2PN Solution to Binary Black Hole Dynamics

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with Tom Colin & Laura Bernard

Théorie, Univers et Gravitation Workshop (Annecy, 2024)

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The system

Inspiraling binary black holes (BBHs) that finally merge

Image credit: www.eoportal.org

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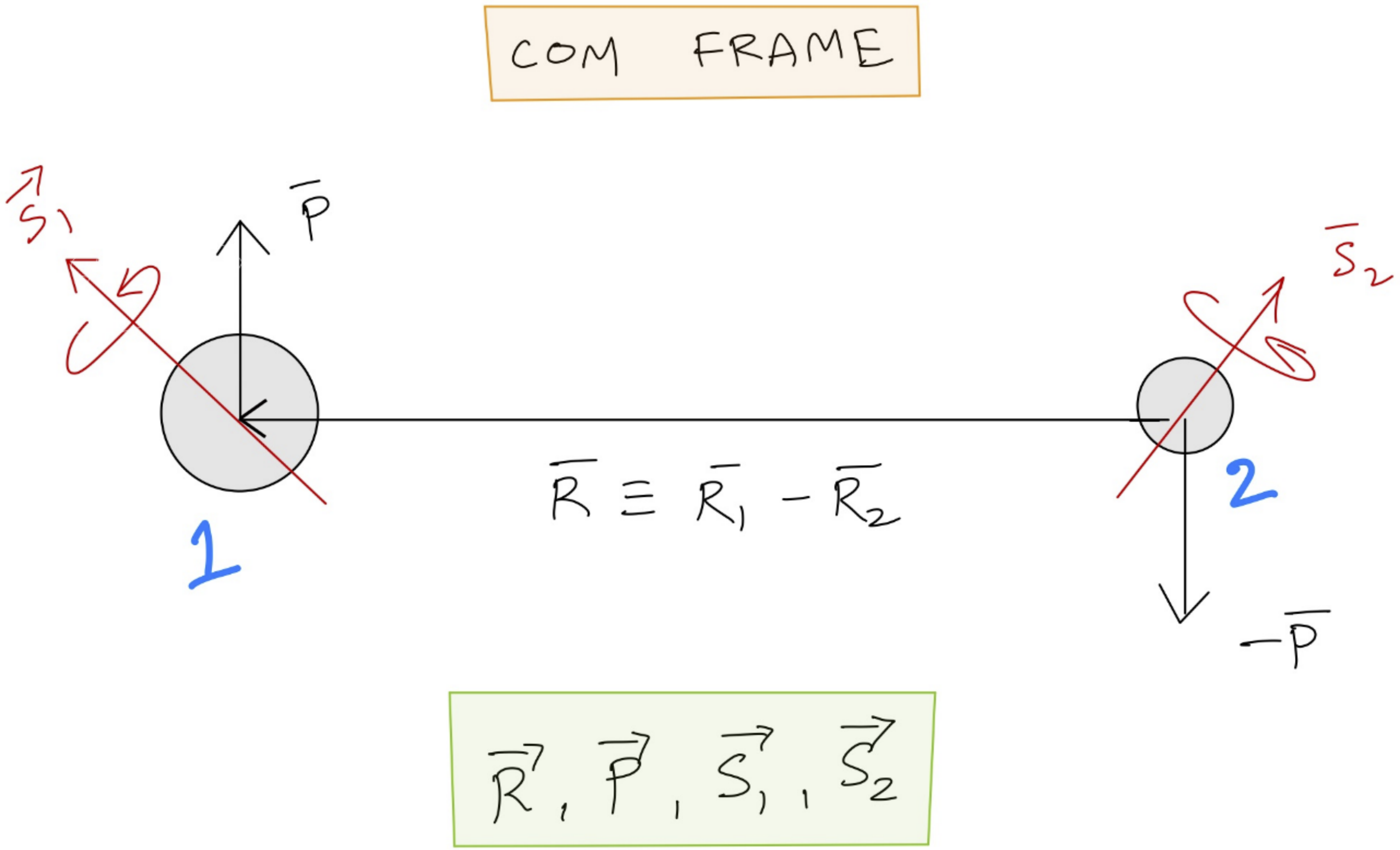
- Arbitrary masses, spins and eccentricity.

Phase space



COM FRAME
$\vec{S}_1$
$\bar{P}$
$\bar{S}_2$
$\bar{R} \equiv \bar{R}_1 - \bar{R}_2$
1
2
$-\bar{P}$
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- Why analytically?
  - Deepens understanding by exposing mathematical structure.
  - Numerical integrations: slow and error-prone. [1703.03967: Chatziioannou, Klein, Yunes]

Hamiltonian

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$$+ H_{1.5PN}(\overrightarrow{R}, \overrightarrow{P}, \vec{S}_1, \vec{S}_2) + H_{2PN}(\overrightarrow{R}, \overrightarrow{P}, \vec{S}_1, \vec{S}_2)$$

