Comprendre l’Infiniment Grand - Introduction to Cosmology - Part II

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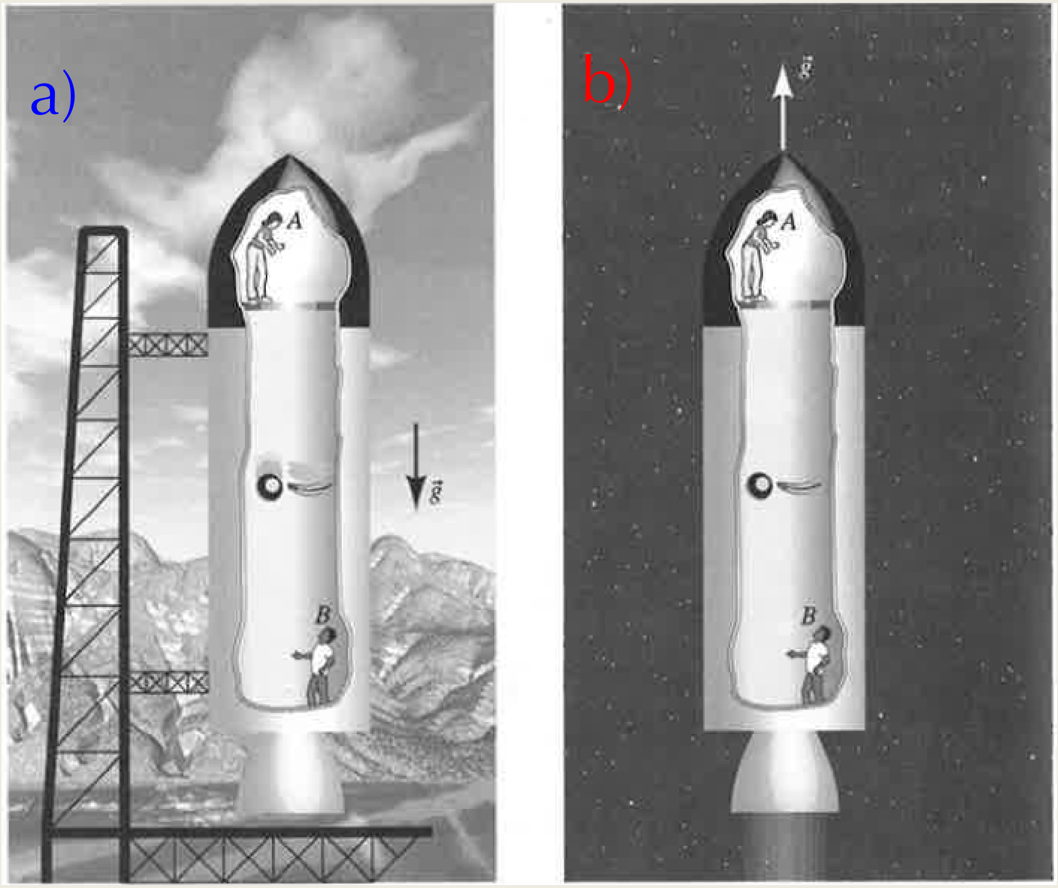
Summary of Part I

Equivalence principle

a) $m_i a = m_g\, g \Rightarrow$
the lead ball and the feather experience the same Acceleration
$\Rightarrow m_i = m_g$ and $a=g$

b) they have the same constant speed but appear with the same acceleration

- uniform gravitational field
  = uniform acceleration

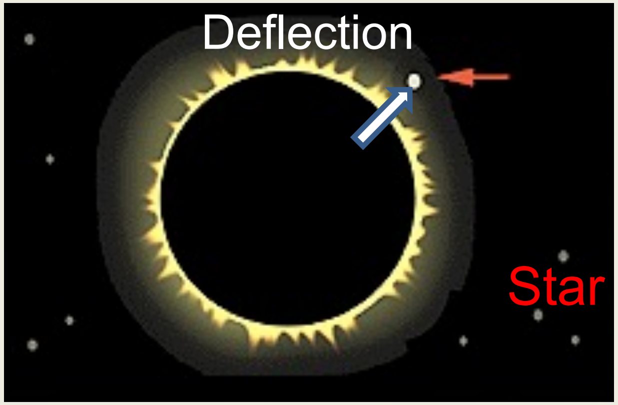


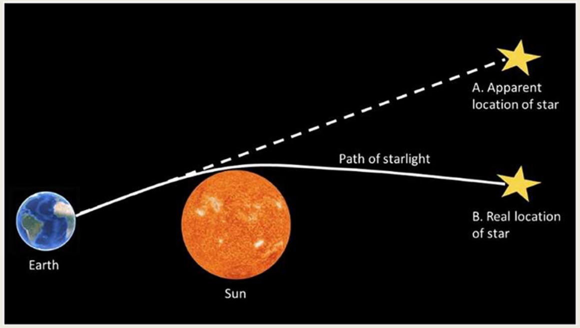
James B. Hartle

study effect of acceleration ⇒ study gravitation

Light rays are bent – Clocks and Gravitation

- In 1919: Arthur Eddington observes light deviation by the sun during a solar eclipse:
  - 1.75 arc second = 8.5 μrad as predicted by Einstein
  - Twice the deflection predicted by first computation (Eq. principle alone)





-Times run slower in a gravitational field !

$$\boxed{\Delta t_B = \left(1 - \frac{\Phi_A - \Phi_B}{c^2}\right)\Delta t_A}$$

At the surface of a star: $\phi_A = -GM/R$ and far away: $\phi_B = 0$

$$\Delta t_\infty = \left(1 + \frac{GM}{Rc^2}\right)\Delta t_*$$

Cosmology - Part II

1. **Geometry of the Universe**
   - Curved spacetime – Metric
   - Cosmological principles
   - FLWR metric

