Memory effects from symmetries

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Based on
arXiv: 2406.07106, JCAP
In collaboration with Jean-Philippe Uzan

Context and Motivations

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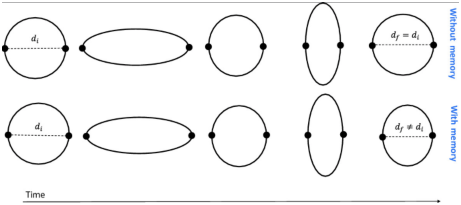

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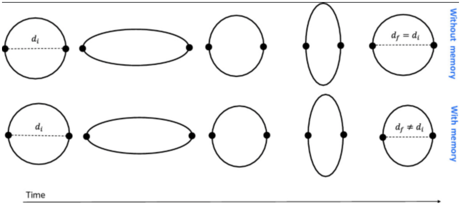

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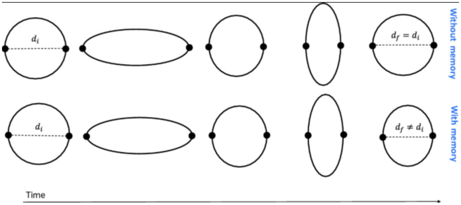

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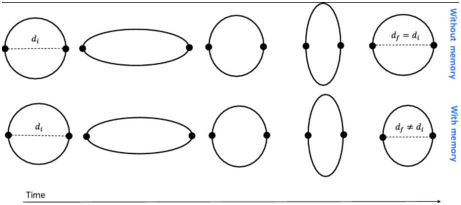

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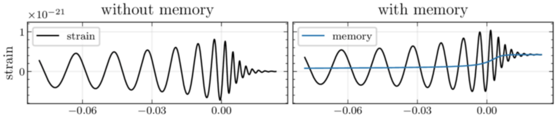

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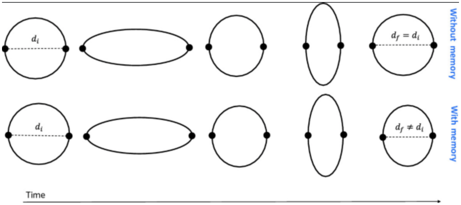

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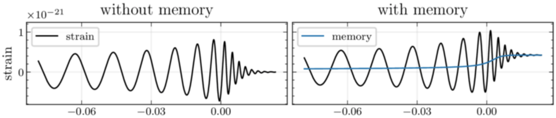

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History of memory effects in short:

- Linearized gravity: displacement memory [Zel'dovich and Polnarev '74]
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Relation between memory and the symmetries of spacetime

- Associated to the degeneracy of the vacuum in any gauge theory
- Gravitational memory effects $\leftrightarrow$ flux-balance laws $\leftrightarrow$ symmetries of open systems

$$\delta Q = \mathcal{F} \tag{1}$$

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Memory effects tells us about the explicit and hidden symmetries of spacetime and vice-versa

- New memories recently identified: Spin memory, centered-of-masse memory, gyroscope [Pasterski, Strominger, Zhiboedov '16] [Nichols '18][Seraj, Oblak '23]
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How can we relate memory effects to explicit or hidden symmetries of spacetimes?

Symmetries of the geodesic deviation equation

Geodesic deviation

- Consider two nearby curves $\bar{X}^\mu(\tau) := X^\mu(\tau, \sigma = 0)$ and $X^\mu(\tau, \sigma)$
- Their relative distance expands as folllows

$$\Delta X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma) - \bar{X}^\mu(\tau,0) = \sigma N^\mu(\tau) \tag{2}$$

$$+ \sigma^2 \left(B^\mu - \bar{\Gamma}^\mu{}_{\alpha\beta} N^\alpha N^\beta \right)(\tau) + \mathcal{O}(\sigma^3) \tag{3}$$

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$$\boxed{\frac{D^2 N^\mu}{\mathrm{d}\tau^2} = \bar{R}^\mu{}_{\alpha\beta\gamma} \bar{u}^\alpha \bar{u}^\beta N^\gamma} \tag{4}$$

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- Lagrangian formulation

$$L[N] = \frac{1}{2} \bar{g}_{\mu\nu} \bar{u}^\alpha \bar{u}^\beta \nabla_\alpha N^\mu \nabla_\beta N^\nu - \frac{1}{2} \bar{R}_{\mu\nu\alpha\beta} \bar{u}^\mu \bar{u}^\alpha N^\nu N^\beta \tag{5}$$