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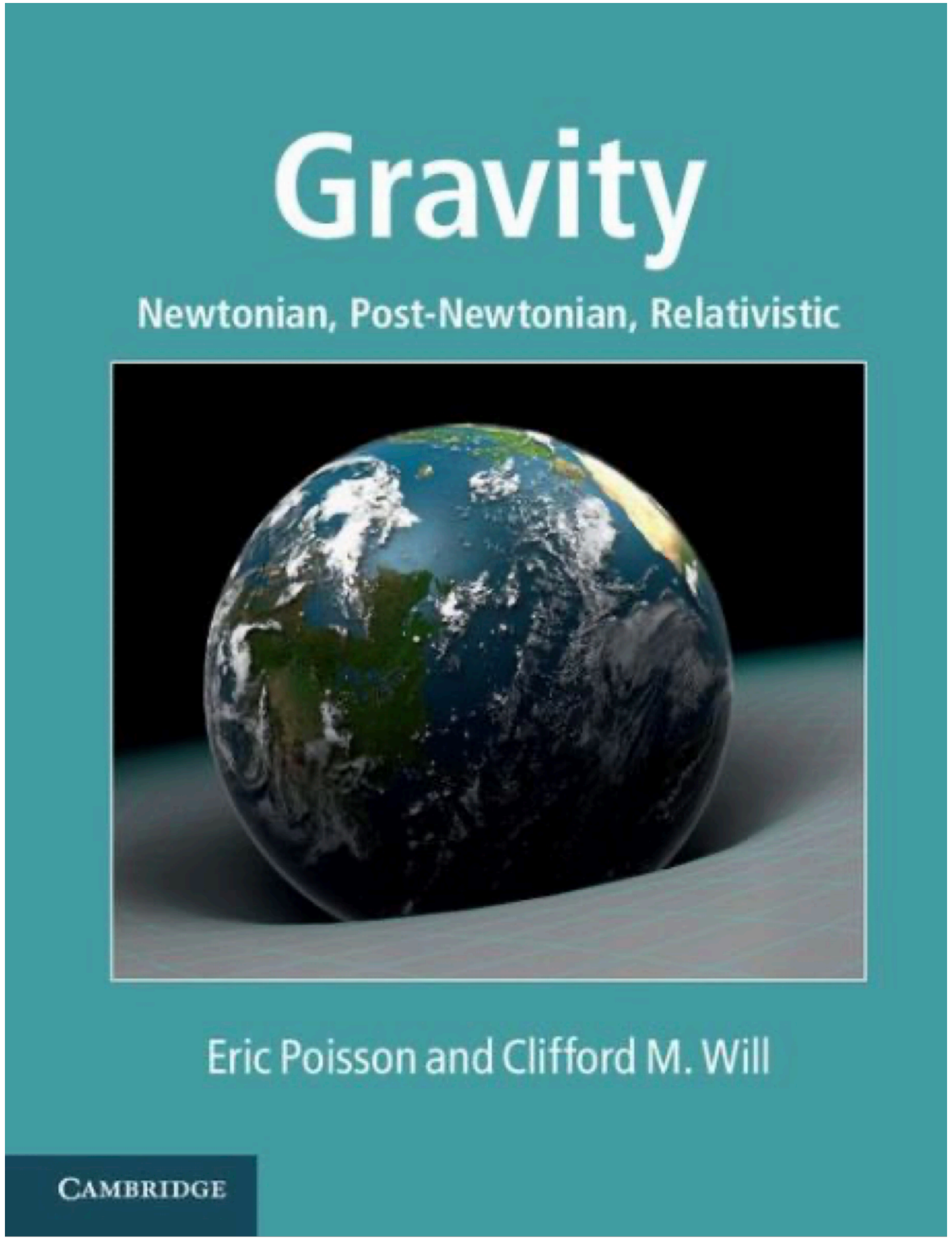
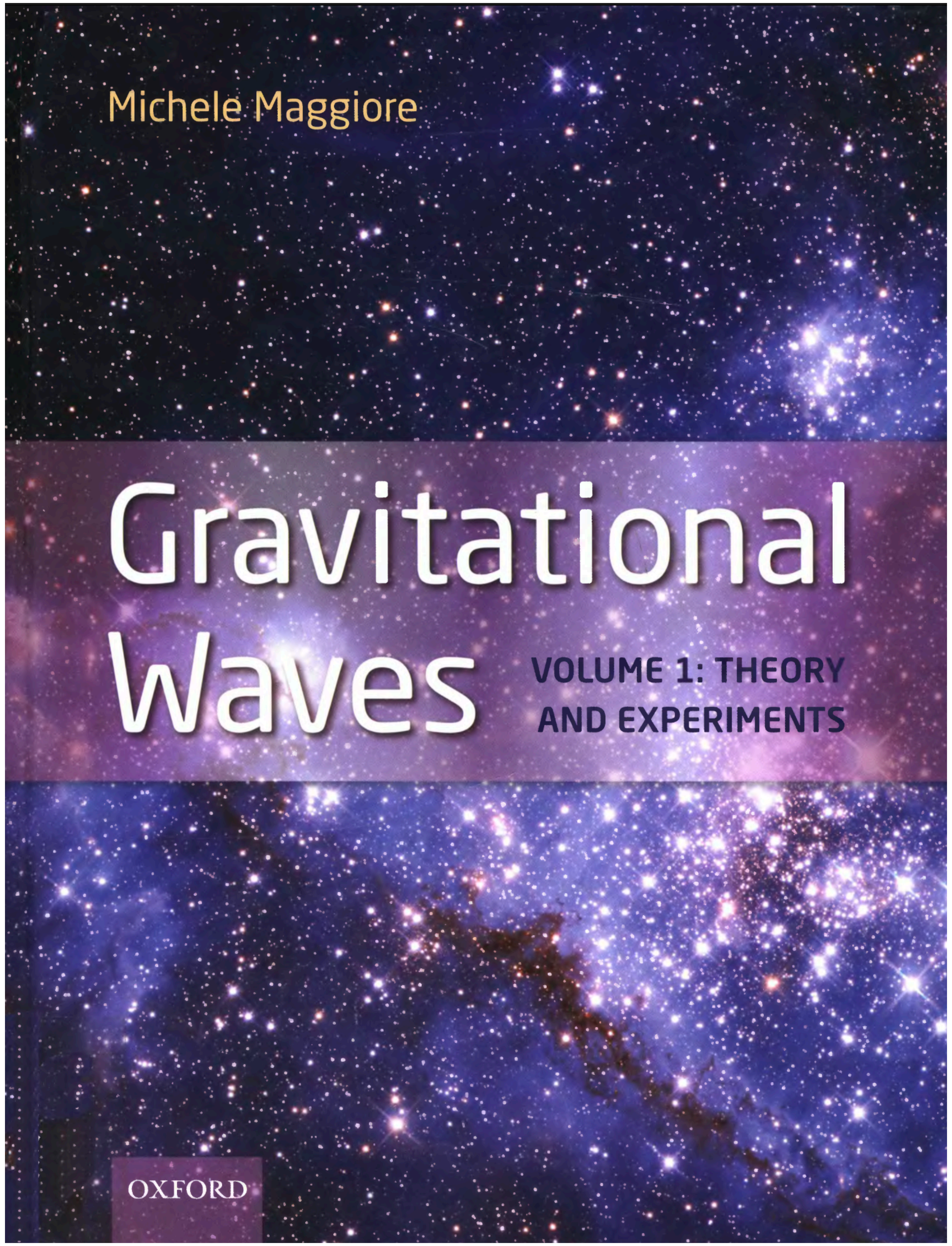