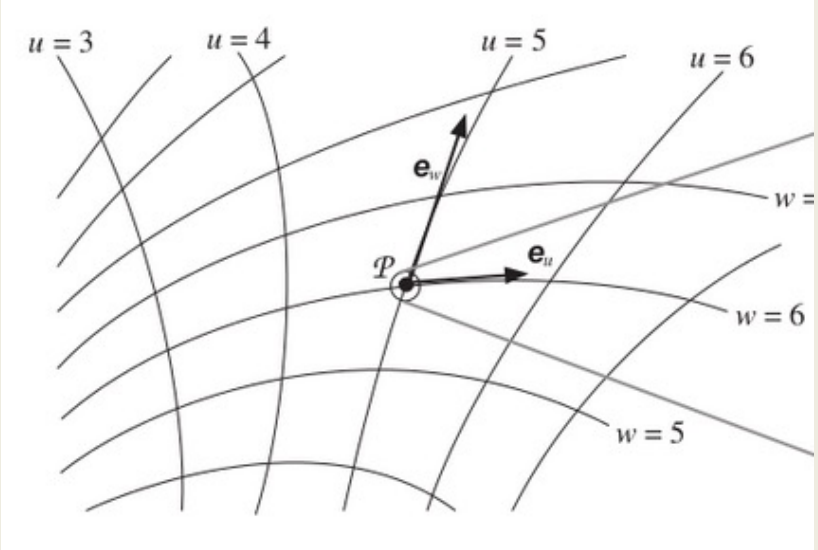
2. **Expansion of the Universe**
   - Cosmological redshift
   - Friedman equation

3. **History of the Universe**

1) Geometry of the Universe

From 3D space to 4D spacetime

1) In the usual 3D Euclidian



- Define a coordinate system
  x^i = a labeling of space
  ex plan (x,y) or (r,ϕ)

- We can measure distances with a ruler: $dS^2 = \mathbf{g_{ij}(x)}\, dx^i\, dx^j$

- The metric $g_{ij}(x)$ alone totally defines the geometry

- but $dS^2 = dr^2 + r^2 d\phi^2$ and $dS^2 = dx^2 + dy^2$: same geometry
  we mean $(dx)^2$ and not $d(x^2)$! length² not surface

2) We generalize to a non-Euclidian 4D spacetime

Curved spacetime – Metric

- We generalize the 3D metrics to 4D in special relativity

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$$

- We generalize in GR with non constant terms $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

- The equivalence principle tells, $\Delta\tau_B = \left(1 - \frac{\Phi_A - \Phi_B}{c^2}\right) \Delta\tau_A$

- GR : for a weak and static field, the metric is :

$$ds^2 = \underbrace{\left(1 + \frac{2\Phi(x)}{c^2}\right)}_{\text{equivalence principle}} c^2 dt^2 - \underbrace{\left(1 - \frac{2\Phi(x)}{c^2}\right)}_{\text{GR}} (dx^2 + dy^2 + dz^2)$$

fixed object $\Delta\tau^2 = \frac{ds^2}{c^2} = \left(1 + \frac{2\Phi}{c^2}\right) dt^2 \Rightarrow \Delta\tau_B = \left(1 - \frac{\Phi_A - \Phi_B}{c^2}\right) \Delta\tau_A$

Homogenous and isotropic

- **Cosmological principle**
- Universe isotropic + homogeneous on large scales
- Universe looks the same whoever and wherever you are

- **Isotropic** (on large scales)
- CMB very isotropic
- X ray background, radio galaxies

- **Homogeneous**
- Test with 3D galaxy surveys
- Only at large scales…. >Mpc

FLRW metric

- Homogeneous and isotropic ⇒

Friedmann, Lemaitre, Robertson, Walker metric

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

- Isotropic: spherical coordinates $dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$

- Homogeneous: scale factor $R(t)$ due to expansion, it does not depend on (r, θ, ϕ)

- Dimensionless scale factor : $a(t)=R(t) / R(t_0)$

  now $a(t_0) = 1$ **index 0, means today**

  in the past $a(t) < 1$

  Big Bang $a(t) = 0$

FLRW metric

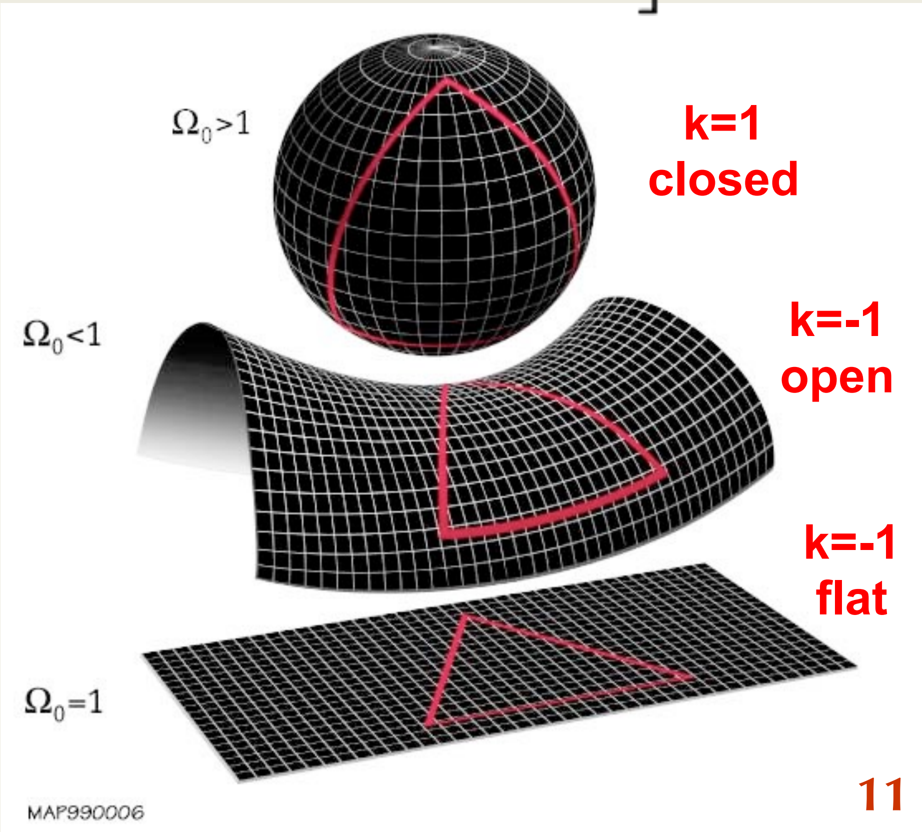
Friedmann, Lemaitre, Robertson, Walker metric