- Admit a hidden symmetry given by

$$\delta N^\mu = \bar{K}^\mu{}_{\alpha_1 \dots \alpha_p} \bar{u}^{\alpha_1} \dots \bar{u}^{\alpha_p} \qquad \rightarrow \qquad \delta L = \nabla_\alpha \left(N_\mu \bar{u}^\alpha \bar{u}^\beta \nabla_\beta \delta N^\mu \right) \tag{6}$$

where $\bar{K}^\mu{}_{\alpha_1 \dots \alpha_p}$ is a rank-p (conformal) Killing tensor (similar to higher spin generators for the Laplacian) : $\nabla_{(\mu} K_{\nu\alpha_1 \dots \alpha_p)} = \xi_{(\mu} h_{\nu\alpha_1 \dots \alpha_p)}$ [Caviglia, Zordan, and Salmistraro '82][**BA** '24]

Integrability of the geodesic deviation equation (GDE)

- Consider the GDE associated to a timelike or null reference geodesic

$$\frac{D^2 N^\mu}{\mathrm{d}\tau^2} = \bar{R}^\mu{}_{\alpha\beta\gamma}\bar{u}^\alpha \bar{u}^\beta N^\gamma \qquad \text{with} \qquad \bar{u}^\alpha \bar{u}_\alpha = \epsilon \qquad \bar{u}^\alpha \nabla_\alpha \bar{u}^\mu = 0 \tag{7}$$

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$$\nabla_{(\mu} K_{\alpha\beta)} = \xi_{(\mu} g_{\alpha\beta)} \qquad \text{with} \qquad \nabla_{(\mu}\xi_{\nu)} = \Omega g_{\mu\nu} \tag{8}$$

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- For radiative spacetime, relate memory effects to explicit and hidden symmetries of spacetime [Caviglia, Zordan, and Salmistraro '82][**BA** '24]

Application to the simplest non-linear gravitational wave model

Application to pp-waves

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Useful toy model for studying memory effects outside the asymptotically flat framework

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with

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- In the following, we focus on i) vacuum configurations and ii) polarized waves such that

$$T_{uu} = 0 \qquad A_{12} = 0 \tag{14}$$

What are the explicit and hidden symmetries of this vacuum gravitational plane wave ?

Symmetries of vacuum gravitational plane wave

Conformal isometries of pp-waves

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where we have introduced

$$H^{ij}(u_0, u) \equiv \int_{u_0}^{u} A^{ij}(w)\mathrm{d}w$$

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  where we have introduced

$$H^{ij}(u_0, u) \equiv \int_{u_0}^{u} A^{ij}(w)\mathrm{d}w$$

- If we look for a CKV for any A_{ij}, only one solution: HKV for $\Omega = 2$

$$Z^\alpha \partial_\alpha = 2v\partial_v + x^i \partial_i \, . \qquad (15)$$

Conformal isometries of pp-waves

- Isometries of pp-waves have been studied long time ago: 5d isometry group [Souriau '73] [Sippel '86][Maartens '91][Horvathy '17]
- Three translations

$$\mathcal{N}^\alpha \partial_\alpha = \partial_v \,, \qquad P_+^\alpha \partial_\alpha = \partial_x \qquad P_-^\alpha \partial_\alpha = \partial_y$$

- Two carrollian boosts

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- Admit a rank-2 Killing tensor : Koutras theorem

$$\boxed{K_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = 2v\mathrm{d}u^2 - u(2\mathrm{d}u\mathrm{d}v + A_{ij}\mathrm{d}x^i\mathrm{d}x^j) + A_{ij}x^j\mathrm{d}u\mathrm{d}x^i \,.} \qquad (16)$$

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- Symmetries holding fro any wave profile A_{ij}

Integrating the geodesic motion from the symmetries

Algebraic integration of the geodesic flow

- Phase space for geodesic motion:

$$\{v,p_v\}=\{u,p_u\}=1\,,\qquad \{x^i,p_j\}=\delta^i{}_j\,.\qquad H=p_up_v+\frac{1}{2}A^{ij}p_ip_j=\frac{\epsilon}{2}\,.\quad \epsilon=\{0,-1\} \tag{17}$$

- Conserved charges

$$\xi^\alpha\partial_\alpha\to\mathcal{O}=\xi^\alpha p_\alpha \qquad K_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\to K=K^{\mu\nu}p_\mu p_\nu \tag{18}$$