Gravity
Newtonian, Post-Newtonian, Relativistic
Eric Poisson and Clifford M. Will
CAMBRIDGE


Michele Maggiore
Gravitational Waves
VOLUME 1: THEORY AND EXPERIMENTS
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- **2019:** 1.5PN solution. [Cho, Lee: 1908.02927]
- **2021:** 1.5PN solution (using action-angle variables). [S.T., Cho, Stein: 2110.15351]

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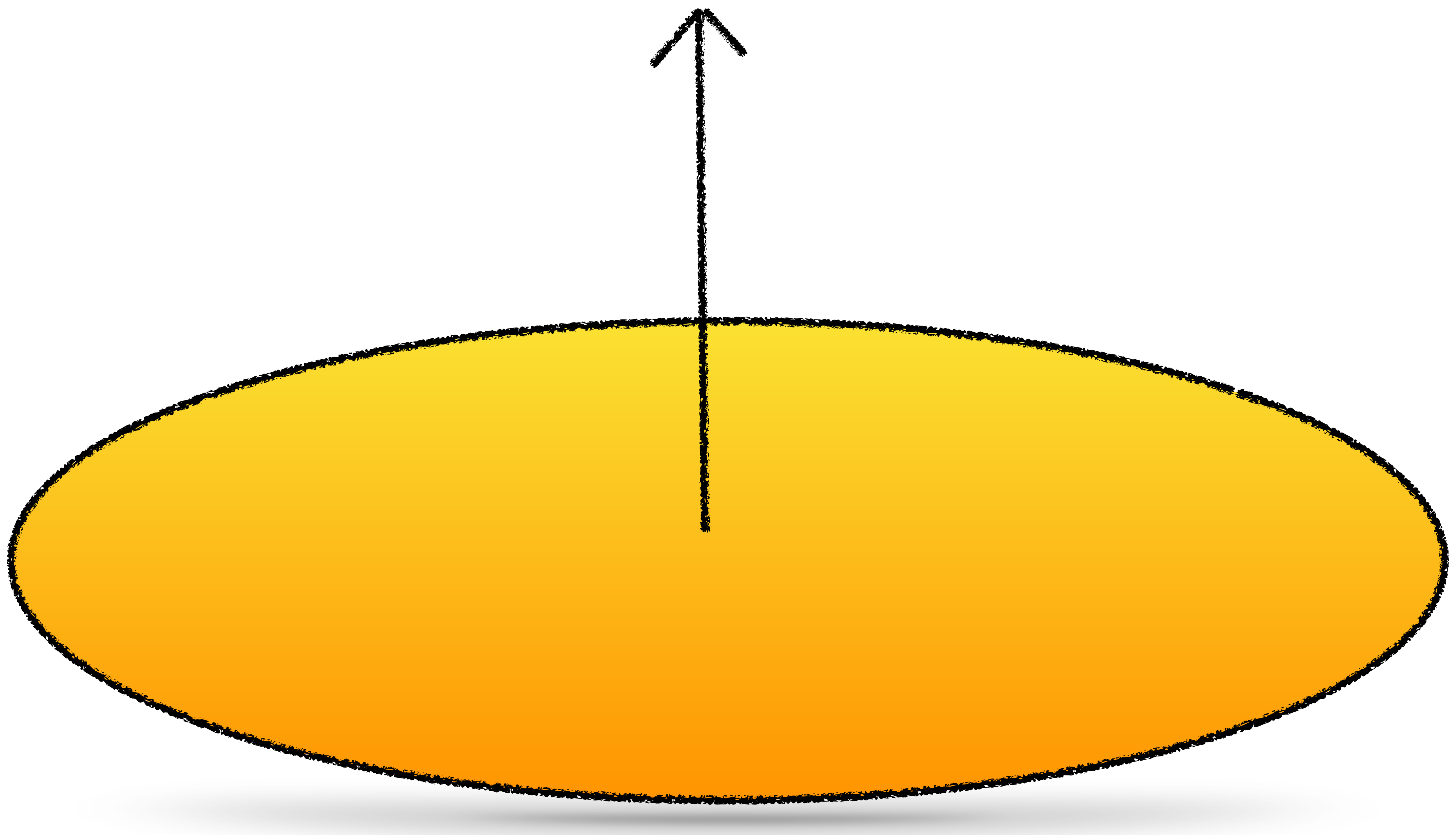