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

k = 1 : spherical geometry
or closed $(\sum\alpha > 180^o)$

k = -1 : hyperbolic geometry
or open $(\sum\alpha < 180^o)$

k = 0 : flat geometry $(\sum\alpha = 180^o)$



Comoving distance

- **Change of coordinates** r =sin χ (k=1, closed)
  r = χ (k=0, flat)
  r = sinh χ (k=-1, open)

$$ds^2 = dt^2 - R^2(t)\left[d\chi^2 + \begin{Bmatrix} \sin^2\chi \\ \chi^2 \\ \sinh^2\chi \end{Bmatrix} \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \begin{Bmatrix} \text{closed} \\ \text{flat} \\ \text{open} \end{Bmatrix}$$

sin → spherical sinh → hyperbolical

- **Distance:**
  - Galaxies remain at χ = cst (up to small local velocities)
  - Physical distance between 2 galaxies : R(t) × **Δ**χ (Mpc)
    increases with the expansion
  - "comoving" distance : R(**t_0**) × **Δ**χ is fixed (comoving Mpc)
    = distance including the expansion up to **$t=t_0$**
    = independent from Universe expansion

2) Expansion of the Universe

Cosmological redshift

- Radial photon

$ds^2 = dt^2 - R^2(t)\, d\chi^2 = 0 \quad \Rightarrow \quad d\chi = dt/R$

$$\chi = \int_{t_e}^{t_r} \frac{dt}{R(t)} = \int_{t_e+\delta t_e}^{t_r+\delta t_r} \frac{dt}{R(t)}$$

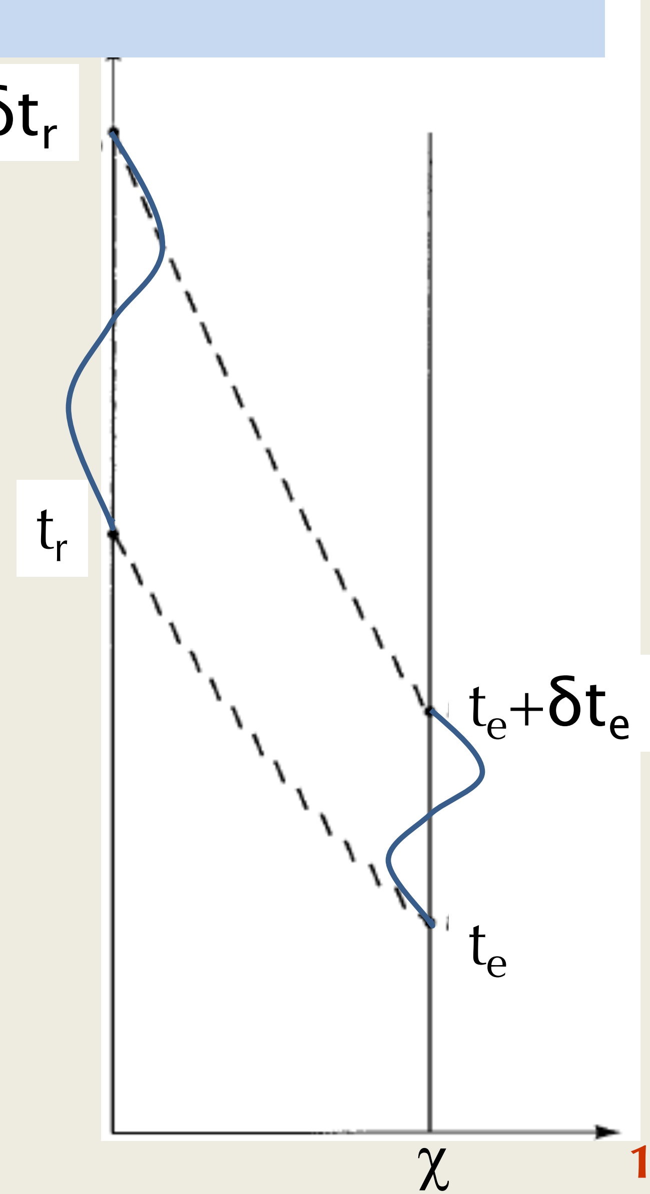
$$\Rightarrow \int_{t_e}^{t_e+\delta t_e} \frac{dt}{R(t)} = \int_{t_r}^{t_r+\delta t_r} \frac{dt}{R(t)}$$

$$\Rightarrow \frac{\delta t_e}{R(t_e)} = \frac{\delta t_r}{R(t_r)}$$

$$1 + z_e \equiv \frac{\lambda_r}{\lambda_e} = \frac{\delta t_r}{\delta t_e} = \frac{R(t_r)}{R(t_e)} \equiv \frac{1}{a(t_e)}$$

$$\boxed{1 + z = \frac{1}{a}}$$

« λ is dilating with the Universe »



Redshift: A fundamental concept in cosmology

- Measuring z $\rightarrow$ scale factor *a* when light emitted

- It is a cosmological redshift, $1+z = 1/a$ can be e.g. z=1000 (at CMB) cannot be interpreted as a simple Doppler effect

- In case of Hubble law ($v=H_0d$), it is locally interpreted as a Doppler effect

• z is also a measurement of time: e.g. CMB occurred at z = 1100 (i.e. when a=0.0009)

Hubble parameter

- Assume $t_e \sim t_0$ (locally) $\Rightarrow$ a ~ 1, small z

$$1 + z = \frac{1}{a} \qquad z = \frac{v}{c} = \frac{1-a}{a} = \frac{\dot{a}\Delta t}{a} \qquad \Rightarrow \qquad v = \frac{\dot{a}}{a}(c\Delta t)$$

$$v = \frac{\dot{a}}{a}D$$ Hubble law with $$H_0 = \frac{\dot{a}(t_0)}{a(t_0)} = H(t_0)$$