- Translations: Since the Hamiltonian does not depend on neither v nor x^i, p_v and p_i are automatically conserved. We denote them as

$$\mathcal{N}=p_v\,,\qquad P_+=p_x\,,\qquad P_-=p_y\,, \tag{19}$$

- Carrolian boost: The conserved charges generating the boosts are given by

$$\mathcal{B}_+=H^{xx}(u)p_x+H^{xy}(u)p_y-p_vx(u)\,, \tag{20}$$

$$\mathcal{B}_-=H^{yy}(u)p_y+H^{yx}(u)p_x-p_vy(u)\,. \tag{21}$$

- Hidden Killing tensor charge: charge coming from the Killing tensor reads

$$\mathcal{K}=K^{\mu\nu}p_\mu p_\nu=p_v\mathcal{Z}-2uH \qquad \text{with} \qquad \mathcal{Z}=2p_vv+p_ix^i\,, \tag{22}$$

- Charge algebra

$$\{P_\pm,\mathcal{B}_\pm\}=\mathcal{N}\,. \tag{23}$$

$$\{P_\pm,\mathcal{K}\}=\mathcal{N}P_\pm\,,\qquad \{B_\pm,\mathcal{K}\}=\mathcal{N}B_\pm\,,\qquad \{\mathcal{N},\mathcal{K}\}=2\mathcal{N}^2\,, \tag{24}$$

- Geodesic motion integrable since $(\mathcal{N},\mathcal{B}_\pm,H)$ are in involution
- Fully algebraic integration: Killing tensor charge enters in the longitudinal motion

Constructing the Fermi coordinates

Adapted Fermi normal coordinates

- Pick up a null geodesic $\bar{\gamma}$ with tangent vector

$$\bar{u}^\mu \partial_\mu = \partial_u \qquad \bar{u}_\mu \mathrm{d}x^\mu = \mathrm{d}v\,. \tag{25}$$

- Introduce a set of adapted Fermi coordinates, X^I, with $I \in \{0, \ldots, 3\}$ related to the initial coordinates x^μ via

$$E^I{}_\mu \equiv \frac{\partial X^I}{\partial x^\mu}\,. \tag{26}$$

- Choose the "time-leg" such that the coordinate X^0 coincides with the affine parameter of the geodesic

$$\bar{E}^0{}_\mu \mathrm{d}x^\mu = \bar{u}_\mu \mathrm{d}x^\mu\,. \tag{27}$$

  Impose that i) the remaining legs be parallel transported along the null geodesic, and ii) the orthogonality relations

$$\bar{u}^\mu \nabla_\mu E^I{}_\nu = 0 \qquad g_{\mu\nu}|_{\bar{\gamma}} = \bar{E}^I{}_\mu \bar{E}^J{}_\nu \eta_{IJ} \tag{28}$$

- In our case, one obtains

$$\dot{E}^i{}_A = -\frac{1}{2} A^{ik} \dot{A}_{kj} E^j{}_A \tag{29}$$

- Analyzing the effects in the Fermi coordinates requires to analytically solve this PT equation
- Constructing the Fermi coordinates is closely related to the so called Penrose limit : exact for pp-wave

- Fermi coordinates : $X^0 = U$, $X^1 = V$ and $X^A = \{X, Y\}$

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- Then, the Taylor expansion of the BJR coordinates $x^\mu(U, V, X^A)$ up to second order reads

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- It follows that the Fermi and BJR coordinates are related by

$$u = U\,, \tag{31}$$

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- The vacuum GPW metric becomes in the Fermi coordinates

$$\boxed{\mathrm{d}s^2 = 2\mathrm{d}U\mathrm{d}V + \delta_{AB}\mathrm{d}X^A\mathrm{d}X^B + H_{AB}(U) X^A X^B \mathrm{d}U^2 \,,} \tag{34}$$

with the wave-profile

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- Known as the Brinkmann coordinates which are the one use to analyze the physical effects
- Classify the memory effects via the symmetries

Classification of memory effects

Classification of memory effects

The three different types of memory effects

- Consider situations for which asymptotically, i.e. for $u < u_0$ and $u > u_f$, one has

$$\zeta^i \neq 0 \qquad \text{and} \qquad \ddot{\zeta}^i = 0\,. \tag{37}$$

- **Velocity Memory (VM)**:
  When the relative velocity in the two asymptotic regions satisfies

$$\boxed{\Delta\dot{\zeta} = \dot{\zeta}_f - \dot{\zeta}_0 \neq 0} \tag{38}$$

  $\rightarrow$ constant shift on the asymptotic value of the relative velocity.

