Astrophysical gravitational background in the PTA band: the (theoretical) variance of the Hellings-Downs curve

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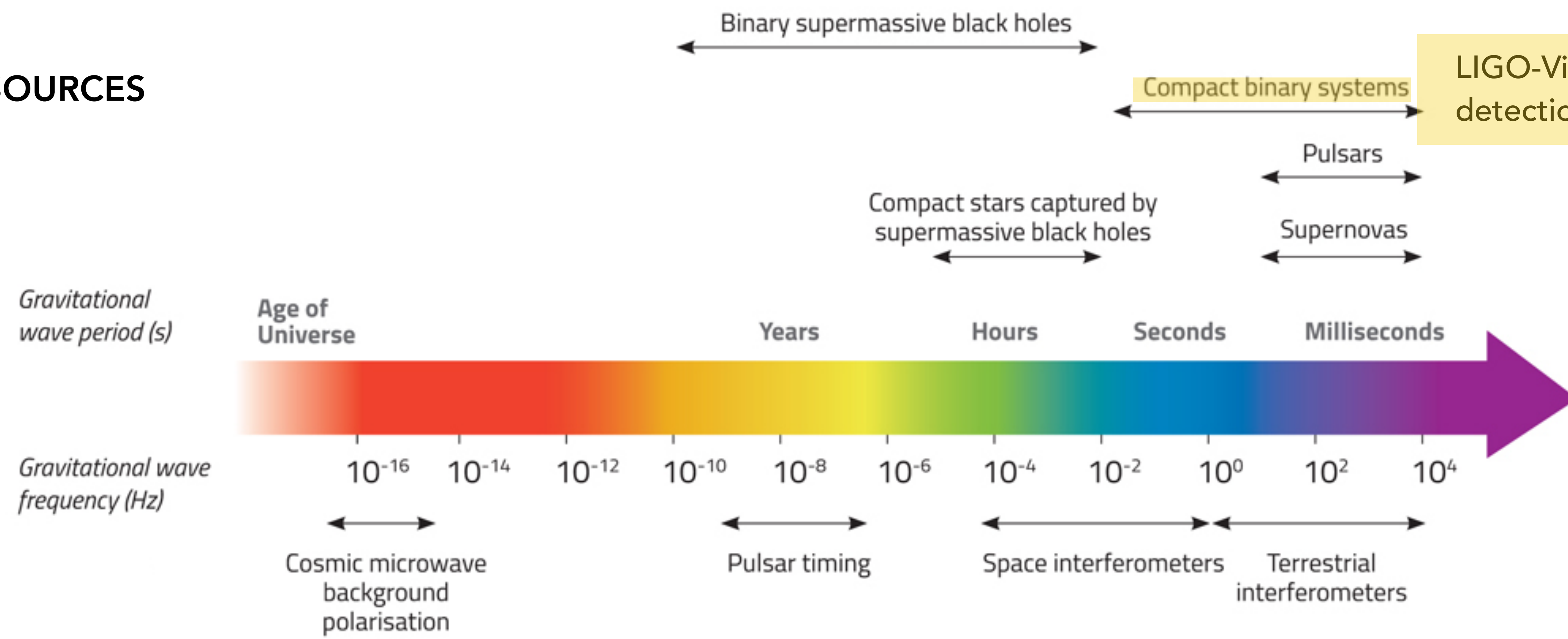
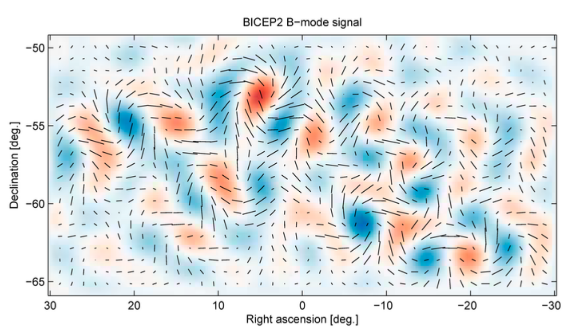
The "new" era of gravitational wave astronomy

SOURCES

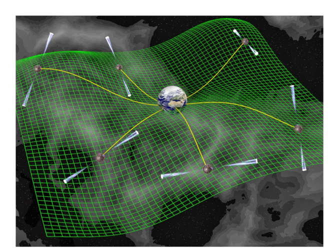
LIGO-Virgo first detections



DETECTORS



CMB polarization



PTAs

First evidence - 3σ of background detection!

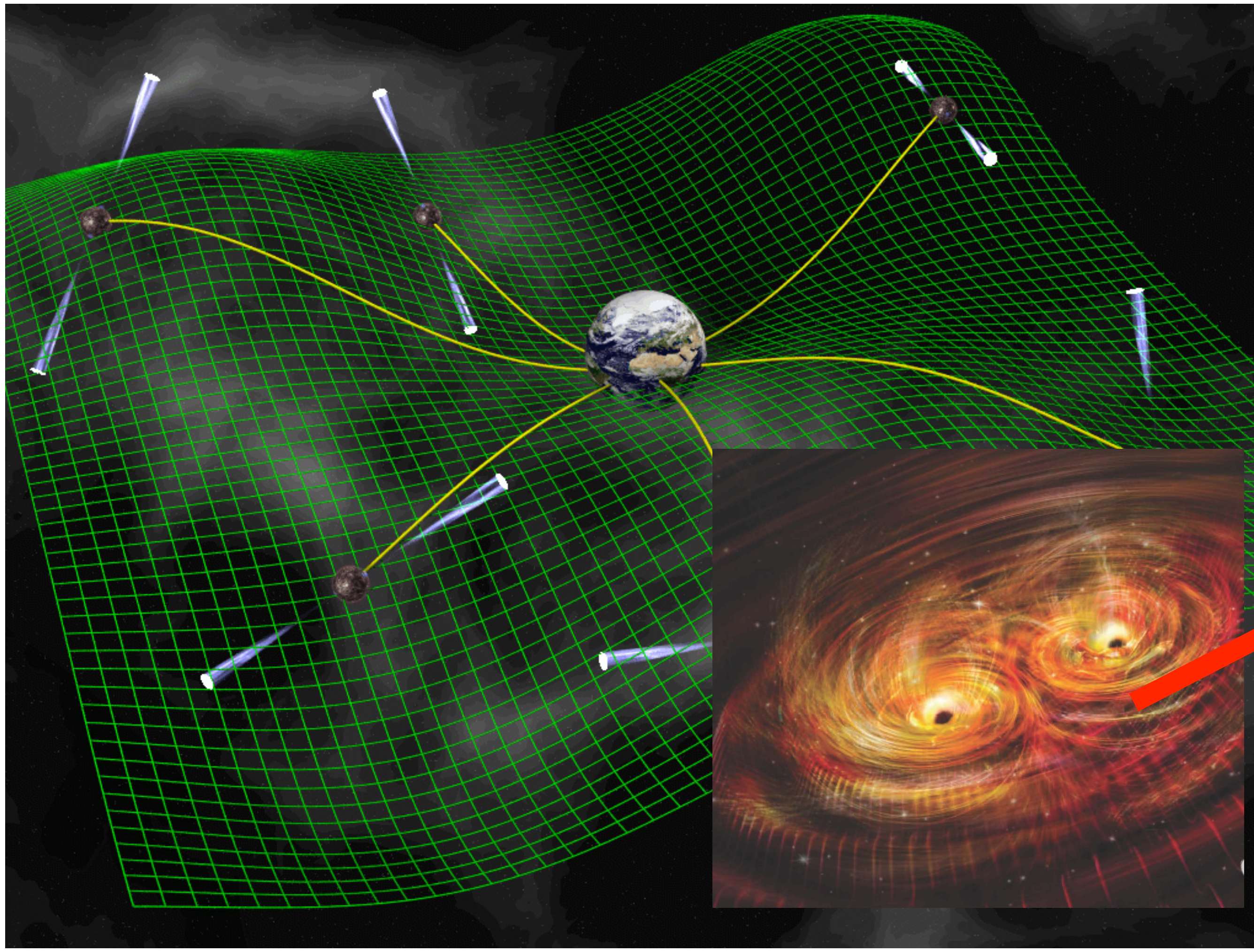
LISA (>2035)

DECIGO
BBO

LIGO, Virgo, LIGO-India, Kagra, ET/CE (>2035)

Expected detection of new astrophysical sources, and **detection of GW background** in various frequency bands

nano Hz band



2023: first evidence of detection!

Possible origin:
supermassive black hole binaries in the inspiralling phase

Pulsar timing arrays: we monitor the period of an array of pulsars.
Perturbations in the period is indication that spacetime Earth-pulsar has been deformed

Detection requires measuring **mostly-quadrupolar correlation** signature between pulsars (the Hellings-Downs curve) *(see later in this talk)*

