = (t, y, \sqrt{\lambda}\phi)$

Canonical quantization

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- and improved stress tensor: ($\mathrm{Tr}\, T = \lambda \det T$ ✓)

$$T_{\alpha\beta} = \eta_{\alpha\beta}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial^\alpha\phi)}\partial_\beta\phi + (\partial_\alpha\partial_\beta + \eta_{\alpha\beta}\partial^2)Y$$

$$Y = \frac{\sqrt{c}}{\sqrt{12\pi}}\phi + \frac{\lambda c}{24}(\partial\phi)^2 + \frac{\pi\lambda^2 c}{48}(\partial\phi)^4 + \frac{\lambda^2\sqrt{\pi c^3}}{48\sqrt{3}}\partial_\alpha\phi\partial_\beta\phi\partial^\alpha\partial^\beta\phi + \ldots$$

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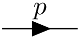
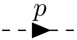
$T\overline{T}$-deformed boundary graviton

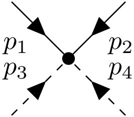
Quantum results

Energy spectrum ✓

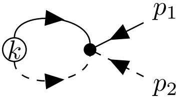
Correlation functions

- Propagators $\langle f'(-p) f'(p)\rangle_0 =$  $= 32\pi G \frac{p_w^2}{p^2}$

$\langle \bar{f}'(-p) \bar{f}'(p)\rangle_0 =$  $= 32\pi G \frac{p_{\bar{w}}^2}{p^2}$

- Vertices  $= \dfrac{\rho_c}{64\pi G}$ and higher orders

- Correlation functions

$\langle \det T(k) \partial_z \phi(p_1) \partial_{\bar{z}} \phi(p_2)\rangle \supset$ 

$$= -\frac{\pi^6 \lambda^2 c_0}{24} \frac{p_{1z}^2}{p_1^2} \frac{p_{2z}^2}{p_2^2} k^4 \left(\frac{2}{\epsilon} + \ln \frac{k^2}{4\pi} + \gamma_E - \frac{9}{10} \right) \delta^{(2)}(k + p_1 + p_2)$$

Requires $\det T \to \det T - \frac{\pi^3 \lambda^2 c}{6} \left(\frac{2}{\epsilon} + \text{finite} \right) \partial_z^2 \partial_{\bar{z}}^2 (\partial_z \phi \, \partial_{\bar{z}} \phi)$

Zamolodchikov's **diverging total derivatives!**

Outline

From $T\overline{T}$ to finite-volume holography

Symmetries

From finite-volume AdS to $T\overline{T}$

Flat space boundary modes: S-matrix & soft theorems

Flat space S-matrix vs. AdS correlators

Gravity in asymptotically flat space: **S-matrix**

LSZ reduction formula

$$S_{p_a,p_b} \sim \int e^{ip_i x_i} \langle 0|\mathcal{T}\phi(x_i)|0\rangle$$

- bulk correlators
- justifies Feynman rules

Flat space S-matrix vs. AdS correlators

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Gravity in AdS: **boundary correlation functions**

Extrapolate dictionary BDHM

$$\langle \mathcal{O}(x_i)\rangle = \lim_{r_i\to\infty} r_i^{\Delta} \langle \phi(r_i, x_i)\rangle$$

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$$\langle \mathcal{O}(x_i)\rangle = \frac{\delta}{\delta\phi_0(x_i)} Z[\phi_0]\Big|_{\phi_0=0}$$

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AFS prescription [Aref'eva, Faddeev, Slavnov]

$$\langle \varphi_- | \hat{S} | \varphi_+ \rangle = \int \mathcal{D}\phi \, e^{iI[\phi,\varphi]}$$

- boundary data: $\varphi_\pm$
- natural language for IR

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AFS prescription

Consider $S[\varphi] \equiv \langle \varphi_- | \hat{S} | \varphi_+ \rangle$ [Arefeva, Feddeev, Slavnov '74]

- **S-matrix** operator $\hat{S} = \lim_{T\to\infty} e^{i\hat{H}_0 T} e^{-2i\hat{H}T} e^{i\hat{H}_0 T}$

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- (scalar field) **coherent states**

$$\hat{\phi}_\pm \, |\varphi\rangle = \varphi_\pm \, |\varphi\rangle \ \ , \quad \hat{\phi}(t, \vec{x}) \equiv \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega_p} (\underbrace{a_{\vec{p}}\, e^{ipx}}_{\hat{\phi}_+} + \underbrace{a^\dagger_{\vec{p}}\, e^{-ipx}}_{\hat{\phi}_-})$$