- **Vanishing Velocity Memory (VM0)**: Subcase corresponding to

$$\boxed{\Delta\dot{\zeta} = \dot{\zeta}_f - \dot{\zeta}_0 = 0} \tag{39}$$

  $\rightarrow$ no velocity memory but still interesting effects on the couple of test particles.

- **Displacement Memory (DM)**: subcase such that

$$\boxed{\dot{\zeta}_f = 0 \qquad \text{and} \qquad \zeta_f \neq \zeta_0} \tag{40}$$

  $\rightarrow$ the relative velocity vanishes in the asymptotic future

Classification of memory effects

A brief look at the conclusions so far

- First work to analyze the memory effects in vacuum gravitational plane wave [Zhang, Duval, Gibbons, Horvathy '17 '18]
- Only exhibits velocity memory effects / displacement memory effect can never occur

The Memory Effect for Plane Gravitational Waves

P.-M. Zhang[1*], C. Duval[2†], G. W. Gibbons[3,4,5‡], P. A. Horvathy[1,4§],

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[2] *Aix Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France*

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[5]*LE STUDIUM, Loire Valley Institute for Advanced Studies, Tours and Orleans France*

(Dated: August 28, 2017)

Abstract

We give an account of the gravitational memory effect in the presence of the exact plane wave solution of Einstein's vacuum equations. This allows an elementary but exact description of the soft gravitons and how their presence may be detected by observing the motion of freely falling particles. The theorem of Bondi and Pirani on caustics (for which we present a new proof) implies that the asymptotic relative velocity is constant but not zero, in contradiction with the permanent displacement claimed by Zel'dovich and Polnarev. A non-vanishing asymptotic relative velocity might be used to detect gravitational waves through the "velocity memory effect", considered by Braginsky, Thorne, Grishchuk, and Polnarev.

Classification of memory effects

A brief look at the conclusions so far

- Very recently, two numerical examples where a displacement occurs have been presented in [Zhang, Horvathy '24]

Displacement within velocity effect

in

gravitational wave memory

P.-M. Zhang[1*], P. A. Horvathy[2,3†],

[1] *School of Physics and Astronomy,*

Sun Yat-sen University, Zhuhai, China

[2] *Laboratoire de Mathématiques et de Physique Théorique,*

Université de Tours, (France)

[3] *Erwin Schrödinger Institute,*

Vienna (Austria)

(Dated: May 29, 2024)

Abstract

Sandwich gravitational waves exhibit the *velocity memory effect* (VM) which however can become, for specific values of the wave parameters, pure *displacement* (DM) as suggested by Zel'dovich and Polnarev. Fixing such a "miraculous" value, the particle trajectory is an (approximate) standing wave characterized by a unique integer m, for which the particle does not absorb any energy from the passing wave. Our statements are illustrated by a simple Gaussian and by the Pöschl-Teller potential as profiles.

- No analytic conditions were provided leaving the question of the classification of the conditions to have a velocity versus a displacement memory open

Memory effects for pulse profiles

Memory effects for pulse profiles

- Focus on the polarized case: $\mathcal{A}_{12} = 0$ / Introduce $\mathcal{A}_i = \mathcal{A}_{ii}$
- Past and future asymptotic behavior of the pulse profile:

$$H_+(u) = \frac{\ddot{\mathcal{A}}_1}{\mathcal{A}_1} = -\frac{\ddot{\mathcal{A}}_2}{\mathcal{A}_2} = 0 \qquad \text{for} \qquad u < u_0 \qquad u > u_f \tag{41}$$

- To analyze the memory, we need the dynamics of the geodesic deviation vector

$$\dot{\zeta}^i(u) = \frac{\dot{\mathcal{A}}_i(u)}{\mathcal{A}_i(u)}\zeta^i(u) + \frac{p_i}{\mathcal{A}_i(u)}\,, \tag{42}$$

$$\ddot{\zeta}^i(u) = \frac{\ddot{\mathcal{A}}_i(u)}{\mathcal{A}_i(u)}\zeta^i(u). \tag{43}$$

- Asymptotic behavior of the relative acceleration

$$\ddot{\mathcal{A}}_i = 0 \qquad \rightarrow \qquad \ddot{\zeta}^i(u) = 0 \qquad \text{for} \qquad u < u_0 \qquad u > u_f \tag{44}$$

- Compute the asymptotic form of the relative distance and velocity $(\zeta_f, \dot{\zeta}_f, \zeta_0, \dot{\zeta}_0)$ in terms of the initial conditions $(p_i, \mathcal{B}_i)$ and the asymptotic wave form $(\mathcal{A}_0, \mathcal{A}_f)$.