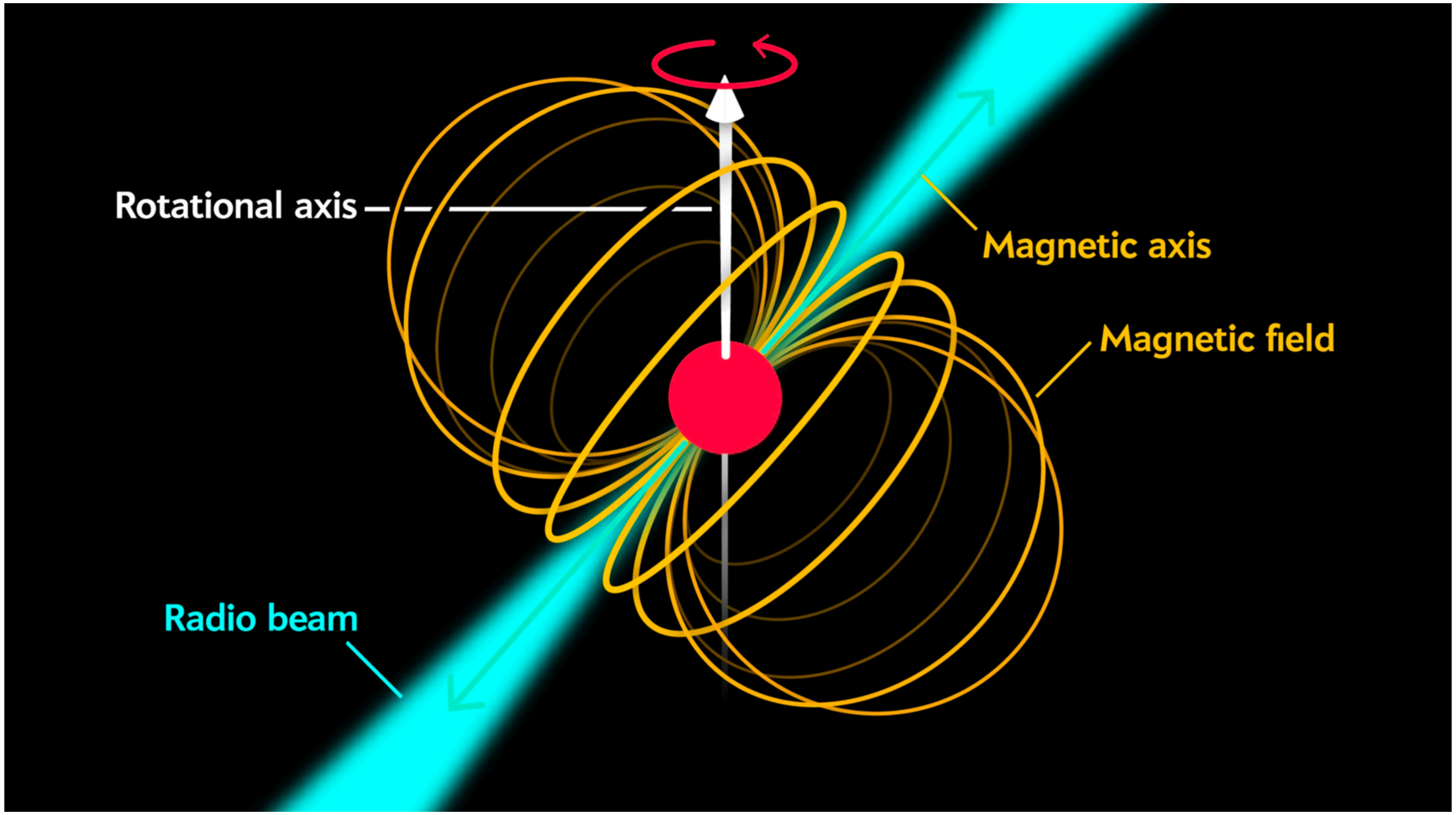
The PTA network

Milliseconds pulsars

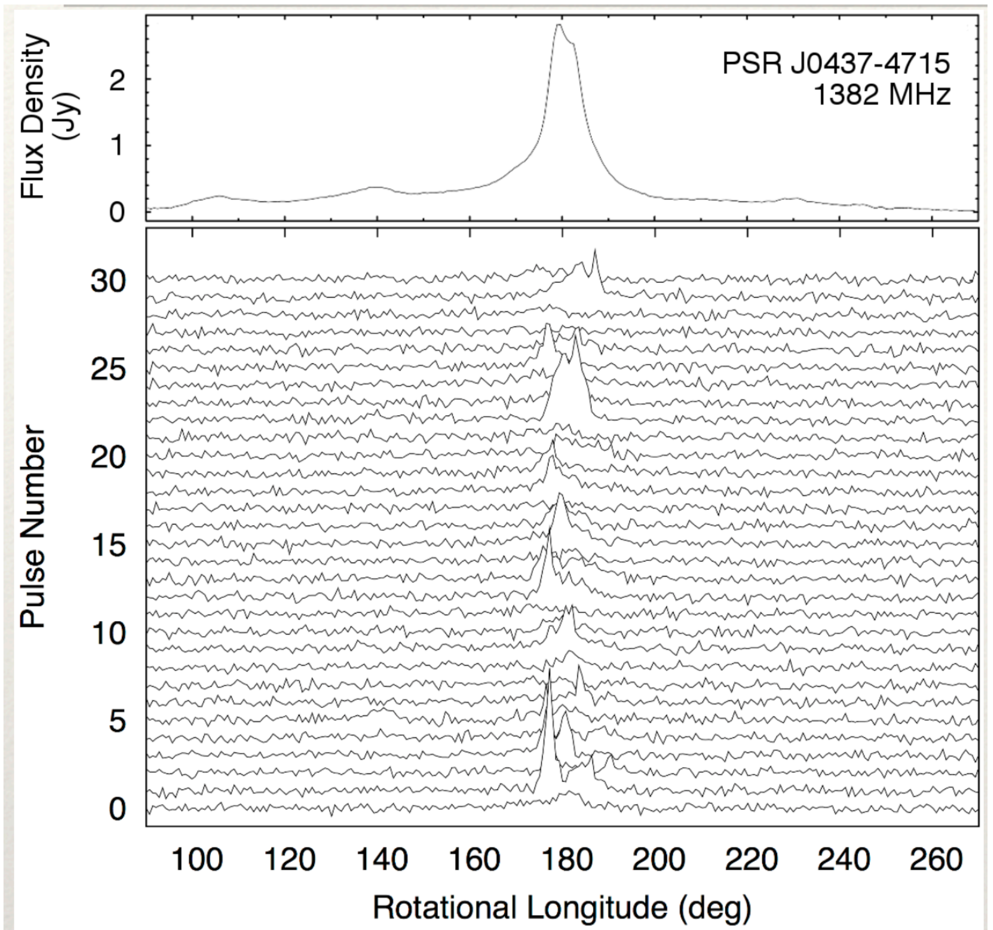
Very old neutron stars, with very stable rotation (millisec)

Often in binaries

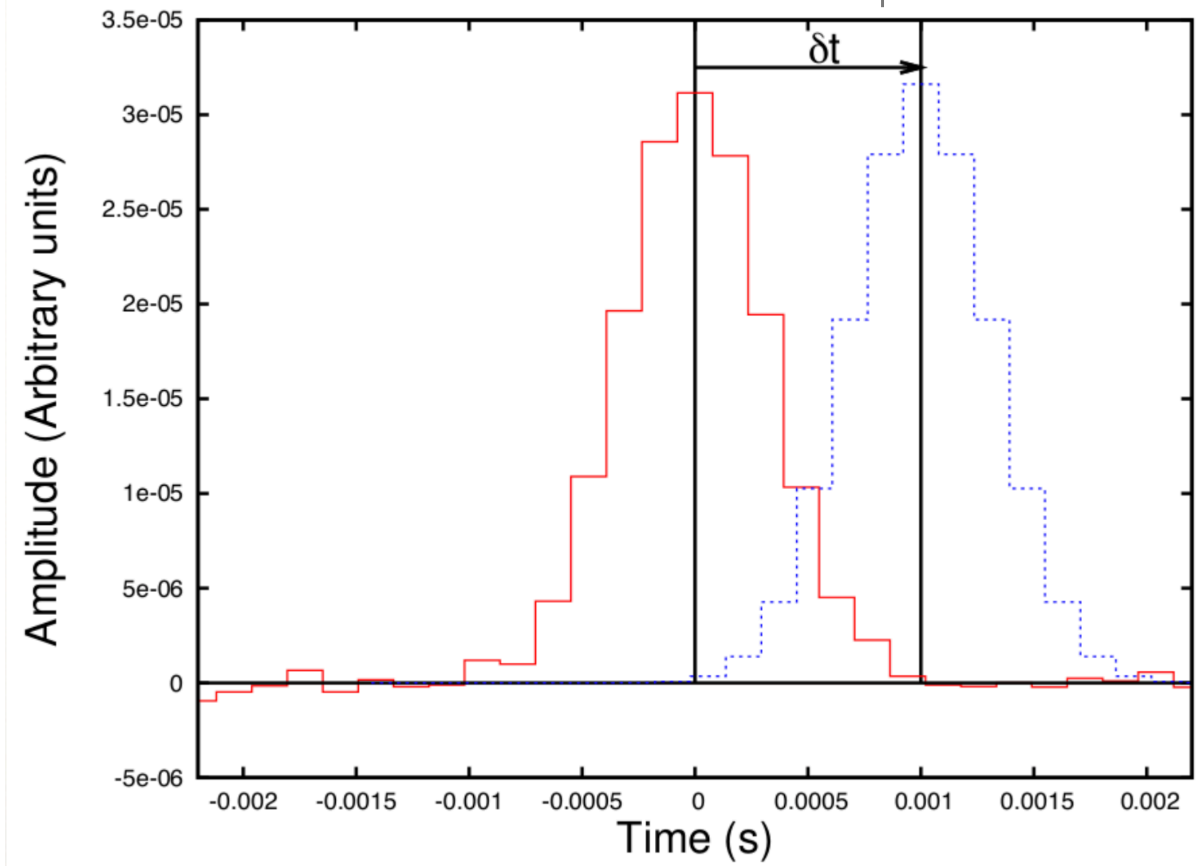
The most stable clocks on the long time scale (decades)



Pulsar timing



S. Burke-Spolar & L. Lentati



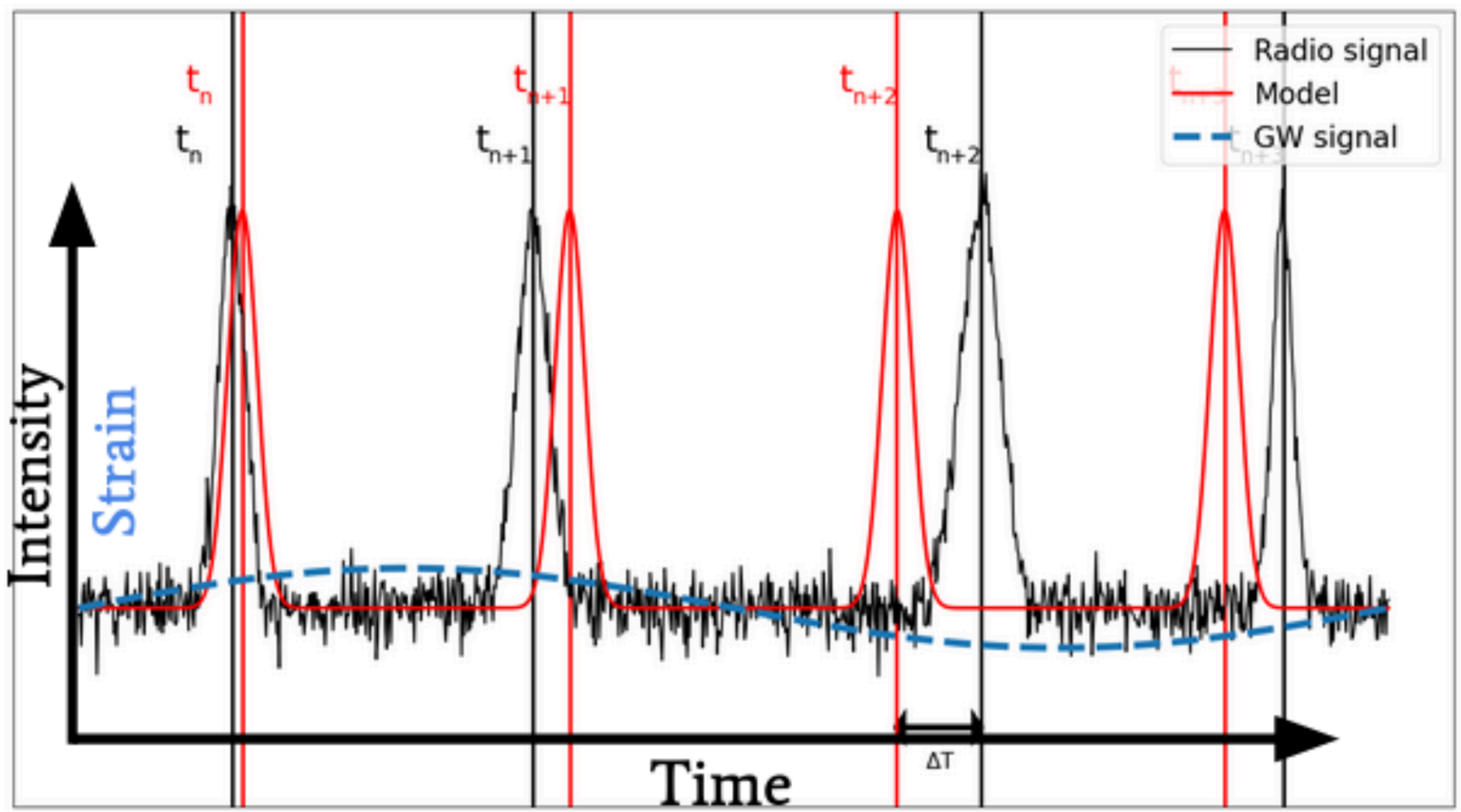
Each observed **radio pulse profile has a lot micro-structure**. If we average over ~hour the (average) profile is very stable.