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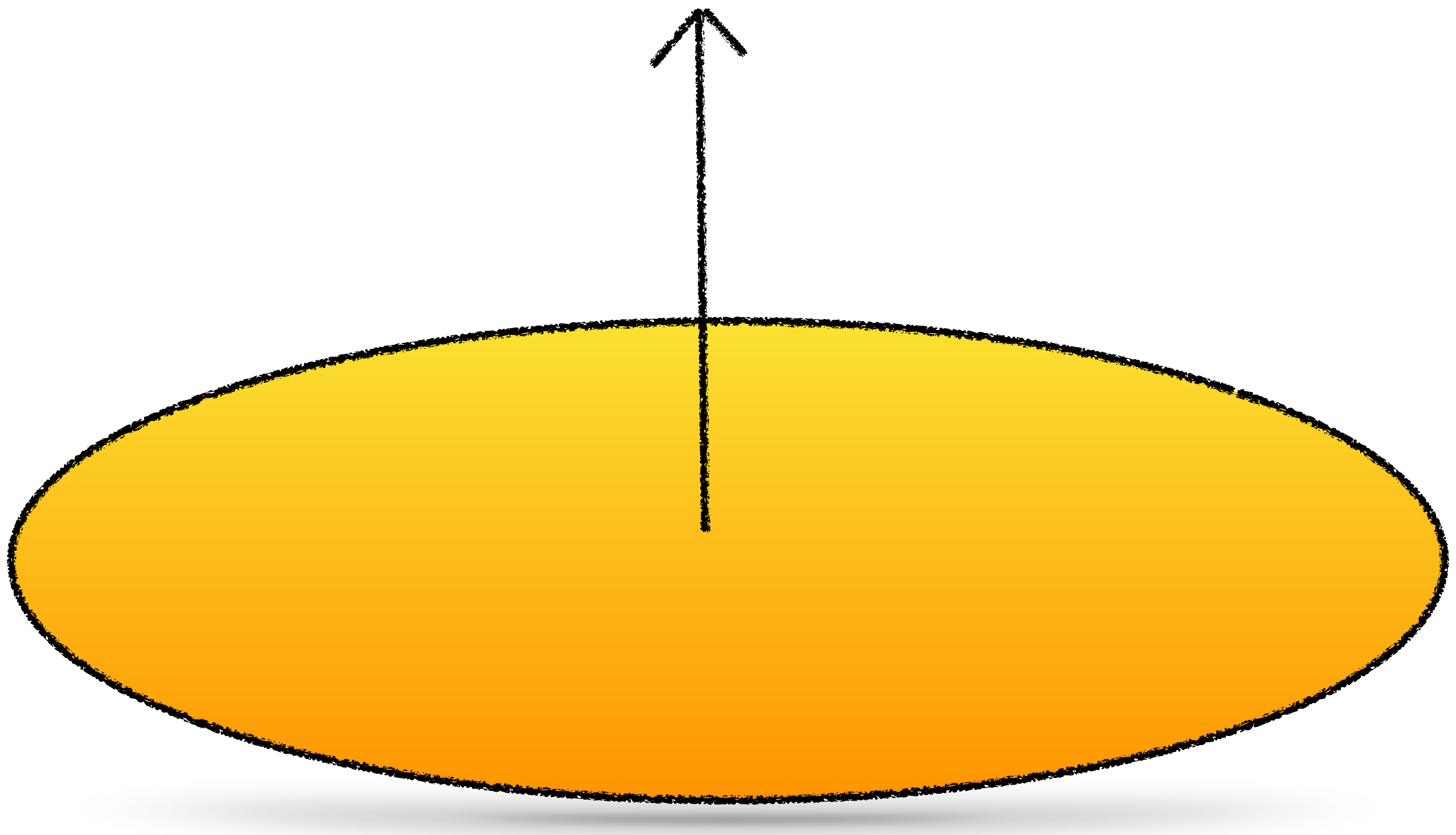