Memory effects for pulse profiles

- **Asymptotic past:** for $u < u_0$, one has

$$\mathcal{A}(u) = \dot{\mathcal{A}}_0(u - u_0) + \mathcal{A}_0\,, \tag{45}$$

$$H(u,u_0) = \int_{u_0}^{u} \frac{1}{\mathcal{A}^2(u)} = \frac{u-u_0}{\mathcal{A}_0\mathcal{A}(u)} \qquad \text{such that} \qquad \mathcal{A}(u)H(u,u_0) = \frac{u-u_0}{\mathcal{A}_0}\,, \tag{46}$$

$$\zeta(u) = \dot{\zeta}_0(u-u_0) + \zeta_0 \tag{47}$$

- **Asymptotic future:** for $u > u_f$, one has

$$\mathcal{A}(u) = \dot{\mathcal{A}}_f(u - u_f) + \mathcal{A}_f\,, \tag{48}$$

$$H(u_0,u) = \int_{u_0}^{u_f} \frac{1}{\mathcal{A}^2(u)} + \int_{u_f}^{u} \frac{1}{\mathcal{A}^2(u)} = H_{0f} + \frac{u-u_f}{\mathcal{A}_f\mathcal{A}(u)}\,. \tag{49}$$

$$\zeta(u) = \dot{\zeta}_f(u-u_f) + \zeta_f\,. \tag{50}$$

- **Results:** relations between $(\zeta_f, \dot{\zeta}_f, \zeta_0, \dot{\zeta}_0)$ the different asymptotic quantities in terms of the initial conditions $(p_i, \mathcal{B}^i)$ and the asymptotic properties of the wave profile $(\mathcal{A}_0, \mathcal{A}_f, H_{0f})$

$$\boxed{\zeta_0 = -\mathcal{B}\mathcal{A}_0\,, \qquad \dot{\zeta}_0 = \frac{p}{\mathcal{A}_0} - \mathcal{B}\dot{\mathcal{A}}_0} \tag{51}$$

$$\boxed{\zeta_f = \mathcal{A}_f(H_{0f}\,p - \mathcal{B}) \qquad \dot{\zeta}_f = \frac{p}{\mathcal{A}_f} + \dot{\mathcal{A}}_f(H_{0f}\,p - \mathcal{B})}\,, \tag{52}$$

Memory effects for pulse profiles

Classification of memories for pulse profiles

- A **VM** occurs under the condition $\dot{\zeta}_f \neq \dot{\zeta}_0$.
  Since pulses profiles implies a constant asymptotic velocities

$$H_+(u) = \frac{\ddot{\mathcal{A}}_1}{\mathcal{A}_1} = -\frac{\ddot{\mathcal{A}}_2}{\mathcal{A}_2} = 0 \qquad \rightarrow \qquad \ddot{\zeta}^i(u) = 0 \qquad \text{for} \qquad u < u_0 \qquad u > u_f \tag{53}$$

  they generically lead to a constant VM for any $(\zeta_0, \dot{\zeta}_0)$ and $(p_i, \mathcal{B}^i, \mathcal{A}_0, \mathcal{A}_f, H_{0f})$ except for special cases.
- A **VM0** occurs in the special case in which $\dot{\zeta}_f = \dot{\zeta}_0$ which implies:

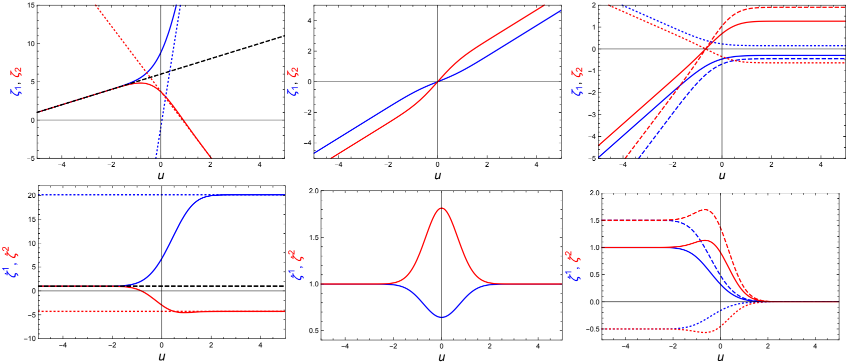
$$\boxed{(\dot{\mathcal{A}}_f - \dot{\mathcal{A}}_0)\mathcal{B} = -p\left[\frac{\mathcal{A}_f - \mathcal{A}_0}{\mathcal{A}_f\mathcal{A}_0} - \dot{\mathcal{A}}_f H_{0f}\right] .} \tag{54}$$