We know the spin of the pulsars, so we can **predict the TOA**. The idea is to measure the time-of-arrival (TOA), and compare to the expected TOA.

The **difference between measured and expected TOA: residuals**

Timing residuals

credit: M. Falxa



$$\delta t = t_{\mathrm{TOA}}^{\mathrm{th}} - t_{\mathrm{TOA}}^{\mathrm{obs}} = \delta t_{\mathrm{errors}} + \delta T_{\mathrm{GW}} + \mathrm{noise}$$

Residuals

Errors in fitting the model

Due to GW

Residual due to GW

Fractional time-delay for pulsar "a": fractional deviation in the periodicity of pulse from pulsar "a" due to GW passing

Earth term *Pulsar term*

$$z_a(t) \equiv \frac{\Delta T_a(t)}{T_a(t)} = \frac{1}{2}\frac{n_a^i n_a^j}{1+\mathbf{n}\cdot\mathbf{n}_a}\left[h_{ij}(t,\mathbf{0}) - h_{ij}(t-\tau_a, \tau_a \mathbf{n})\right]$$

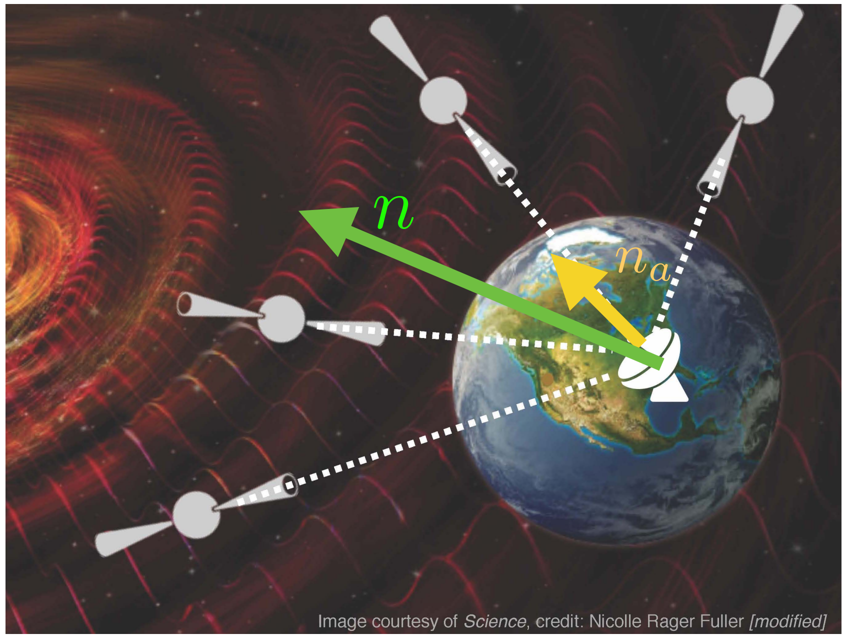
distance to the pulsar

For a SGWB this quantity can be written as

$$z_a(t) = \sum_{A=+,\times}\int_{-\infty}^{\infty} \mathrm{d}f \int \mathrm{d}^2\mathbf{n}\, h_A(f,\mathbf{n}) F_a^A(\mathbf{n}) e^{-2\pi i f t}\left[1 - e^{2\pi i f \tau_a(1+\mathbf{n}\cdot\mathbf{n}_a)}\right]$$

Antenna pattern function in direction **n** (depends on geometry)

$$F_a^A(\mathbf{n}) = \frac{n_a^i n_a^j e_{ij}^A(\mathbf{n})}{2(1+\mathbf{n}\cdot\mathbf{n}_a)}$$



Residual due to GW

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Single-arm LIGO-like detector working in the **long arm** regime $f\tau_a \gg 1$ (regime opposite to LIGO/Virgo)

$$F_a^A(\mathbf{n}) = \frac{n_a^i n_a^j e_{ij}^A(\mathbf{n})}{2(1+\mathbf{n}\cdot\mathbf{n}_a)}$$

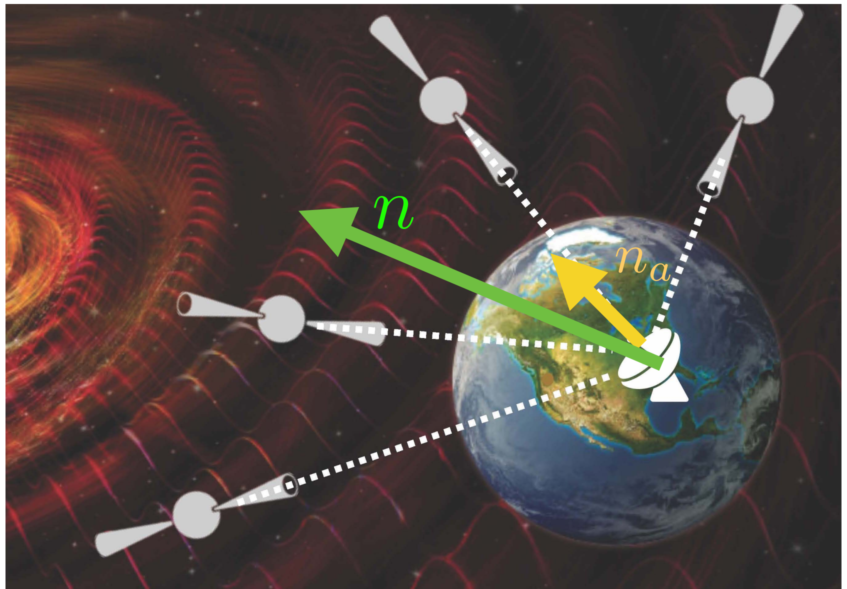
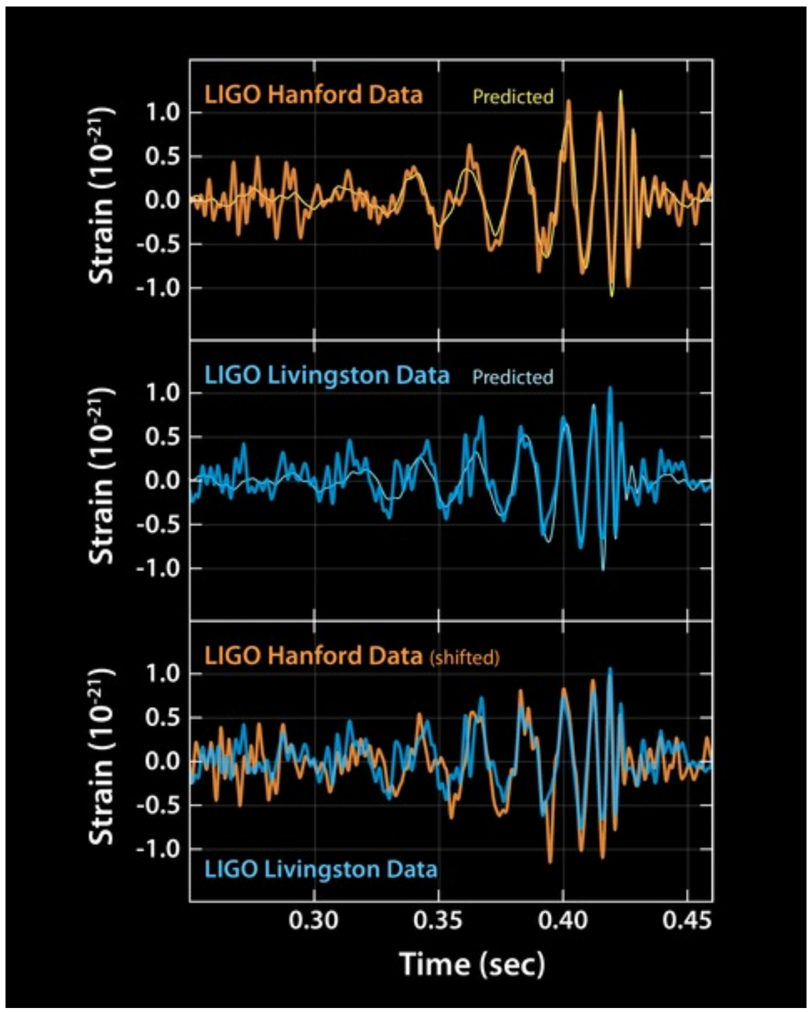