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- Claim: $S[\varphi]$ generates S-matrix elements

$$\langle p_i | \hat{S} | q_j \rangle = \frac{\delta}{\delta\alpha_{\vec{p}_i}} \frac{\delta}{\delta\alpha^\dagger_{\vec{q}_j}} S[\varphi] \Big|_{\varphi=0}$$

Intuition: cf. QM $|\alpha\rangle = e^{\alpha a^\dagger}|0\rangle \Rightarrow \partial_\alpha \langle\psi|\alpha\rangle = \langle\psi|a^\dagger|\alpha\rangle$

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- Path integral version

$$S[\varphi] = \lim_{T\to\infty} \int_{\phi_+(-T)=\varphi_+(-T)}^{\phi_-(T)=\varphi_-(T)} \mathcal{D}\phi\, e^{iI[\phi,\varphi]}$$

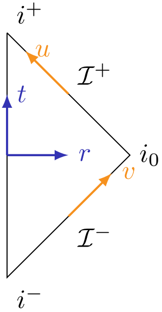
with boundary terms $(\varphi^*_-, \phi_+)_{t_f} - (\varphi^*_+, \phi_-)_{t_i}$ so that $\delta I = 0$

Soft photon theorem

Massless scalar + QED

$$I = \frac{1}{2}\int \mathrm{d}^4x \left(A^\mu \nabla^2 A_\mu + \phi^* D^2 \phi + D^2\phi^*\,\phi\right) + I_{\mathsf{ct}} + I_{\mathsf{ghost}} + I_{\mathsf{bdy}}$$

$$I_{\mathsf{bdy}} \supset (\mathcal{A}_-^{\mu *}, A_\mu)_{t_f} + (\varphi_-^*, \phi)_{t_f} + (\varphi_-, \phi^*)_{t_f}$$



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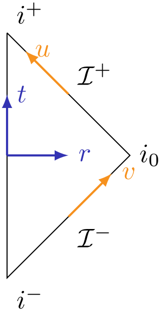

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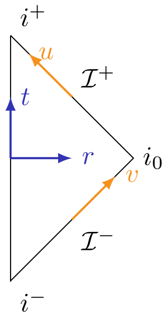
$$\boxed{S[\varphi, \mathcal{A}_\mu] \overset{\checkmark}{=} S[e^{iq\lambda_0}\varphi, \mathcal{A}_\mu + \partial_\mu \lambda_0]}$$



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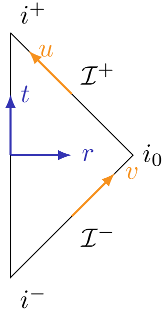
- **Infrared modes** of the photon

$$\partial_z N \equiv A_z(u=\infty) - A_z(U=-\infty)$$

$$\partial_z \tilde{\lambda} \equiv A_z(u=\infty) + A_z(U=-\infty)$$

are “canonical conjugates”:

$$[\partial_w N(w,\bar{w}), \tilde{\lambda}(z,\bar{z})] = \frac{i}{4\pi}\frac{1}{w-z}$$



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$$\boxed{S[\varphi, \mathcal{A}_\mu] \overset{\checkmark}{=} S[e^{iq\lambda_0}\varphi, \mathcal{A}_\mu + \partial_\mu\lambda_0]}$$

- **Infrared modes** of the photon are "canonical conjugates":

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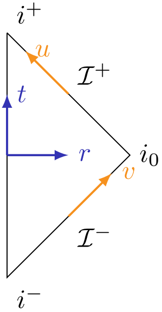

Soft photon theorem

Massless scalar + QED

$$I = \frac{1}{2}\int \mathrm{d}^4x \left(A^\mu \nabla^2 A_\mu + \phi^* D^2\phi + D^2\phi^*\,\phi\right) + I_{\text{ct}} + I_{\text{ghost}} + I_{\text{bdy}}$$

$$I_{\text{bdy}} \supset (\mathcal{A}_-^{\mu *}, A_\mu)_{t_f} + (\varphi_-^*, \phi)_{t_f} + (\varphi_-, \phi^*)_{t_f}$$