- A **DM** occurs when $\dot{\zeta}_f = 0$ implying additionally that

$$\boxed{\dot{\mathcal{A}}_f\mathcal{B} = -p\left(\frac{1}{\mathcal{A}_f} + \dot{\mathcal{A}}_f H_{0f}\right) .} \tag{55}$$

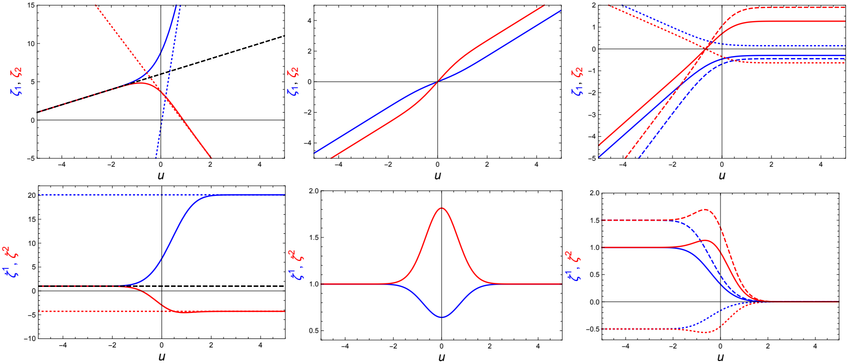
- Lead to a finer classification depending on $\dot{\mathcal{A}}_f = \dot{\mathcal{A}}_0$ or $\dot{\mathcal{A}}_f \neq \dot{\mathcal{A}}_0$.
- Let us see some examples.

Memory effects for pulse profiles



- Profiles of the relative displacement (ζ_1, ζ_2) (upper line) and relative velocity $(\dot{\zeta}_1, \dot{\zeta}_2)$ (lower line) for $H_+ = \mathrm{e}^{-u^2}$ with initial conditions that ensures $\dot{\mathcal{A}}_f - \dot{\mathcal{A}}_0 \neq 0$.

Memory effects for pulse profiles



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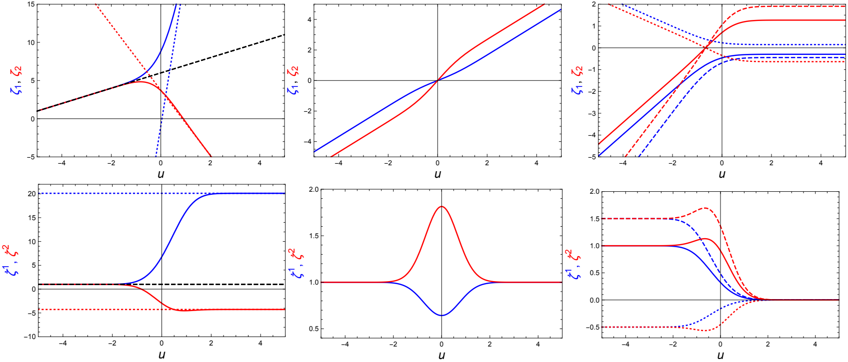
- Left panel: assume $p_i = (1, 1)$ and $\mathcal{B}_i = (-1, -1) \to$ clear non vanishing VM, so that $\dot{\zeta}_f \neq \dot{\zeta}_0$. Projected motion. Longitudinal position can be different so no colliding trajectories.

Memory effects for pulse profiles



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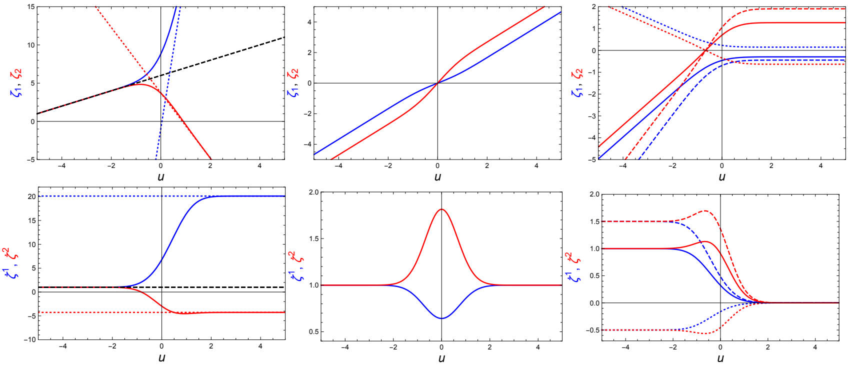
Memory effects for pulse profiles