Image courtesy of *Science*, credit: Nicolle Rager Fuller *[modified]*

Resolvable signal vs stochastic background

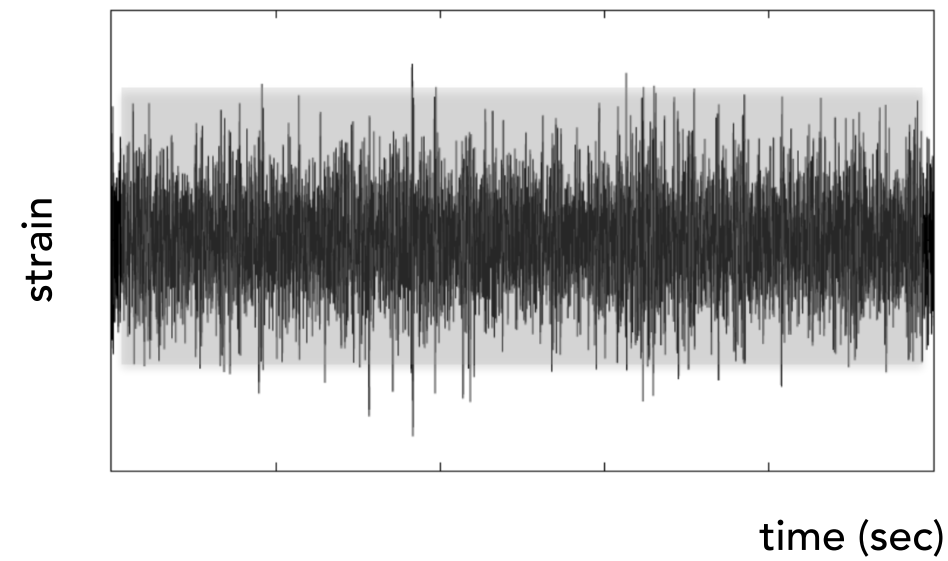
GW from resolvable event



LVC

quantity directly observable is the strain

stochastic background: incoherent superposition of GW signals



"waveform" similar to noise

typically within instrumental noise threshold

↓

two detectors needed to apply matched filtering techniques

what is measured is a strain square

Hellings-Downs curve

To claim a detection, one has to **correlate the time delay from two different pulsars**

$$z_a(t) \propto \sum_{A=+,\times} \int df \int d^2\mathbf{n}\, h_A(f,\mathbf{n}) F_a^A(\mathbf{n})$$ Time delay from pulsar "a"

$$\overline{z_a z_b} \equiv \frac{1}{T}\int_{-T/2}^{T/2} z_a(t) z_b(t)$$ Correlation

The expectation value is computed using that only waves from same direction and polarisation are correlated

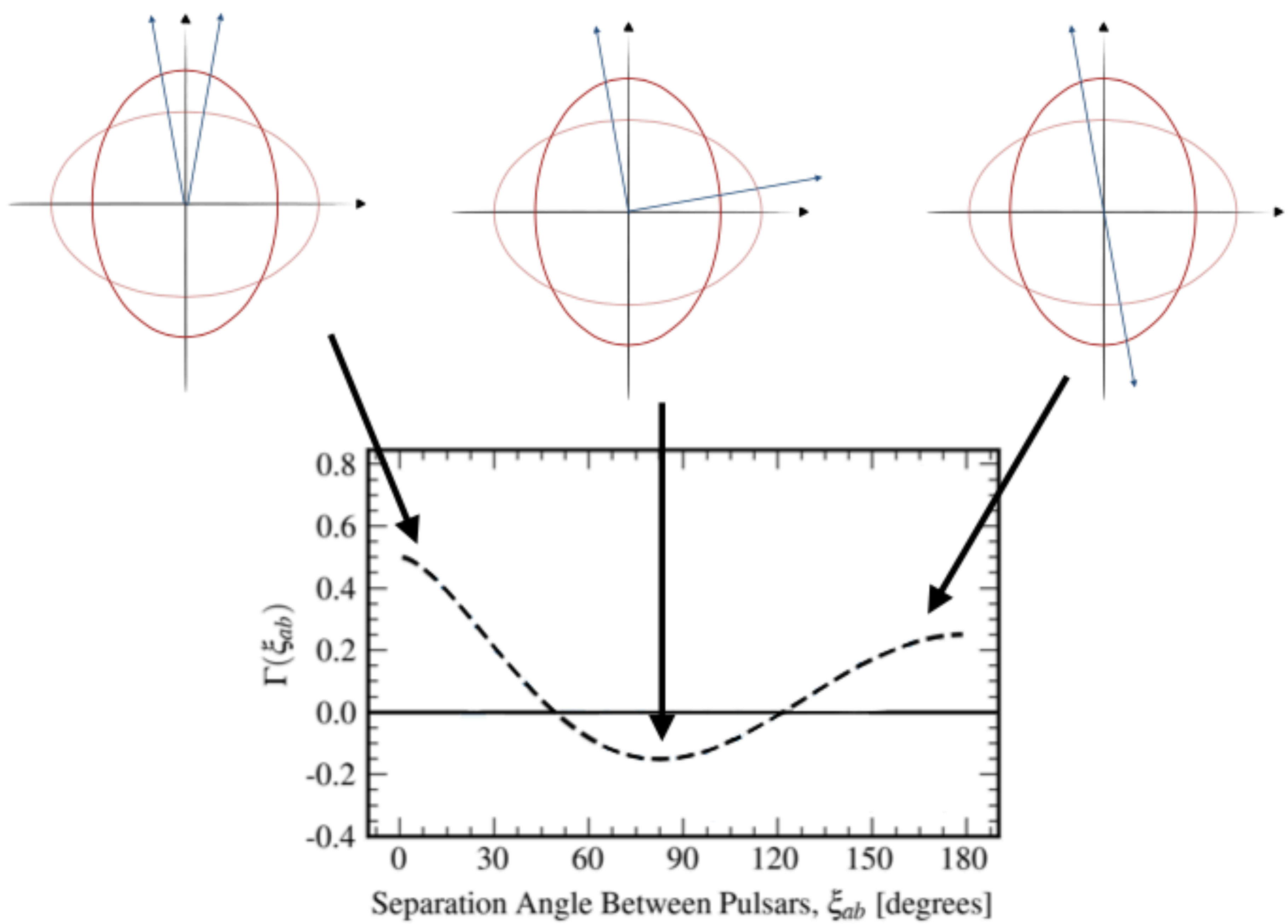
$$\langle h_A(f,\mathbf{n}) h_B(f',\mathbf{n}')\rangle \propto S_h(f)\delta_{AB}\delta(f-f')\delta(\mathbf{n},\mathbf{n}')$$

Pulsar separation

$$\overline{\langle z_a z_b \rangle} \propto C(\theta_{ab}) \int df\, S_h(f)$$ quadratic in strain

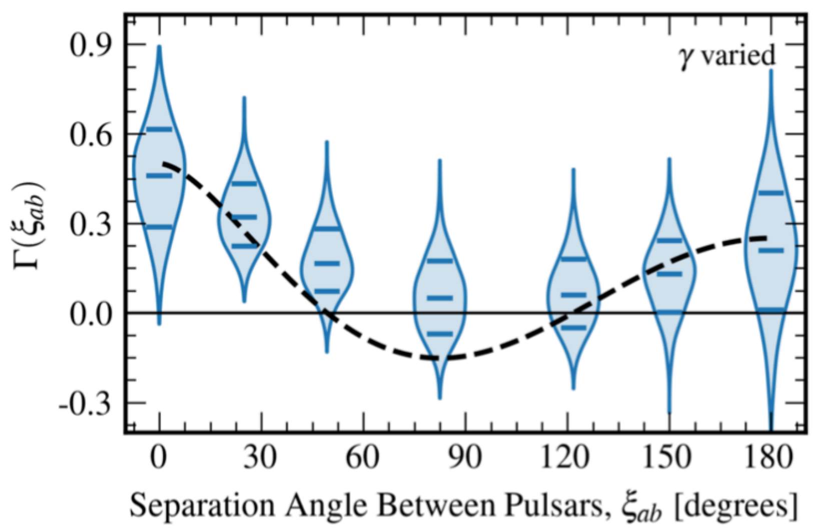
Hellings-Downs function

Smoking gun to claim that a given observed time delay is due to GW



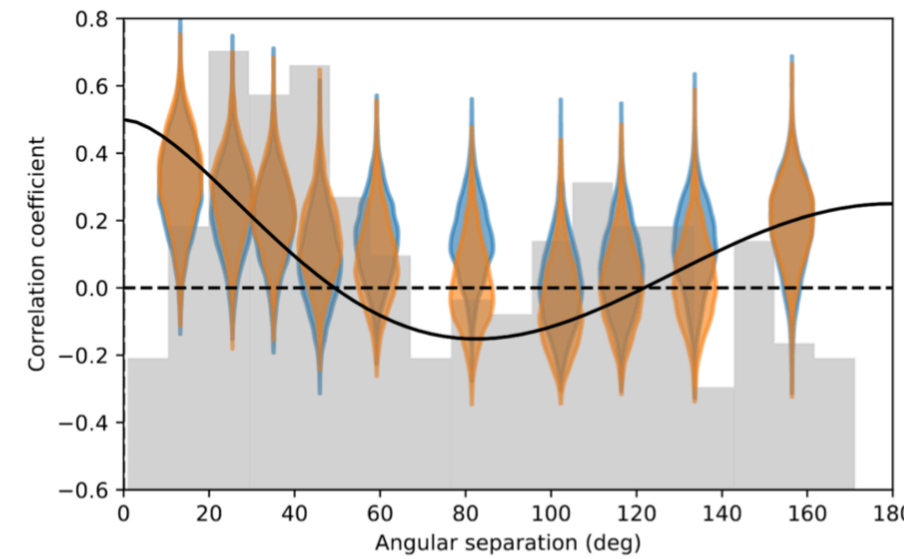
Evidence for the Hellings-Downs curve

NANOGrav



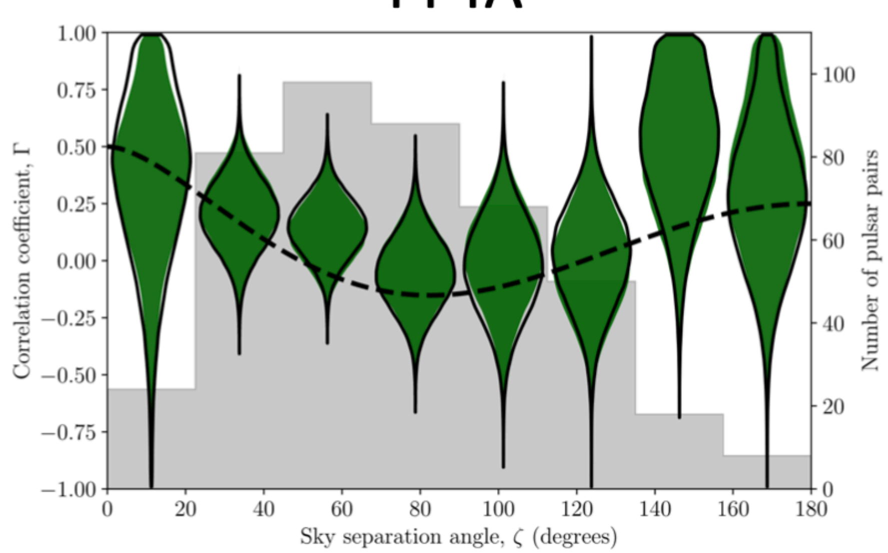
[arXiv:2306.16213]