- data on $\mathcal{I}^\pm$: $r \to \infty$ at fixed $u(v) = t \pm r$,
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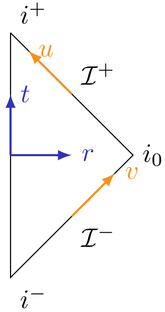
$$\langle\text{out}|[\partial_z N, \hat{S}]|\text{in}\rangle = \left(\sum_{k\ \text{in}} \frac{q_k}{z-z_k} - \sum_{k\ \text{out}} \frac{q_k}{z-z_k}\right)\langle\text{out}|\hat{S}|\text{in}\rangle$$



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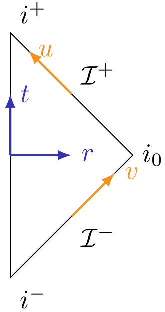

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Conclusions and outlook

Conclusions

- Locality in gravity**?**
- $T\overline{T}$: irrelevant but solvable deformation
  - $\rightsquigarrow$ finite-size gravitational systems in less than 3 dimensions
  - preserves $\infty\#$ symmetries, non-local corrections
- boundary modes of gauge theories ($\supset$ gravity) $\rightsquigarrow$ e.g. soft theorems

Conclusions and outlook

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  - preserves $\infty\#$ symmetries, non-local corrections
- boundary modes of gauge theories ($\supset$ gravity) $\rightsquigarrow$ e.g. soft theorems

Future directions

- full QFT $T\overline{T}$ symmetries?
- well-defined (unique) correlation functions?
- non-perturbative effects: imaginary energies?
- $\Lambda > 0$?
- gravitational & subleading soft theorems
- IR finite S-matrix à la Faddeev–Kulish?

рахмат
danke
謝謝
spas
ngiyabonga
teşekkür ederim
Баярлалаа
mersi
kia ora
barka
welalin
tack
mahalo
tapadh leat
спасибо
faafetai lava
vinaka
спасибі
blagodaram
dank je
misaotra
matondo
paldies
grazzi
хвала
kiitos
dankie
nanni
nandri
dhanyavad
thank you
gracias
asante
manana
obrigada
tenki
bedankt
bayarlalaa
gràcie
hvala
mauruuru
köszönöm
akun dankon
aciū
djiere dieuf
tau
mochchakkeram
murakoze
enkosi
dziękuję
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mamnun
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gratias ago
gràcies
sulpáy
taiku
go raibh maith agat
chokrane
sobodi
dėkuji
sukriya
kop khun krap
sagolun
obrigado
mèsi
didi madloba
ありがとう
arigatô
takk
dakujem
trugarez
najis tuke
tanemirt
grazie
terima kasih
rahmet
kam sah hamnida
rahmat
diolch
dhanyavadagalu
shukriya
mercé
мерси
তোমাকে ধন্যবাদ
감사합니다
xièxie
ευχαριστώ
merci

Extra slides

Geometric perturbation theory

back

✂ **AdS_3** BPD perturbation theory: start from empty AdS

$$\tilde{\phi} = \phi + \frac{\alpha}{2}[C(\phi) + D(\phi)] + \dots\ ,$$
$$\tilde{t} = t + \frac{i}{2}[C(\phi) - D(\phi)] + \dots\ ,$$
$$\tilde{\rho} = \rho + \dots \qquad [\alpha = \sqrt{1+\rho_c}]$$

Geometric perturbation theory

back

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- Simplify with field redefinitions at higher orders
- Find:

$$\Omega = \tfrac{c}{24\pi}\int \mathrm{d}\phi\,[(\delta C + \delta C'')' \wedge \delta C - (\delta D + \delta D'')' \wedge \delta D]$$
$$P = \tfrac{c}{24\pi}\int \mathrm{d}\phi\,[(C' + C''')C' - (D' + D''')D']$$
$$H = -\tfrac{c}{6(1+\alpha)} - \tfrac{c}{48\pi\alpha}\int \mathrm{d}\phi\left[(C + C'')'C' - \frac{\rho_c}{\alpha}(C'^2 - C''^2)D' + (C \leftrightarrow D)\right] + \dots$$

Chern–Simons formulation

back

First order formulation

[Achucarro–Townsend '86, Witten '88]

- Frame field $e^a \equiv e^a_\mu \, \mathrm{d}x^\mu$ and spin connection $\omega^a = \omega^a_\mu \, \mathrm{d}x^\mu$ with $g_{\mu\nu} = e^a_\mu \, \eta_{ab} \, e^b_\nu$