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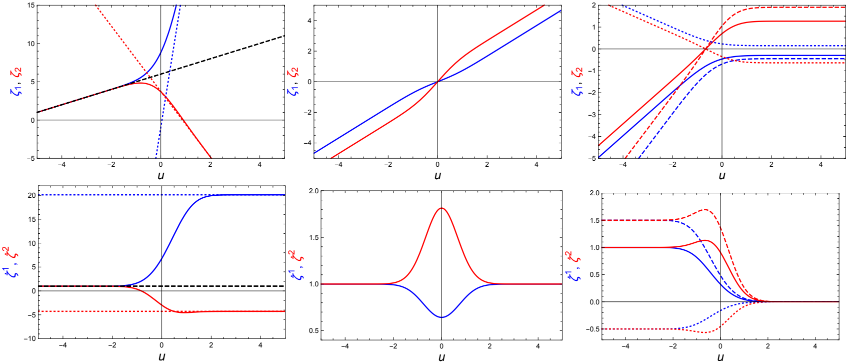
- Middle: $p_i \neq (0, 0)$ and B_i is tuned to get a VM0 $\rightarrow$ a vanishing VM for which $\dot{\zeta}_f = \dot{\zeta}_0$.
- Right: $p_i = 1$ (Solid), $p_i = 1.5$ (Dahed) and $p_i = -0.5$ (Dotted) and B_i is tuned to $\dot{\zeta}_0 = p/\mathcal{A}_0 - \mathcal{B}\dot{\mathcal{A}}_0 \rightarrow$ pure constant DM

Memory effects for pulse profiles



- Profiles of the relative displacement (ζ_1, ζ_2) (upper line) and relative velocity $(\dot{\zeta}_1, \dot{\zeta}_2)$ (lower line) for $H_+ = \mathrm{e}^{-u^2}$ with initial conditions that ensures $\dot{\mathcal{A}}_f - \dot{\mathcal{A}}_0 \neq 0$.
- Left panel: assume $p_i = (1,1)$ and $\mathcal{B}_i = (-1,-1) \rightarrow$ clear non vanishing VM, so that $\dot{\zeta}_f \neq \dot{\zeta}_0$. Projected motion. Longitudinal position can be different so no colliding trajectories.
- Middle: $p_i \neq (0,0)$ and B_i is tuned to get a VM0 $\rightarrow$ a vanishing VM for which $\dot{\zeta}_f = \dot{\zeta}_0$.
- Right: $p_i = 1$ (Solid), $p_i = 1.5$ (Dahed) and $p_i = -0.5$ (Dotted) and B_i is tuned to $\dot{\zeta}_0 = p/\mathcal{A}_0 - \mathcal{B}\dot{\mathcal{A}}_0 \rightarrow$ pure constant DM

New type of memory (middle column) simply switches the projected position of the two particles !

Conclusion and perspectives

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- Study the memories of extended quadrupolar bodies described by Dixon's theory

$$\frac{Dp^{\mu}}{\mathrm{d}\tau} = -\frac{1}{2} R^{\mu}{}_{\nu\alpha\beta} v^{\nu} S^{\alpha\beta} - \frac{1}{6} J^{\alpha\beta\gamma\delta} \nabla^{\mu} R_{\alpha\beta\gamma\delta} \tag{56}$$

$$\frac{DS^{[\mu\nu]}}{\mathrm{d}\tau} = 2p^{[\mu} v^{\nu]} + \frac{4}{3} R^{[\mu}{}_{\alpha\beta\gamma} J^{\nu]\alpha\beta\gamma} \tag{57}$$

$\rightarrow$ quasi-conserved charges and Killing-Yano symmetries [Compere, Druart '23]: are there new memories to identify ?