EPTA+InPTA



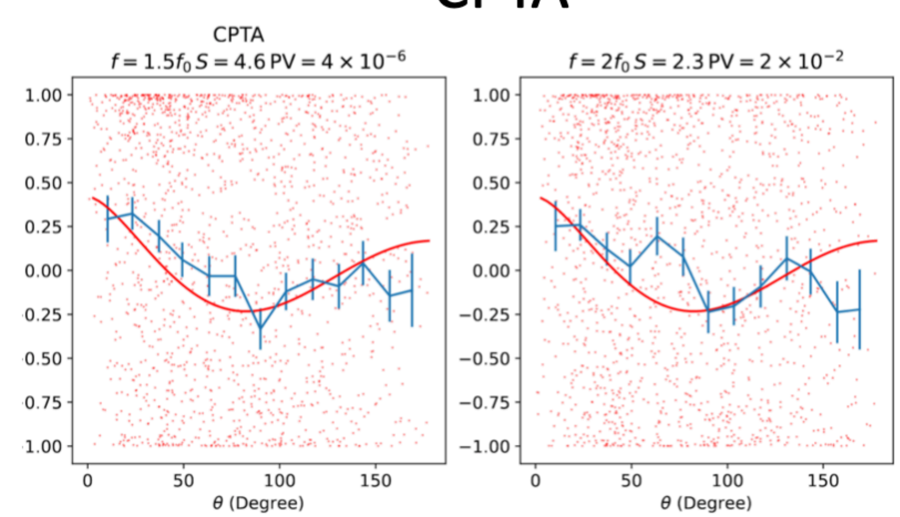
[arXiv:2306.16214]

PPTA



[arXiv:2306.16215]

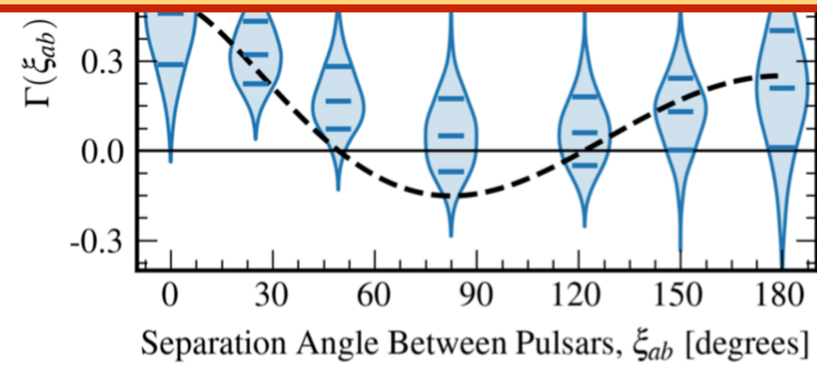
CPTA



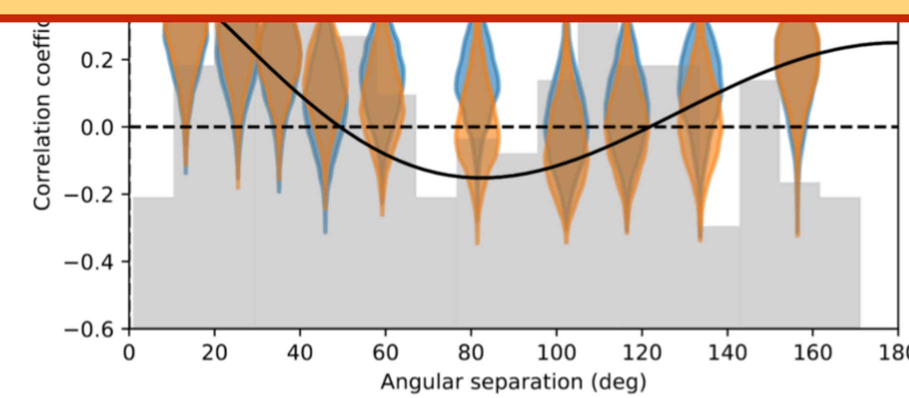
[arXiv:2306.16216]

Evidence for the Hellings-Downs curve

If we had **noise free measurements**: how well do we expect our data to follow the HD curve?

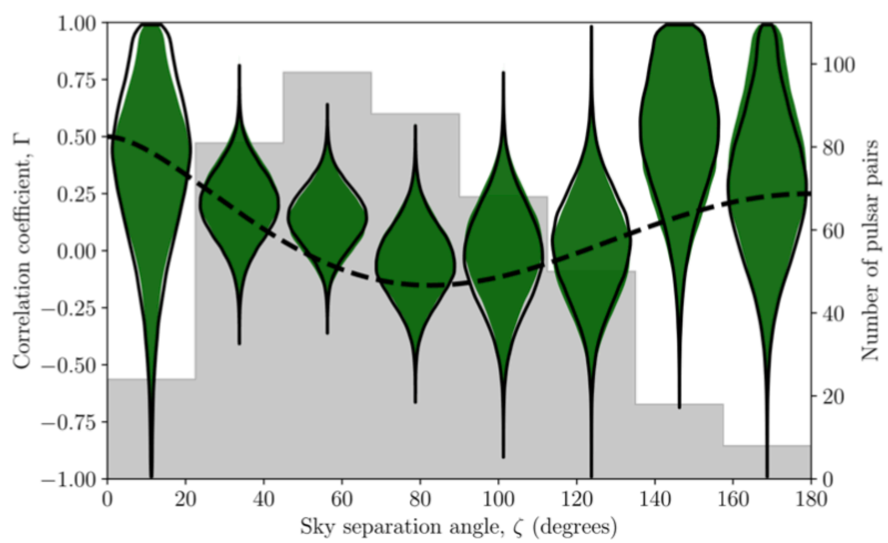


[arXiv:2306.16213]



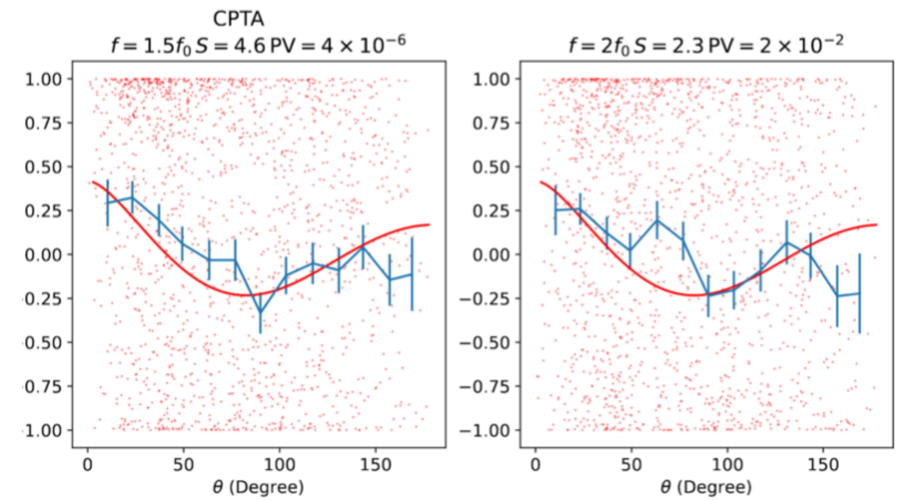
[arXiv:2306.16214]

PPTA



[arXiv:2306.16215]

CPTA



[arXiv:2306.16216]

The variance of the Hellings-Downs curve

PTA experiments operate in a regime where the time average over observation time does not allows sampling many realizations of a SGWB

↓

Two layers of stochasticity: realisation of GW sources, and of source location

Cosmic variance

Clustering variance

These two variances affect any 2-point function of GW: strain, energy density and HD correlation

I'll present derivation of cosmic variance for the wave 2-point function (computation for HD curve follows)

Cosmic variance on GW strain

Strain square *s*: time average of wave square

$$s = \overline{h_{ab}h^{ab}} = \frac{1}{T}\int dt\, h_{ab}(t)h^{ab}(t)$$

For any realisation

$$s = \sum_A \sum_{A'} \int df \int df' \int d\mathbf{n} \int d\mathbf{n}' \,\mathrm{sinc}\left(\pi(f-f')T\right) e^A_{ab}(\mathbf{n}) e^{ab}_{A'}(\mathbf{n}') h_A(f,\mathbf{n}) h_{A'}(f',\mathbf{n}')$$

Mean and variance

$$\langle s \rangle = \sum_A \int df\, S_h(f) \int d\mathbf{n}\, e^A_{ab}(\mathbf{n}) e^{ab}_A(\mathbf{n})$$

$$\sigma_s^2 = \langle s^2 \rangle - \langle s \rangle^2 = \int df \int df' \mathrm{sinc}^2\left(\pi(f-f')T\right) S_h(f) S_h(f')$$

High frequency limit (LISA and LVC)

Why is this variance negligible in the LVC and LISA bands?

$$\sigma_s^2 = \langle s^2 \rangle - \langle s \rangle^2 = \int df \int df' \mathrm{sinc}^2\left(\pi(f - f')T\right) S_h(f) S_h(f')$$

$$\Delta f \sim \frac{1}{T}$$

Frequency resolution

Different regimes

$f \approx \Delta f$ — **narrow-band regime**: all sources effectively emit at the same frequency

$f \to$ nHz; $\Delta f \to \frac{1}{\mathrm{yr}} \sim 10^{-6}\mathrm{Hz}$

PTA

$f \gg \Delta f$ — **broad-band regime**: sources emit at distinct frequencies

LIGO and LISA

High frequency limit (LISA and LVC)

Why is this variance negligible in the LVC and LISA bands?