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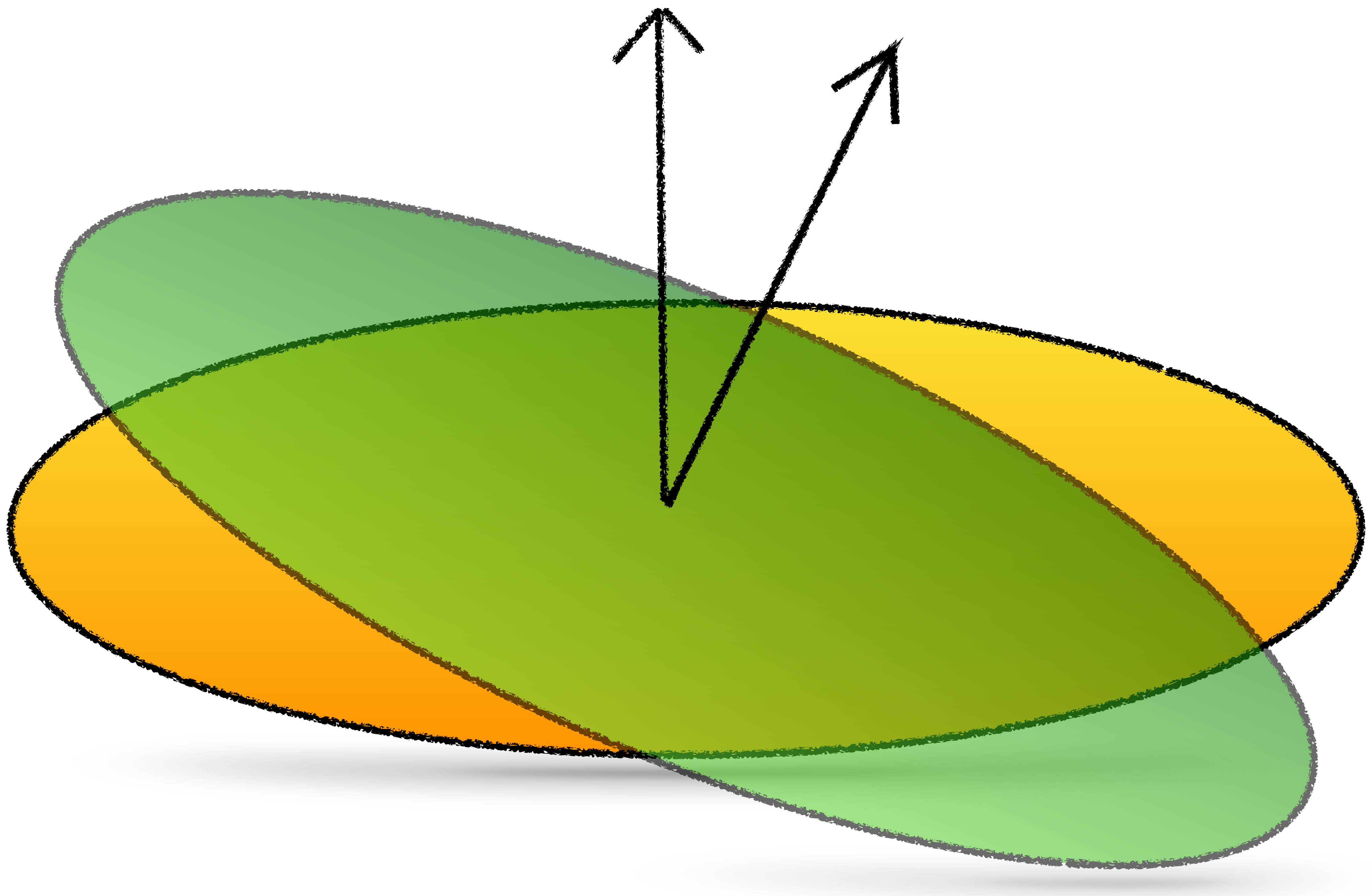