- Hubble parameter $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$

- H_0 is not very precisely measured, we define

$$h \equiv \frac{H_0}{100\ (\mathrm{km/s})/\mathrm{Mpc}} \approx 0.7$$

- cosmological results in units like h^{-1}Mpc
  numerical result independent of h

Thermodynamic

- a volume V including a fixed number of particles (i.e. galaxies !)

$$d E = - P\, dV \qquad\qquad E = \rho V$$

- the physical volume is $V = a^3(t)\, V_{com}$ **(V_{com} = comoving volume)**

$d_t (\rho\, a^3 V_{com}) = -P\, d_t (a^3 V_{com})$ but $V_{com} = cst = V_0$

$$\boxed{d_t\, [\rho(t)\, a^3(t)] = -P(t)\, d_t\, [a^3(t)]}$$

matter, radiation

- **Matter:** $d_t\,[\rho\, a^3] = -P\, d_t\,[a^3]$

  Galaxies may be approximated as a pressure-less gas: galaxies have no velocity relative to the overall expansion

  $$\Longrightarrow d_t\,[\rho_m\, a^3] = 0$$

  $$\rho_m\,(t) = \rho_m\,(t_0)\, a^{-3}(t)$$

- **Pure radiation** (black body) Stefan's law: $\rho_r = g\dfrac{\pi^2}{30}\dfrac{(k_B T)^4}{(\hbar c)^3}$

  Thermodynamics: $P_r = (1/3)\,\rho_r$

  $$d_t\,[\rho\, a^3] = -(1/3)\,\rho\, d_t\,[a^3] \Rightarrow 4\rho a^3 d(a) + a^4 d(\rho) = 0$$

  $\rho_r\,(t) = \rho_r\,(t_0)\, a^{-4}(t)$ $\quad$ a^{-3} for volume

  a^{-1} since $E \propto \lambda^{-1}$

  $T(t) = T(t_0) / a(t)$

Vacuum

- "Vacuum is not empty"
  virtual particle-antiparticle pairs

- Results in a vacuum energy density constant in space and time

$$d_t\,[\rho\, a^3] = -P\, d_t\,[a^3] \quad \Rightarrow \quad \rho\, d_t\,[a^3] = -P\, d_t\,[a^3]$$

$$P_v = -\,\rho_v = cst < 0$$

- Vacuum pressure is negative !
- Vacuum energy equivalent to cosmological constant or a form of dark energy: $\rho_v = \Lambda/(8\pi G)$ in Einstein equation

Friedman equation

- Einstein Eq => $$\boxed{\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi\rho}{3}}$$ (Friedmann Eq.)

- Critical density today for which the Universe is flat (k=0)

$t=t_0$: $$\frac{8\pi\rho_c}{3} = \left(\frac{\dot{R}}{R}\right)_0^2 = \left(\frac{\dot{a}}{a}\right)_0^2 = H_0^2$$

$$\boxed{\rho_c = \frac{3H_0^2}{8\pi}} = 1.88 \times 10^{-29} h^2 \text{ g/cm}^3$$ ~5 protons / m^3

- We introduce $$\Omega_m \equiv \frac{\rho_m(t_0)}{\rho_c}, \qquad \Omega_r \equiv \frac{\rho_r(t_0)}{\rho_c}, \qquad \Omega_v \equiv \frac{\rho_v(t_0)}{\rho_c}$$

$\Omega_T = \Omega_m + \Omega_r + \Omega_v = \rho_0 / \rho_c$ (Ω_x, at $t=t_0$, should be Ω^0_x)

$$\rho(a) = \rho_m(t_0)a^{-3} + \rho_r(t_0)a^{-4} + \rho_v(t_0)$$
$$= \rho_c(\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_v) \ ,$$

Friedman equation

$\rho_0 = \rho_c \, \Omega_T$

- $$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi\rho}{3} \xrightarrow{t=t_0} \frac{k}{R_0^2} = \frac{8\pi\rho_0}{3} - H_0^2 = H_0^2(\Omega_T - 1)$$