$$\sigma_s^2 = \langle s^2 \rangle - \langle s \rangle^2 = \int df \int df' \mathrm{sinc}^2\left(\pi(f-f')T\right) S_h(f) S_h(f')$$

$$\mathrm{sinc}^2\left(\pi(f-f')T\right) \rightarrow \frac{\delta(f-f')}{T}$$

Variance is negligible
For LVC/LISA

$$\frac{\sigma_s}{\langle s \rangle} \propto \frac{1}{\sqrt{fT}} \ll 1$$

Different regimes

$f \approx \Delta f$ (f: nHz; Δf: $\frac{1}{\mathrm{yr}} \sim 10^{-6}\mathrm{Hz}$)

narrow-band regime: all sources effectively emit at the same frequency

PTA

$f \gg \Delta f$

broad-band regime: sources emit at distinct frequencies

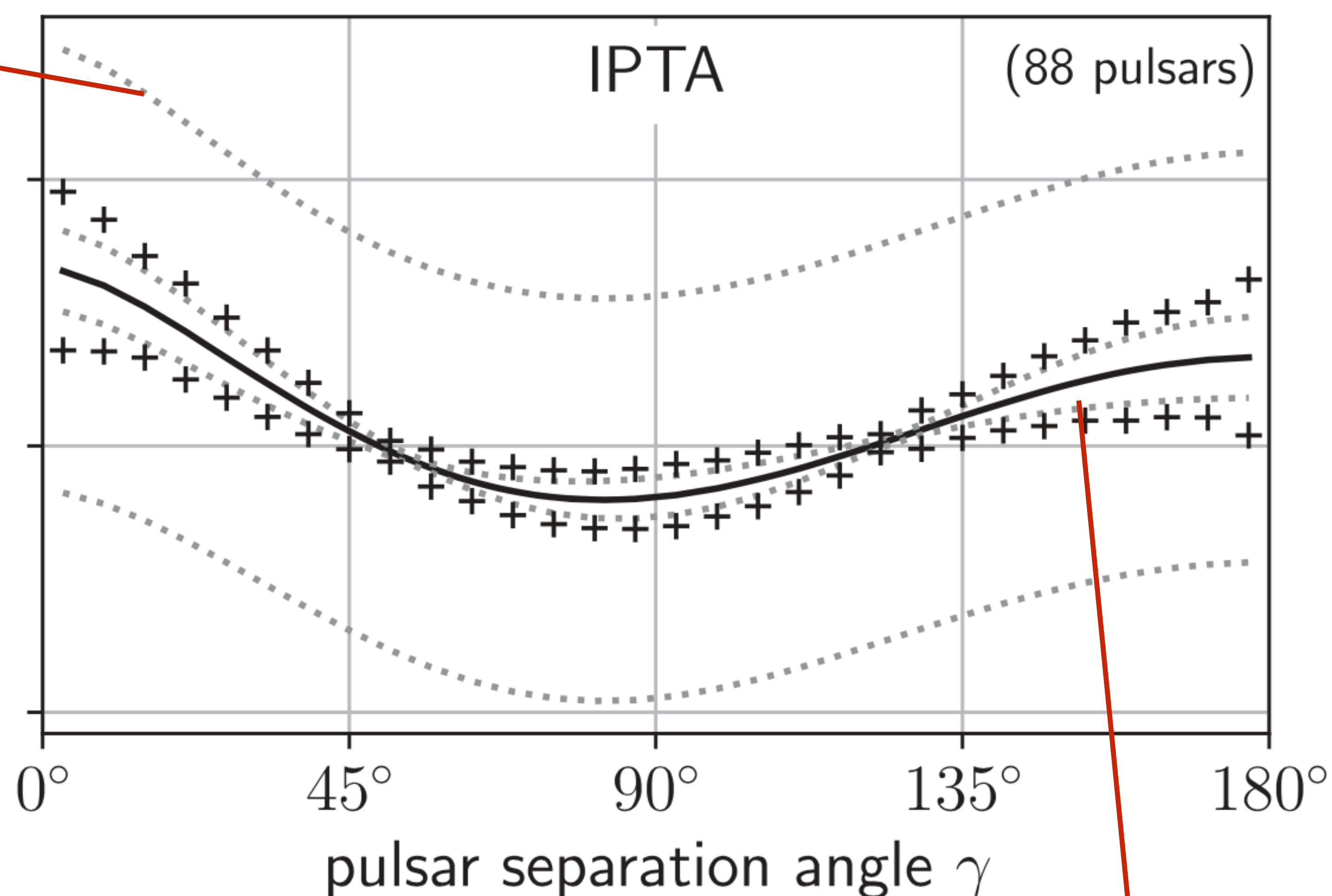
LIGO and LISA

Cosmic variance on the HD curve

With same computations, one can estimate the cosmic **variance of the HD curve** (there is just additional complication due to instrument antenna pattern functions)

one pulsar pair for angular separation

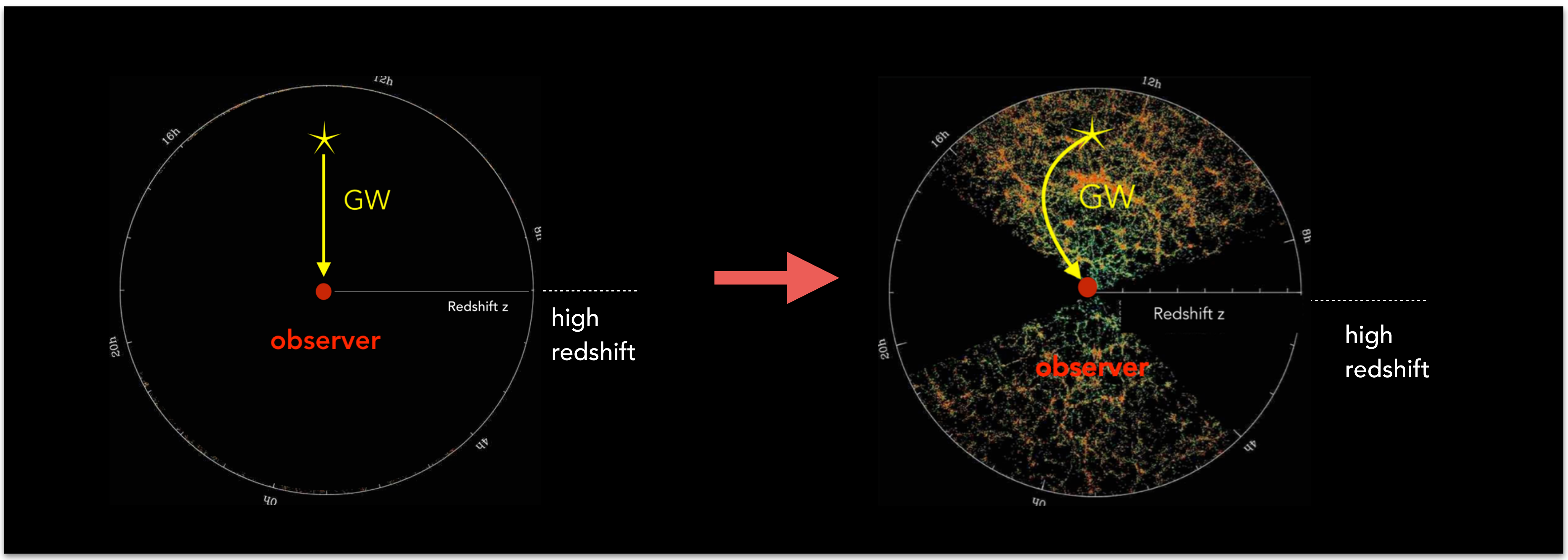
Allen and Romano 2023



Infinite number of pairs per separation (irreducible limit)

Which is the effect of source clustering on the Hellings-Down curve?

Hellings-Downs curve found under the assumption that sources are isotropically distributed: however, in fact they will follow the large scale structure distribution



Which is the effect of clustering on the Hellings-Downs curve (mean and variance)?

Effect of clustering on the HD curve

In an inhomogeneous universe, the **galaxy density distribution** can be treated as a random field

$\xi_g(\mathbf{n}, z)$	Galaxy density field at a given position
$\rho_g(\mathbf{n}, z)$	One specific realisation of this field

So far: special (isotropic) realisation of this field

$$\langle h_A^*(f, \mathbf{n}) h_B(f', \mathbf{n}') | \xi_g(\mathbf{n}, z) = \bar{\rho}_g(z) \rangle = \frac{1}{2} \delta_{AB} \delta(f - f') S_h(f) \frac{\delta^2(\mathbf{n}, \mathbf{n}')}{4\pi}$$

Conditional ensemble average (isotropic galaxy distribution)

Isotropic spectral density

Conditional ensemble average

In an inhomogeneous universe, the **galaxy density distribution** can be treated as a random field

$\xi_g(\mathbf{n}, z)$	Galaxy density field at a given position
$\rho_g(\mathbf{n}, z)$	One specific realisation of this field

Generalisation

$$\langle h_A^*(f, \mathbf{n}) h_B(f', \mathbf{n}') | \xi_g(\mathbf{n}, z) = \bar{\rho}_g(z) \rangle = \frac{1}{2} \delta_{AB} \delta(f - f') S_h(f) \frac{\delta^2(\mathbf{n}, \mathbf{n}')}{4\pi}$$

Conditional ensemble average (isotropic galaxy distribution)

$\bar{\rho}_g(z)$ → $\rho_g(z, \mathbf{n})$

Some anisotropic realisation

Isotropic spectral density

$S_h(f)$ → $S_h(f, \mathbf{n})$

Anisotropic spectral density

Two-steps ensemble average

Two layers of stochasticity: GW sources and their distribution
Calculations of **HD and its variance** are now done in two steps:

Step 1: we fix a galaxy density realisation of the universe and take the conditional average

$$\langle h_A^*(f,\mathbf{n})h_B(f',\mathbf{n}')|\xi_g(\mathbf{n},z)=\rho_g(z,\mathbf{n})\rangle$$

Step 2: we take ensemble averages over all density realisations

$$\langle\langle h_A^*(f,\mathbf{n})h_B(f',\mathbf{n}')\rangle\rangle \equiv \langle\langle h_A^*(f,\mathbf{n})h_B(f',\mathbf{n}')|\xi_g(\mathbf{n},z)=\rho_g(z,\mathbf{n})\rangle\rangle_{\xi_g}$$

Total ensemble average

Conditional average (over GW realisations)

Average over all density distributions

Grimm, Pijnenburg, Cusin, Bonvin 2024

Final results

There is **no effect of clustering on the mean** (HD curve)
But there is a **contribution to the variance**

$$\text{clustering variance} = \int \mathrm{d}z \int \mathrm{d}z'\, \mathcal{A}(z,z') \int \mathrm{d}\mathbf{n} \int \mathrm{d}\mathbf{n}'\, \zeta_g(\mathbf{n}\cdot\mathbf{n}', z, z')\, \chi_{ab}(\mathbf{n})\chi_{ab}(\mathbf{n}')$$