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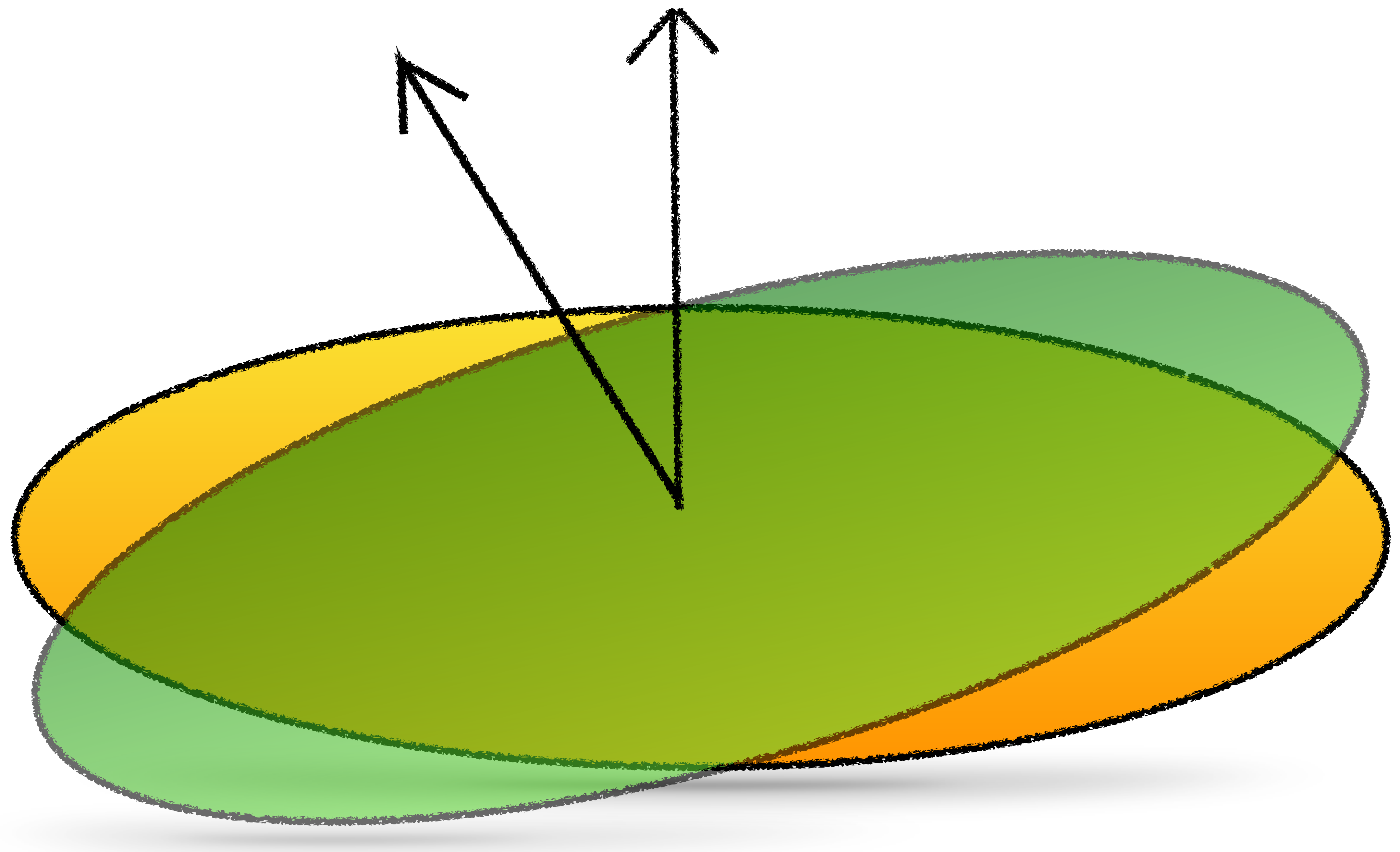