Chern–Simons formulation

back

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- Form $sl(2,\mathbb{R})$ gauge connections:
  $[J_a, J_b] = \epsilon_{abc} J^c$ and $\mathrm{Tr}(J_a J_b) = \eta_{ab}/2$

$$A = (\omega^a + e^a) J_a \qquad \bar{A} = (\omega^a - e^a) J_a$$

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- Einstein–Hilbert action

$$S_{\mathrm{EH}} = S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[\bar{A}] + (\mathrm{bdy})$$

with $k = 1/4G$ and

$$S_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int \mathrm{Tr}_{sl(2)}(A \wedge \mathrm{d}A + \tfrac{2}{3} A \wedge A \wedge A)$$

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- Boundary terms [Llabres '19]

$$S_{\mathrm{bdy}} = \frac{k}{4\pi} \int_{\partial\mathcal{M}} \mathrm{Tr}[A \wedge \bar{A} - L_0 (A - \bar{A}) \wedge (A - \bar{A})]$$

[frame $A^0, \bar{A}^0 \propto n_\mu \mathrm{d}x^\mu$, $\partial_a n^a = 0$]

Chern–Simons formulation

back

Phase space

- A_0 and $\bar{A}_0$ appear as Lagrange multipliers $\rightsquigarrow \tilde{\mathcal{F}} = \tilde{\bar{\mathcal{F}}} = 0$
- solution: $\tilde{\mathcal{F}} = g^{-1}\tilde{\mathrm{d}}g$ and $\tilde{\bar{\mathcal{F}}} = \bar{g}^{-1}\tilde{\mathrm{d}}\bar{g}$
- Gauss parameterization:
  $g = e^{F L_1}\left(\frac{\beta}{\sqrt{\rho}}\right)^{2L_0} e^{\Psi L_{-1}}$ and $\bar{g} = e^{\bar{F} L_1}\left(\frac{\bar{\beta}}{\sqrt{\rho}}\right)^{2L_0} e^{\bar{\Psi} L_{-1}}$

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Find:

$$S = -\frac{k}{\pi}\int_{\partial\mathcal{M}} \mathrm{d}t\,\mathrm{d}y \left[\partial_{\bar{w}}(\beta F')\frac{\beta'}{\beta^2 F'} + \partial_w(\bar{\beta}\bar{F}')\frac{\bar{\beta}'}{\bar{\beta}^3\bar{F}'} - \frac{1}{\rho_c}(\beta^2 F' - 1)(\bar{\beta}^2\bar{F}' - 1)\right]$$

with boundary conditions

$$2e^+ = \frac{1}{\sqrt{\rho_c}} = \frac{\beta^2 F'}{\sqrt{\rho_c}} + \frac{\sqrt{\rho_c}}{\bar{\beta}}\left(\frac{\bar{\beta}'}{\bar{\beta}^2\bar{F}'}\right)', \quad 2e^- = \frac{-1}{\sqrt{\rho_c}} = \frac{\bar{\beta}^2\bar{F}'}{\sqrt{\rho_c}} + \frac{\sqrt{\rho_c}}{\beta}\left(\frac{\beta'}{\beta^2 F'}\right)'$$

Chern–Simons formulation — Perturbation theory

back

Perturbation theory: Solve boundary conditions perturbatively

$$\beta = 1 + f + f_2 + \mathcal{O}(f^3)\ , \qquad F' = 1 - (f + \tfrac{\rho_c}{2}\bar{f}'') + \mathcal{O}(f^2, \bar{f}^2)$$
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Choice of $f_n, \bar{f}_n$ in terms of $f, \bar{f}$:

- only lowest order kinetic term in action
- only first derivatives of $f, \bar{f}$ in action
- up to n derivatives in $f_n, \bar{f}_n$: nonlocality**?**

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$\rightsquigarrow$ find action

$$S = \frac{1}{32\pi G}\int \mathrm{d}t\,\mathrm{d}y\,\Big[f'\partial_{\bar{w}}f + \bar{f}'\partial_w\bar{f} + \tfrac{\rho_c}{2}f'^2\bar{f}'^2\Big(1 + \tfrac{\rho_c}{4}(f'^2 + \bar{f}'^2) + \tfrac{\rho_c^2}{16}(f'^4 + 3f'^2\bar{f}'^2 + \bar{f}'^4) + \dots\Big)\Big]$$

and stress tensor (on the plane)

$$4GT_{ww} = -2f'' - f'^2 + \rho_c f'''\bar{f}' + \dots\ ,$$
$$4GT_{w\bar{w}} = -\rho_c f''\bar{f}'' - \tfrac{\rho_c}{2}(f'^2\bar{f}'' + f''\bar{f}'^2) + \dots$$