$$\frac{8\pi\rho_c}{3} = H_0^2$$

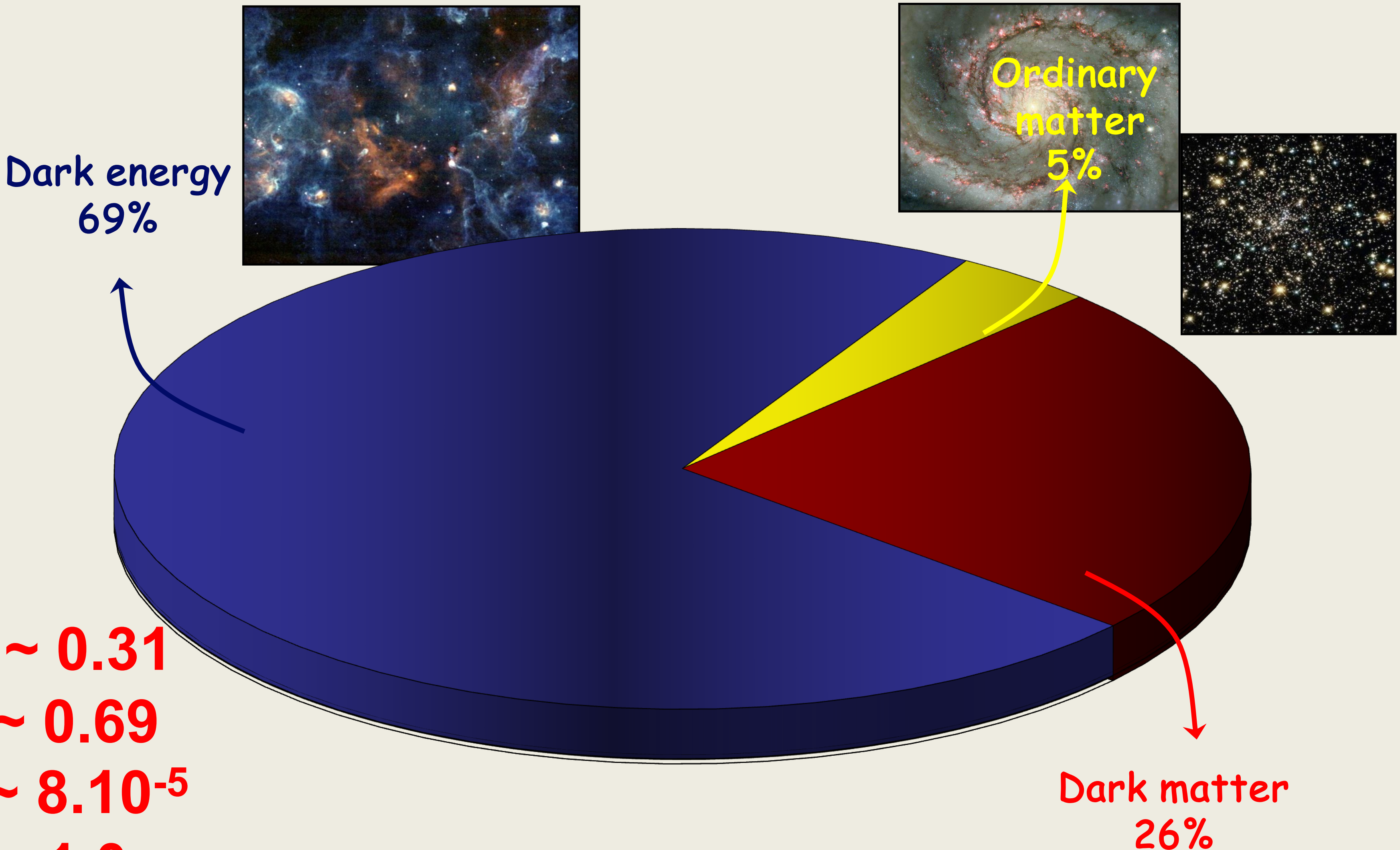
$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi\rho}{3} - H_0^2(\Omega_T - 1)\left(\frac{R_0}{R}\right)^2 = H_0^2\left[\frac{\rho(a)}{\rho_c} + (1-\Omega_T)a^{-2}\right]$$

- $$\begin{aligned}\rho(a) &= \rho_m(t_0)a^{-3} + \rho_r(t_0)a^{-4} + \rho_v(t_0) \\ &= \rho_c(\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_v)\ ,\end{aligned}$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_v + (1-\Omega_T)a^{-2}\right]}$$

Simplification: for a flat Universe (k=0 ⇒ 1- Ω_T =0)

Content of the Universe



$\Omega_m \sim 0.31$

$\Omega_v \sim 0.69$

$\Omega_r \sim 8.10^{-5}$

$\Omega_T \sim 1.0$

3) History
of the Universe

Age of the Universe

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_v + (1-\Omega_T)a^{-2}\right]$$

- many quantities may be computed from this equation by expressing in terms of 'a' and $\dot{a}/a$

- e.g. the age of the universe : $$dt = \frac{dt}{da}da = \frac{da}{\dot{a}} = \frac{da}{a(\dot{a}/a)}$$

$$t = H_0^{-1}\int_0^1 \frac{da}{a\left[\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_v + (1-\Omega_T)a^{-2}\right]^{1/2}}$$

- $$H_0 = 70(km/s)/Mpc = \frac{70km/s}{10^6 \times 3.262 \times 1an \times 300000km/s}$$

$H_0^{-1} = 14.10^9$ years

Age of the Universe

- Note: - our Universe is flat (k=0 ⇒ Ω_T =1)
  - one may often neglect $\Omega_r = 9\ 10^{-5}$ (Ω_m=0.3, Ω_v=0.7)

- Simplification of the equation:

$$t = H_0^{-1} \int_0^a \frac{da}{a\left(\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_v\right)^{1/2}}$$

- Universe with just matter $\Omega_m \approx 1$

$$t(a) = H_0^{-1} \int_0^a \frac{da}{a^{-1/2}} = H_0^{-1} \int_0^a a^{1/2} da = \frac{2}{3} H_0^{-1} a^{3/2}$$