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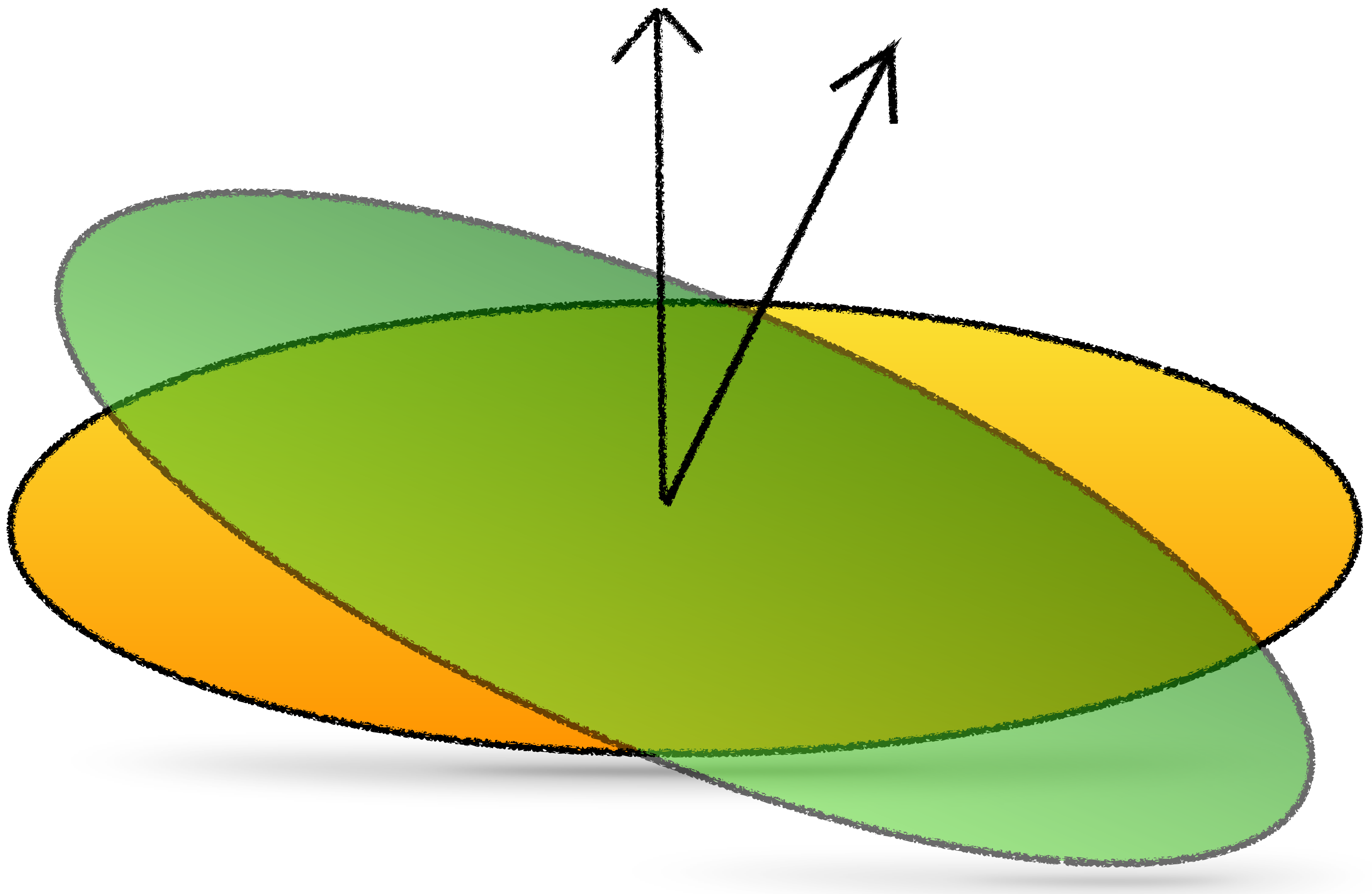

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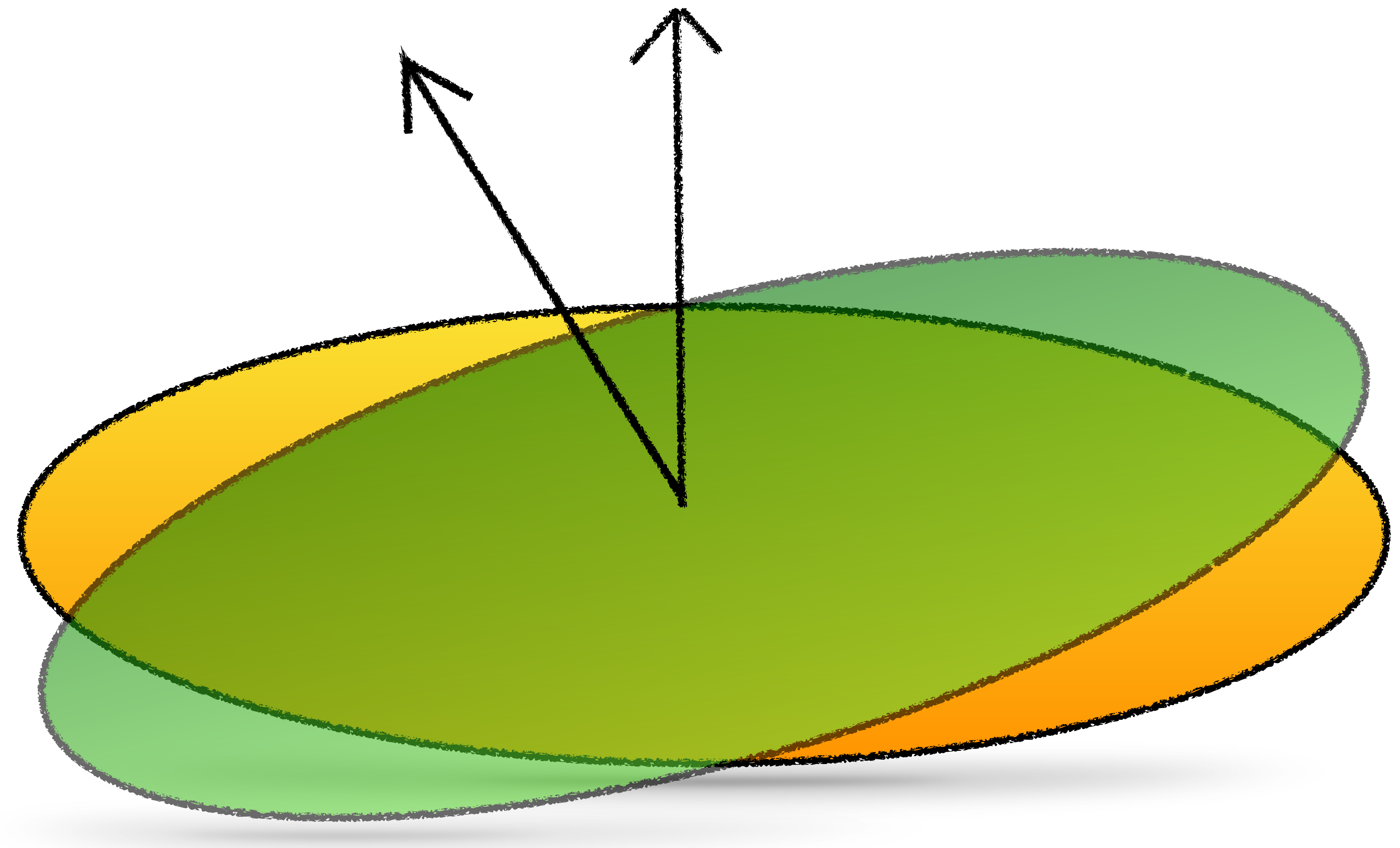