Astro function

Galaxy correlation function

Geometrical functions

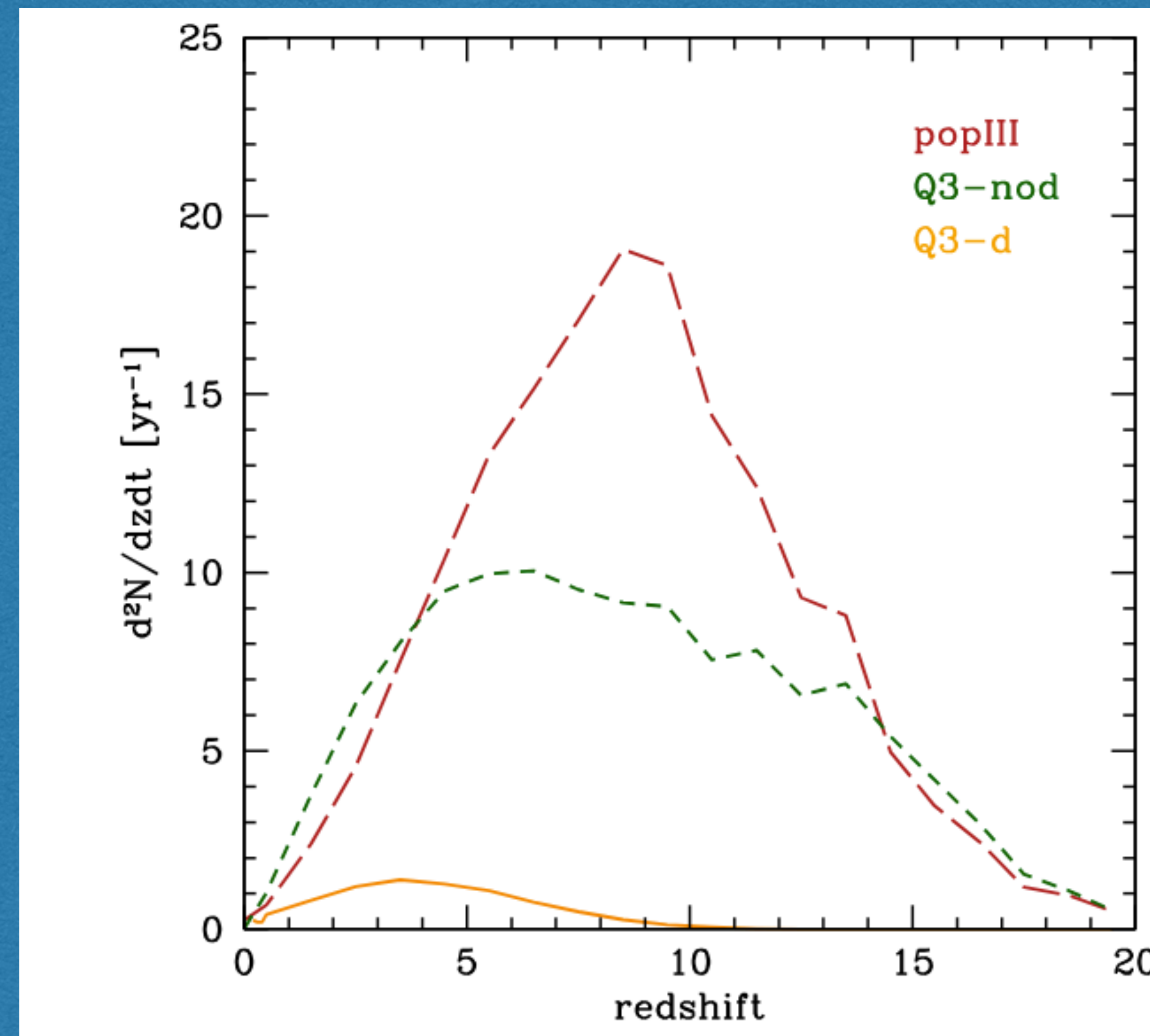
Black hole formation and evolution model
Mass distribution
Redshift distribution
Merger rate
….

Depend on pulsar positions (pusar separation ab)

Grimm, Pijnenburg, Cusin, Bonvin 2024

Astrophysical model

(catalogues **Klein et al. 2016**)



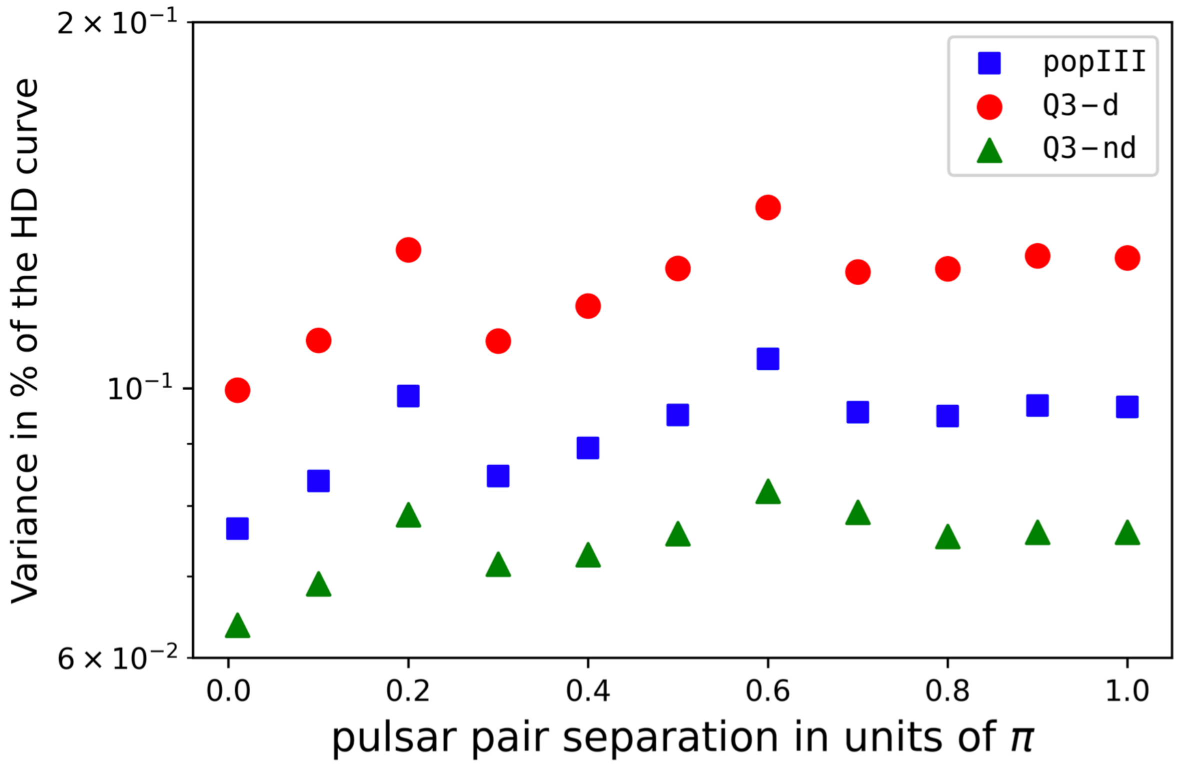
Different astrophysical scenarios *(catalogues Klein et al. 2016)*

Heavy seeds following collapse of protogalactic discs at z 10-15 (both with/without delay)

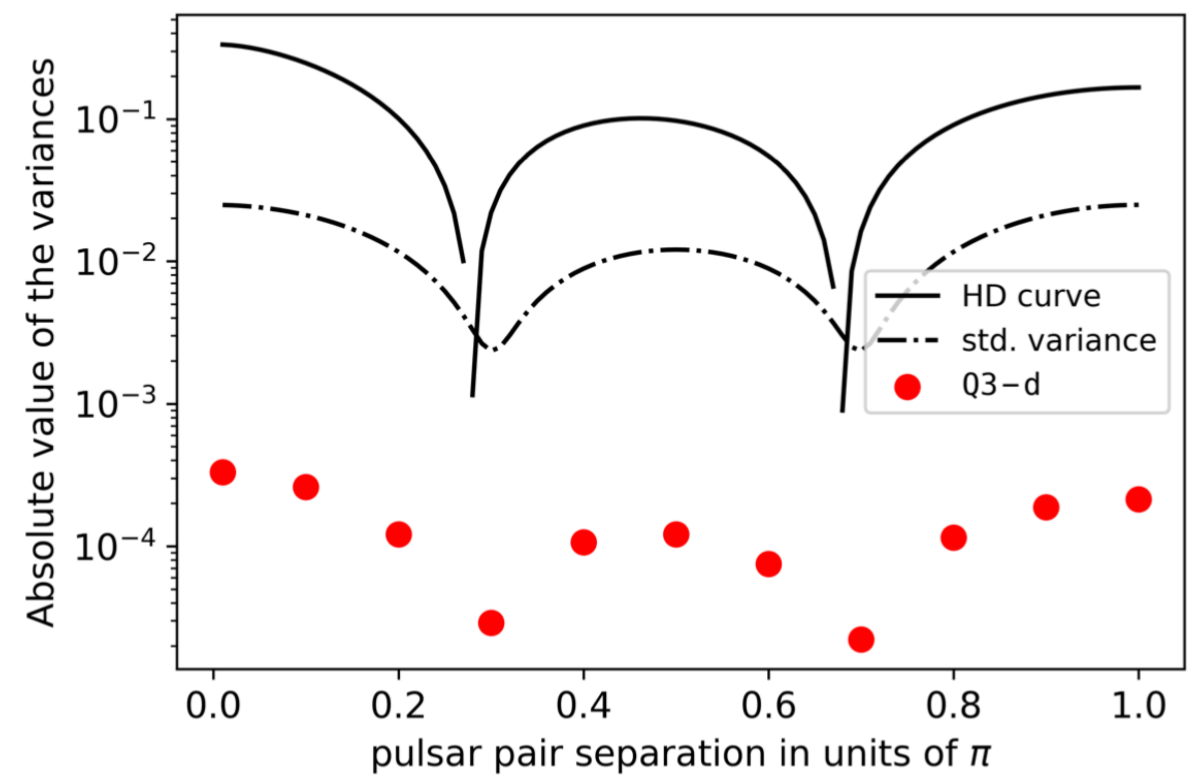
Light seeds following collapse of pop III stars at high redshift

Numerical results

Irreducible contribution (averaging over infinite number of pulsars per angular separation)