T ~9.10^9 years, incompatible with the age of the first stars in MW

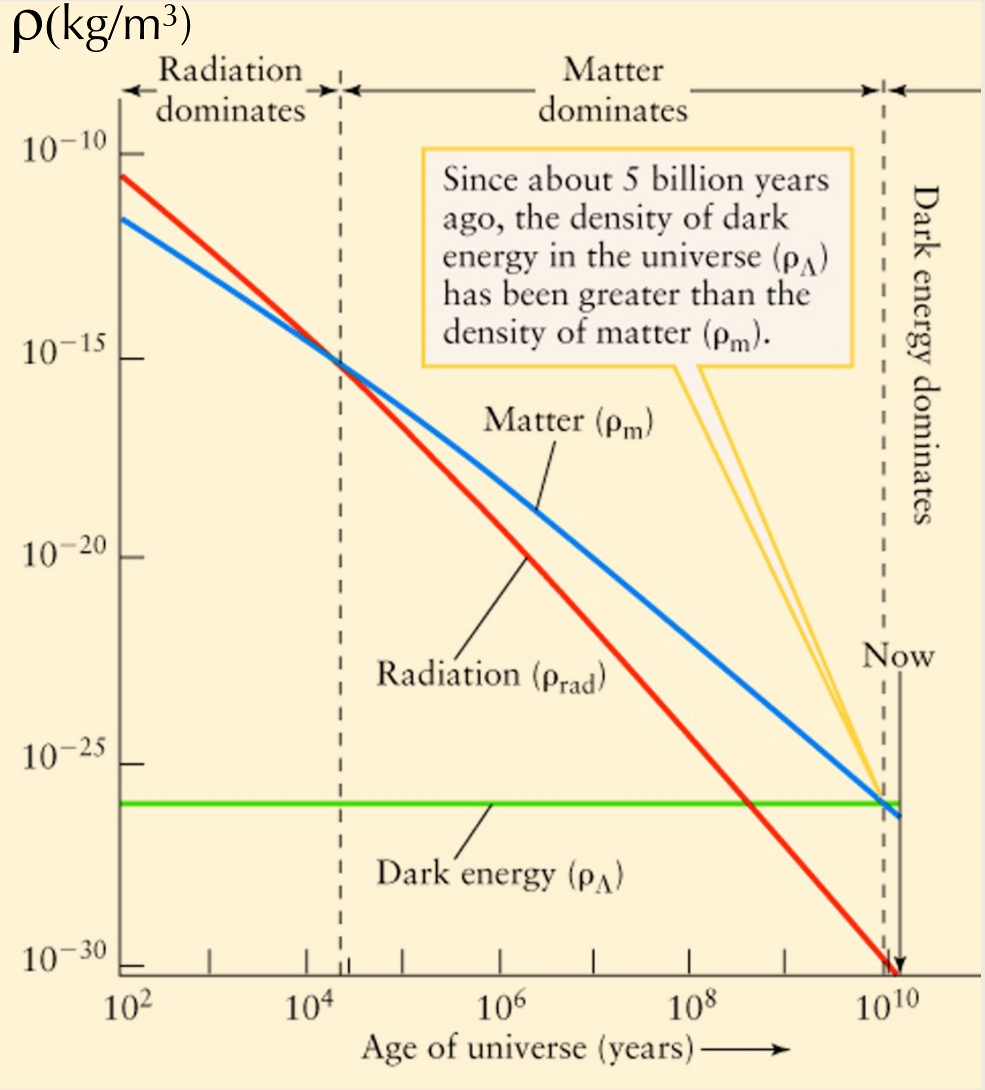
Epochs of the universe

$$\rho(a) = \rho_{crit}\left(\Omega_v + \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4}\right)$$

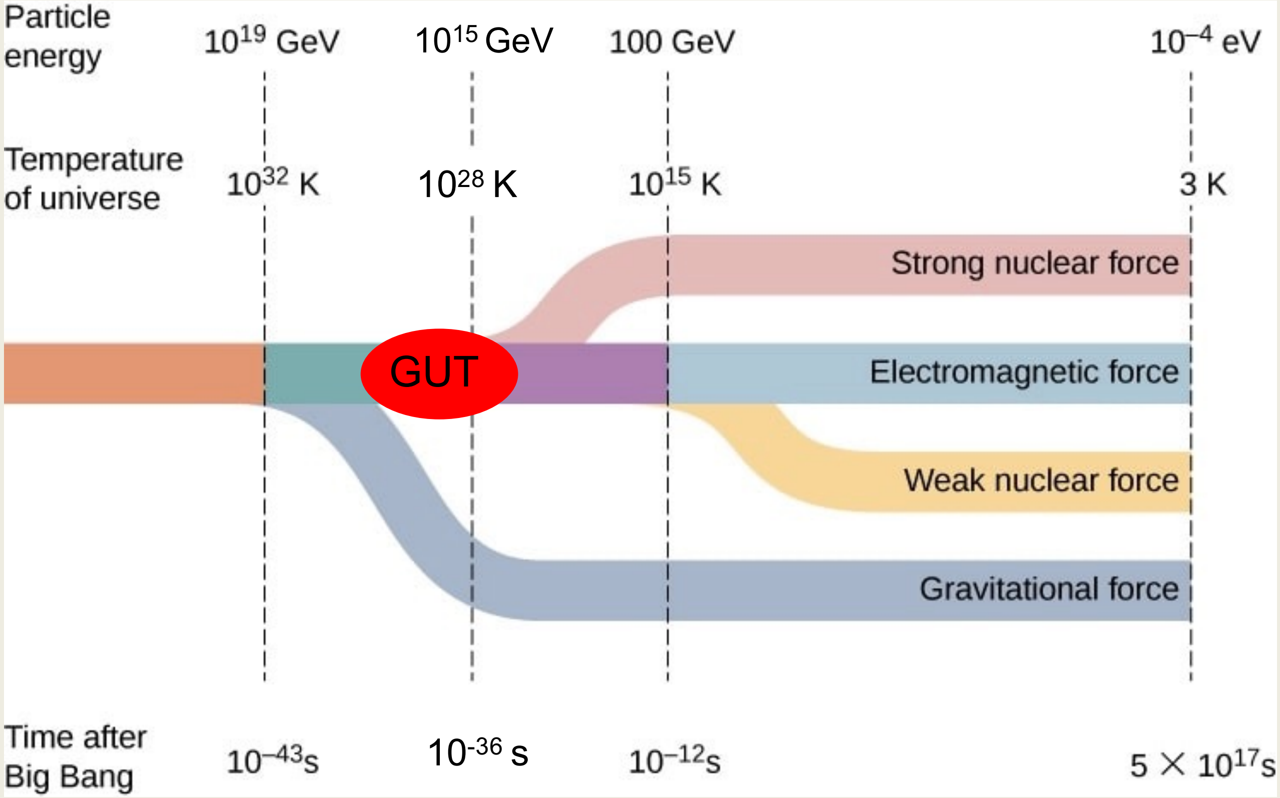
- beginning
  ($'a'$ very small)
  radiation dominates

- then mater dominates

- "recently" vacuum (or dark energy) dominates



Unification of forces



- EW force ~ 100 GeV
- Grand unification Theory (GUT) ~10^{15} GeV at 10^{-36}s after BB
- Theory of Everything ~10^{19} GeV (Planck scale)

Evolution of homogeneous Universe

Governed by coupled differential equations :

- Friedmann equation for a(t)