Thank you

Hidden symmetry of pp-waves

- Koutras theorem:
  if a spacetime admits both a gradient Killing vector and a HKV then it also admits a non-trivial rank-2 Killing tensor (KT) which generates an additional symmetry.
  With the gradient KV and the HKV given by

$$\xi_\alpha \mathrm{d}x^\alpha = (\partial_\alpha \Phi)\mathrm{d}x^\alpha \qquad Z_\alpha \mathrm{d}x^\alpha \tag{58}$$

  the KT is explicitely given by

$$K_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \left[Z_{(\mu}\xi_{\nu)} - \Phi g_{\mu\nu}\right]\mathrm{d}x^\mu \mathrm{d}x^\nu . \tag{59}$$

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Killing tensor for pp-waves

- For the pp-wave, we have a gradient KV and a HKV given by

$$\mathcal{N}_\alpha \mathrm{d}x^\alpha = \mathrm{d}u \qquad Z^\alpha \partial_\alpha = 2v\partial_v + x^i \partial_i \tag{60}$$

  hence a KT

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Penrose limit

- Concretely, pick up a null geodesic γ and construct null Fermi coordinates $X^A = (U, V, X^i)$ with $i \in (1, 2)$ adapted to the region around the geodesic

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- Also powerful to compute the memory effects explicitly

A new symmetry of vacuum gravitational plane wave

Symmetries of Einstein equation

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$$\left[\frac{\ddot{A}_{11}}{A_{11}} - \frac{1}{2}\left(\frac{\dot{A}_{11}}{A_{11}}\right)^2\right] + \left[\frac{\ddot{A}_{22}}{A_{22}} - \frac{1}{2}\left(\frac{\dot{A}_{22}}{A_{22}}\right)^2\right] = 0\,.$$

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- Invariant under Möbius reparametrization : new solution generating map [JBA, Uzan '24]

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Conformal symmetry of the Raychaudhuri equation

- Non-linear field equation

$$\left[\frac{\ddot{A}_{11}}{A_{11}} - \frac{1}{2}\left(\frac{\dot{A}_{11}}{A_{11}}\right)^2\right] + \left[\frac{\ddot{A}_{22}}{A_{22}} - \frac{1}{2}\left(\frac{\dot{A}_{22}}{A_{22}}\right)^2\right] = 0\,. \quad (67)$$

- Introduce the new fields

$$A_{11} = \frac{1}{\dot{F}} \qquad A_{22} = \frac{1}{\dot{G}}\,. \quad (68)$$

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- Large freedom in the choice of wave-profile solving the Einstein equation
- Symmetry reduced version of a more general structure found later for any null hypersurfaces [Ciambelli, Leigh, Freidel '24]

Algebraic integration of the geodesic flow

- Trivial equation for $\dot{u}$:

$$u = p_v \tau = \mathcal{N}\tau. \tag{74}$$

- Transverse motion: Carrolian boosts allows to write the x and y trajectories as

$$\boxed{x^i(u) = \frac{1}{\mathcal{N}}\left[H^{ij}(u_0, u)p_j - \mathcal{B}^i\right]} \tag{75}$$

- Longitudinal motion: Combining the HKV and the KT charge give v-trajectory

$$v(u) = \frac{1}{2\mathcal{N}^2}\left[\epsilon u - H^{ij}(u_0, u)p_i p_j + p_i \mathcal{B}^i + \mathcal{K}\right]. \tag{76}$$

- Relations between initial conditions and conserved charges

$$v_0 \equiv v(u_0) = \frac{1}{2\mathcal{N}^2}\left[\epsilon u_0 - \mathcal{N} p_i x_0^i + \mathcal{K}\right]. \qquad x_0^i \equiv x^i(u_0) = -\frac{\mathcal{B}^i}{\mathcal{N}}. \tag{77}$$

- 4-velocity

$$u^u = \frac{\mathrm{d}u}{\mathrm{d}\tau} = \mathcal{N} \qquad u^i = \frac{\mathrm{d}x^i}{\mathrm{d}\tau} = A^{ij}p_j \qquad u^v = \frac{\mathrm{d}v}{\mathrm{d}\tau} = \frac{1}{2\mathcal{N}}\left(\epsilon - A^{ij}p_i p_j\right). \tag{78}$$

- We can compute the invariant quantities: expansion, shear and rotation

$$\Theta = \nabla_\mu u^\mu = \mathcal{N}\varrho \qquad \sigma = \sigma_{\mu\nu}\sigma^{\mu\nu} = -\mathcal{N}^2\left[\dot{A}^{ij}\dot{A}_{ij} + \frac{2}{3}\varrho(\varrho + \dot{A}_{ij}p^i p^j) - \frac{1}{9}\varrho^2(p_i p^i)^2\right] \tag{79}$$

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- Classify the solutions and the memory effects using the symmetries of spacetime
- Identify all the explicit and hidden symmetries of a vacuum gravitational plane wave