Grimm, Pijnenburg, Cusin, Bonvin 2024



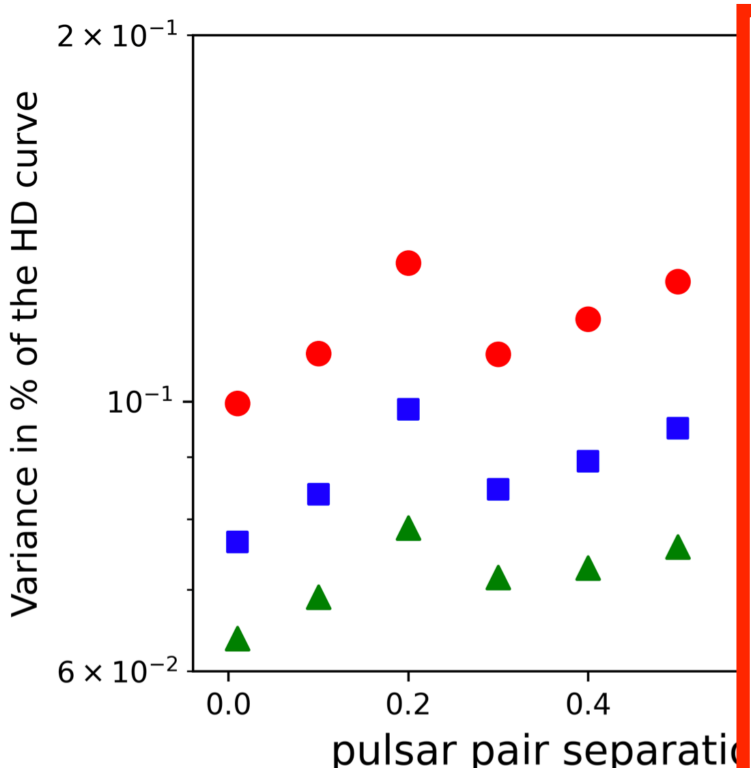
Clustering does not significantly affect HD reconstruction

(cosmic variance due to GW realisations is more relevant)

Shot noise due to poissonian distribution fo sources expected to be in between

Numerical results

Averaging over infinite number of pulsars



Clustering does not sig
(cosmic variance due to
Shot noise due to poiss

Clustering does not significantly affect the HD curve: it gives only contribution to the variance, which is very small.

Instead of treating it as source of contamination, it is more interesting to treat it as **source of signal!**

Angular power spectrum due to clustering

Contribution of clustering of sources on the background energy density

$$\Omega_{\rm GW}(f,\mathbf{n}) = \bar{\Omega}_{\rm GW}(f) + \delta\Omega_{\rm GW}(f,\mathbf{n})$$ *Background energy density*

$$\bar{\Omega}_{\rm GW}(f) = \frac{f}{4\pi\rho_c}\int^{\eta_0} \mathrm{d}\eta\, \mathcal{A}(f,\eta)$$ *Isotropic part*

Astro kernel

$$\delta\Omega_{\rm GW}(f,\mathbf{n}) = \frac{f}{4\pi\rho_c}\int \mathrm{d}\eta\, \mathcal{A}(\eta,f)\left[\delta_G + 4\Psi - 2\mathbf{n}\cdot\nabla v + 6\int^{\eta_0}\mathrm{d}\eta'\dot{\Psi}\right]$$

Anisotropic part

Angular power spectrum due to clustering

Contribution of clustering of sources on the background energy density

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Anisotropic part

Angular power spectrum: two contributions

$$C_\ell^{\rm tot} = C_\ell + N_{\rm shot}$$

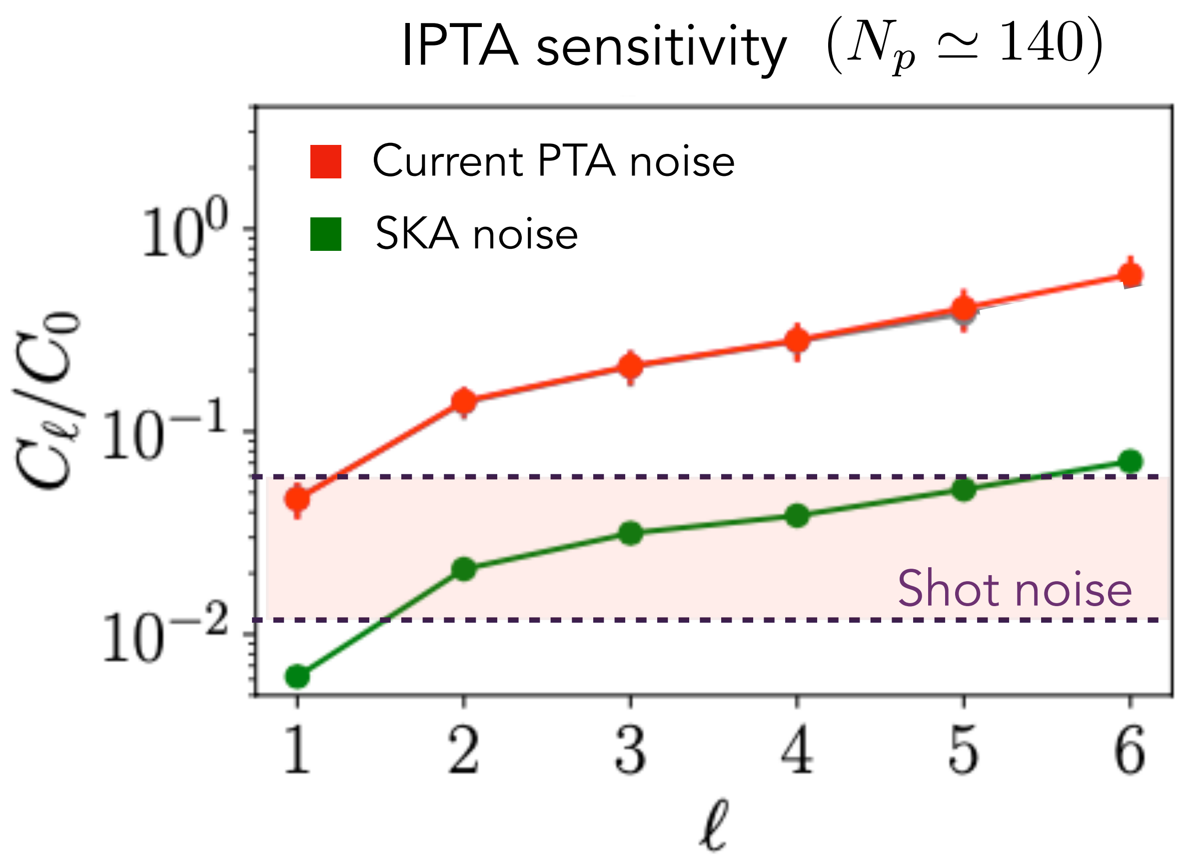
C_ℓ: $\propto \mathcal{A}^2\delta_G^2$ — due to clustering

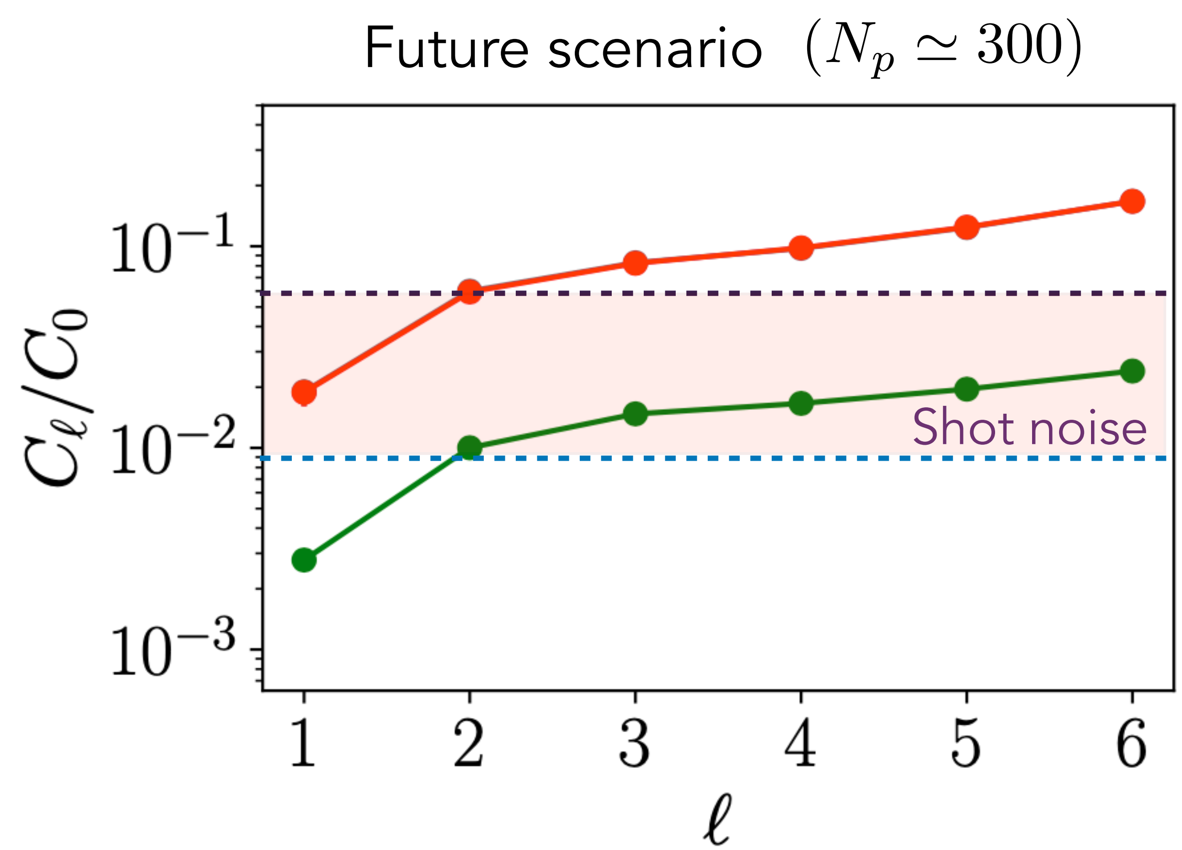
$N_{\rm shot}$: $\propto 1/n_G$ — due to poissonian nature of sources

Predictions (preliminary)

Expected results for angular power spectrum

Noise from Depta et al. 2024





Clustering: to be computed (it will decay as $1/\ell$), expected to be independent of frequency
Shot noise contribution increases as we increase frequency (sources spend less time in band)
Cosmic variance: we expect it to have opposite trend (to get reduced as we increase frequency)

Cusin, Pitrou et al. in prep

Conclusions

PTA work in regime in which cosmic variance due to GW realisations is relevant

Effect of clustering on HD curve can be evaluated with a **2-step average process**: average over GW realisation at fixed matter (galaxy) realisation, and average over matter realisation

Clustering does not affect HD curve, however it gives small contribution to the variance (a few percent of the mean)

Instead of treating clustering as form of contamination it is interesting to treat it as signal: **angular power spectrum** of background energy density: effect of clustering+ shot noise+cosmic variance

Hierarchy between different contributions is frequency dependent: this signal might be **within the reach of future observations** (—>new astrophysical probe!)

Thank you for the attention