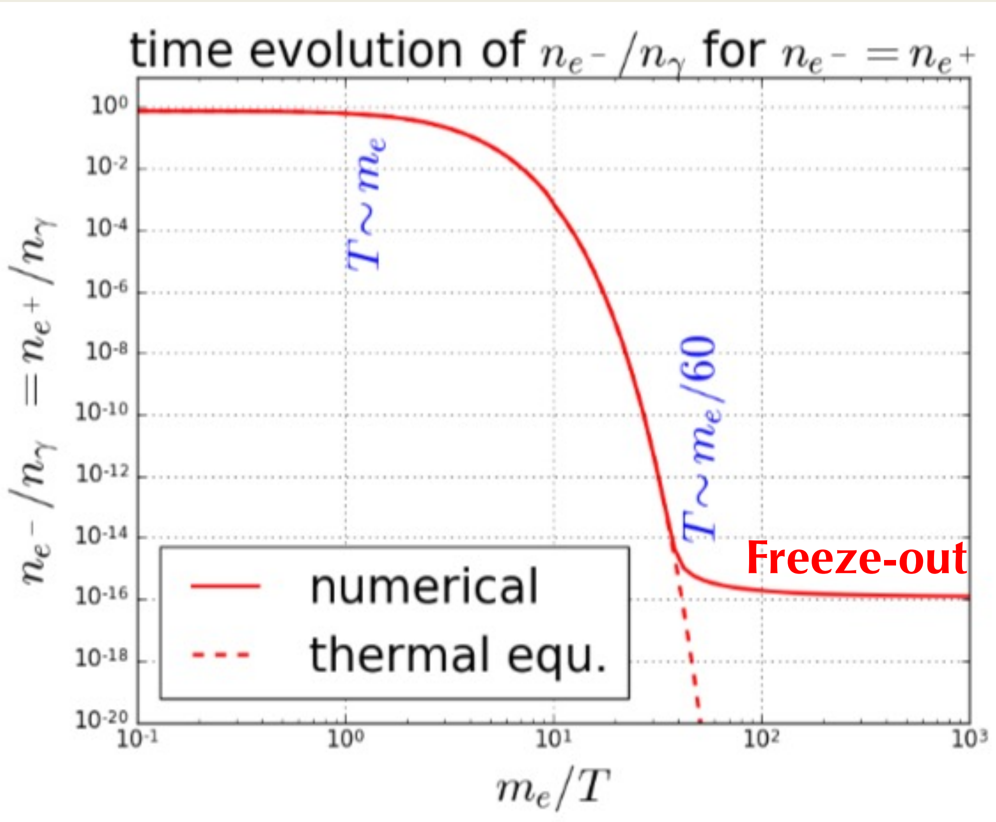
$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{R^2} = \frac{8\pi G\rho}{3}$$

- Boltzmann equations for phase space density of each particle species :

$$\frac{\partial f_i(\boldsymbol{x},\boldsymbol{p},t)}{\partial t} = \text{(Liouville)} + \text{(collision)} + \text{(creation)} + \text{(destruction)}$$

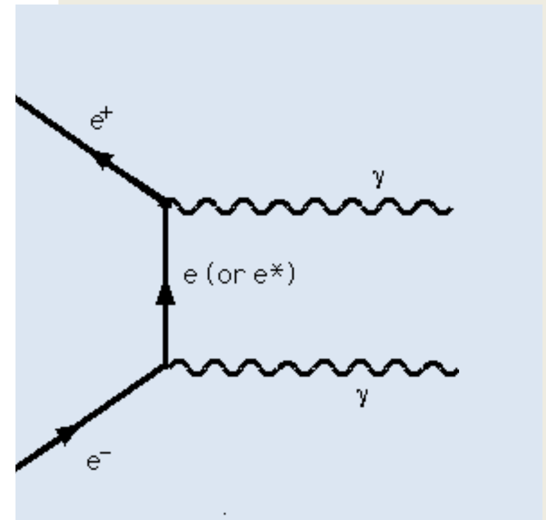
- Solved numerically but intuition by comparing
  - Γ : Number of reactions per particle per unit time
  - H(z) = expansion of Universe

The simplest example, electrons and positrons annihilation



$T \gg m_e$:
$n_e/n_\gamma \to 1$

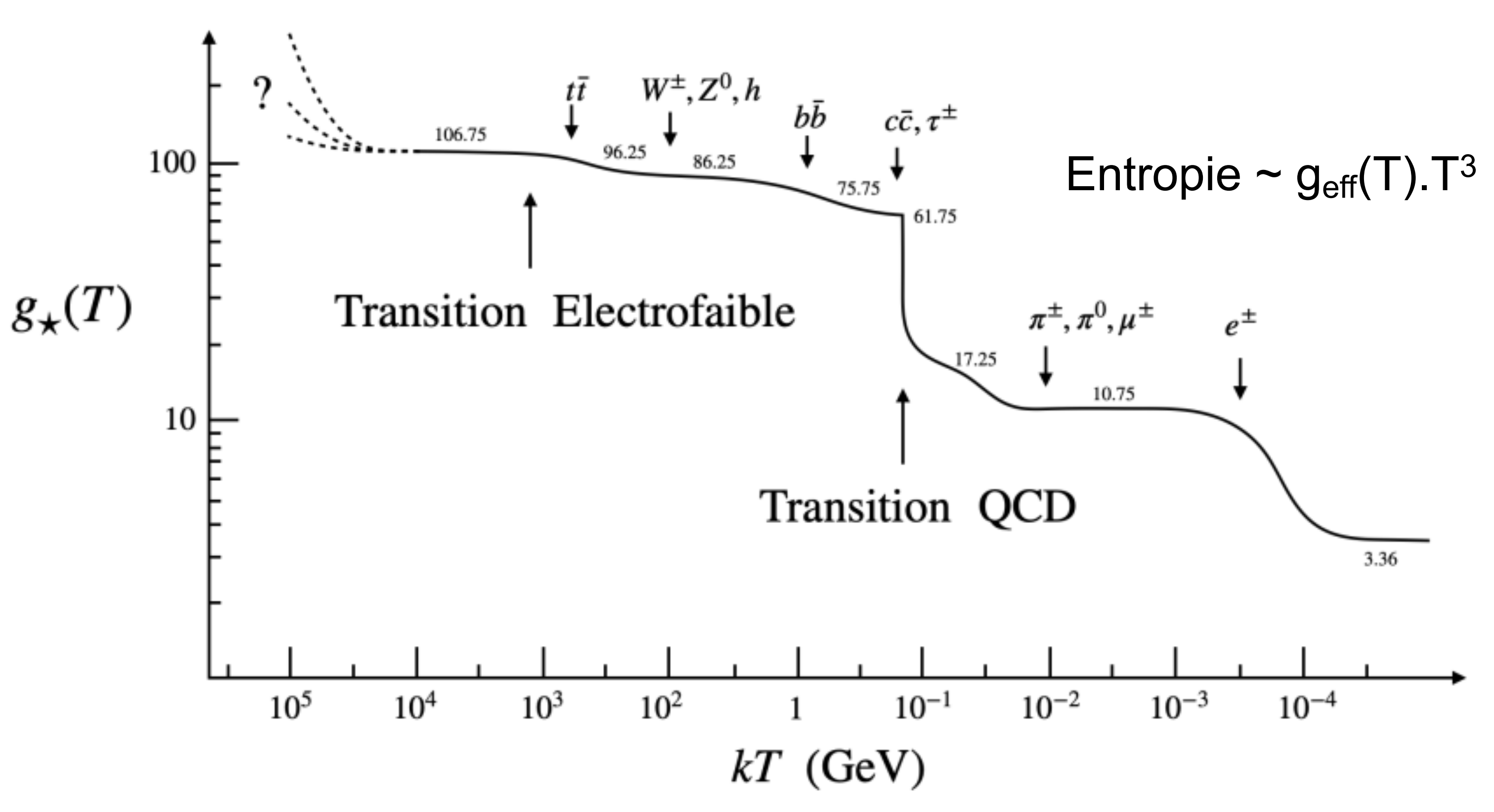
$T \ll m_e$:
$n_e/n_\gamma \to 10^{-16}$



- Freeze-out for $T_{freeze} \sim \frac{m_e}{60} \sim$ **10 keV when Γ~H**
- After the freeze-out, the number of e^+ and e^- is constant
- The density n_e decreases as $1/a^3$
- Caveat: the asymmetry of matter/antimatter is not considered!

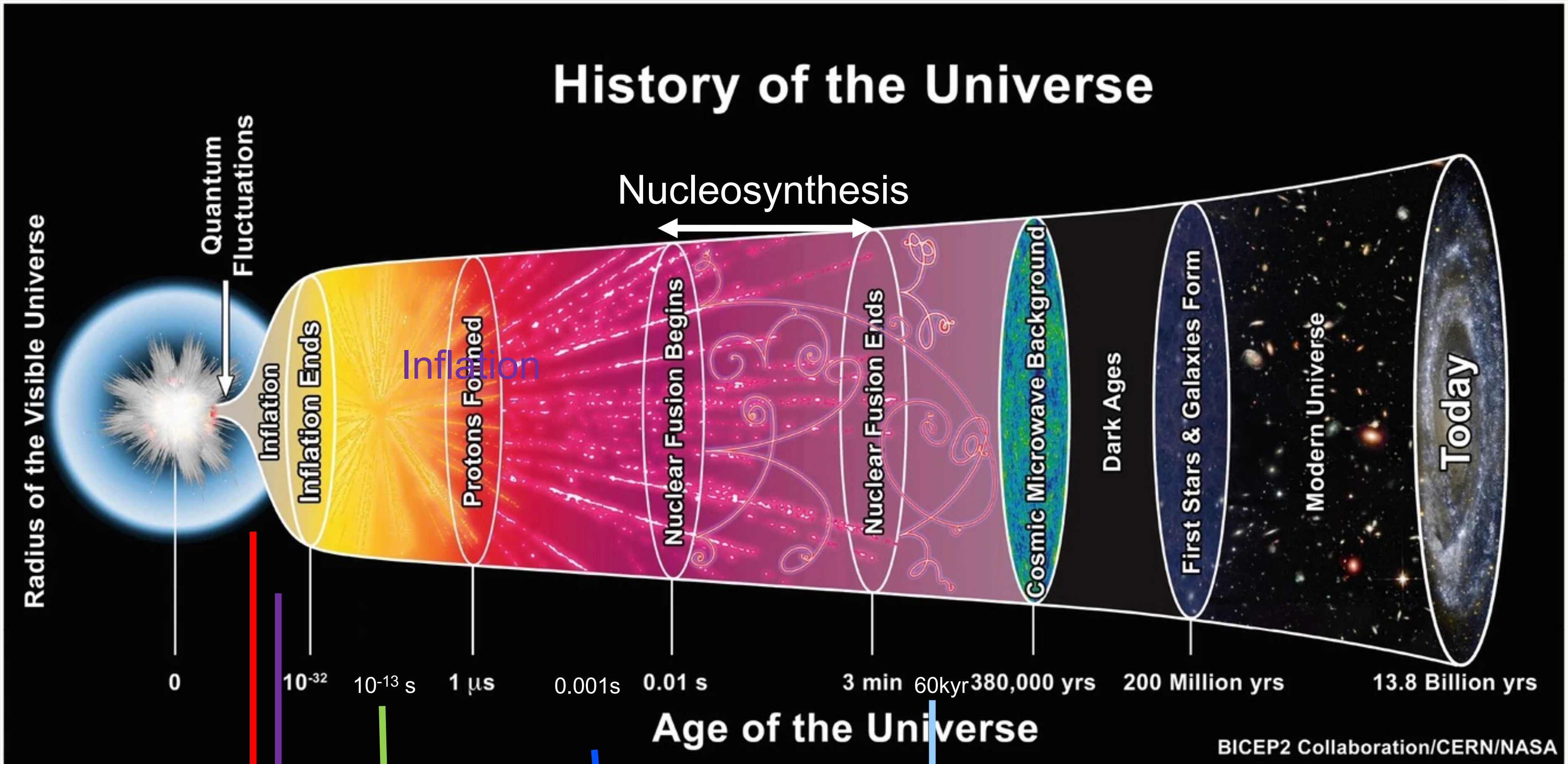
Thermal history with primordial particles

Effective number of spin states ~ Number of degrees of freedom



- See classes of particles physics
- Time of particle physics: age of Universe < 1s

Timeline of Universe history



GUT

Inflation

EW

Neutrinos decoupling

Matter/radiation equality