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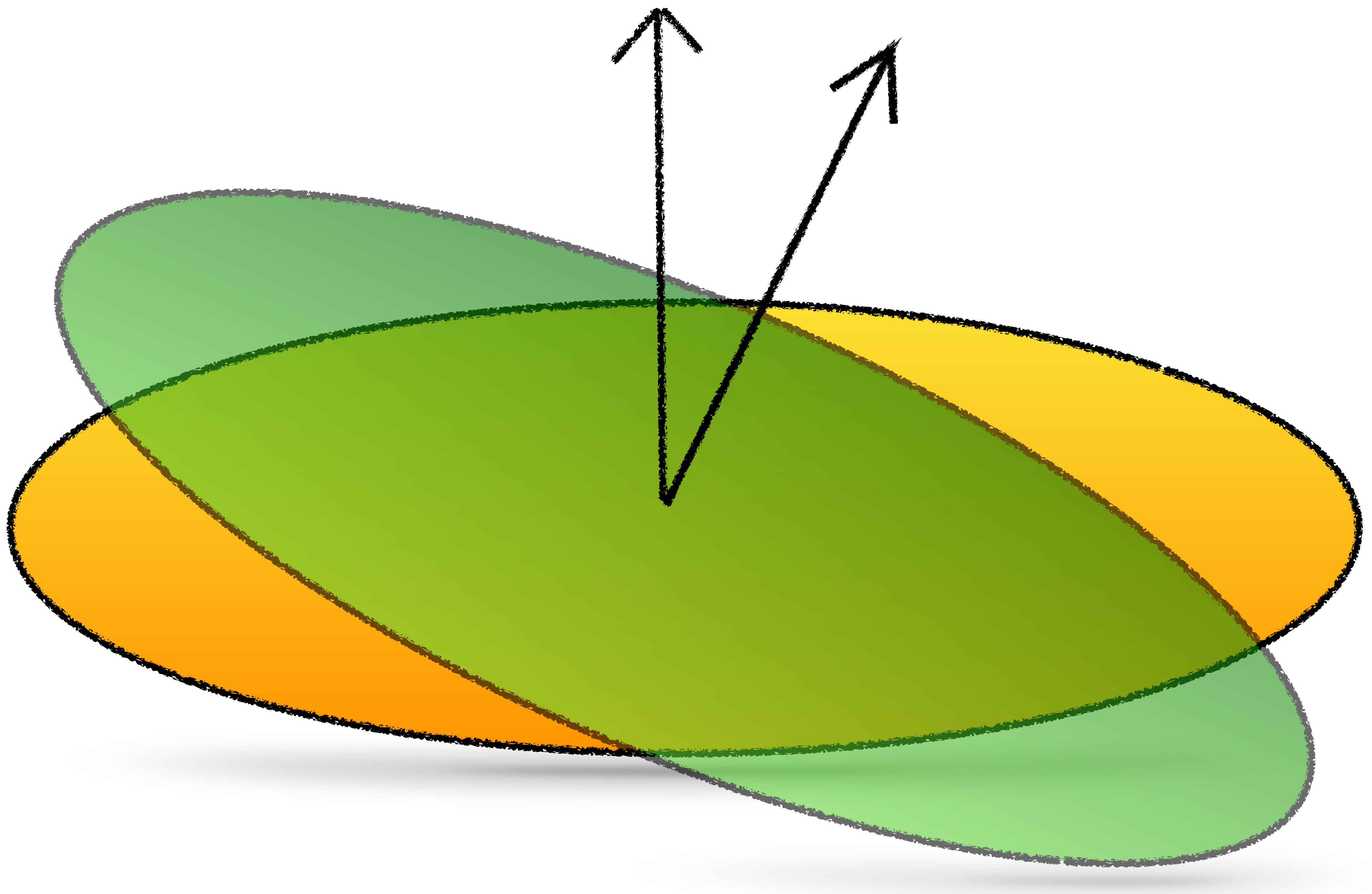

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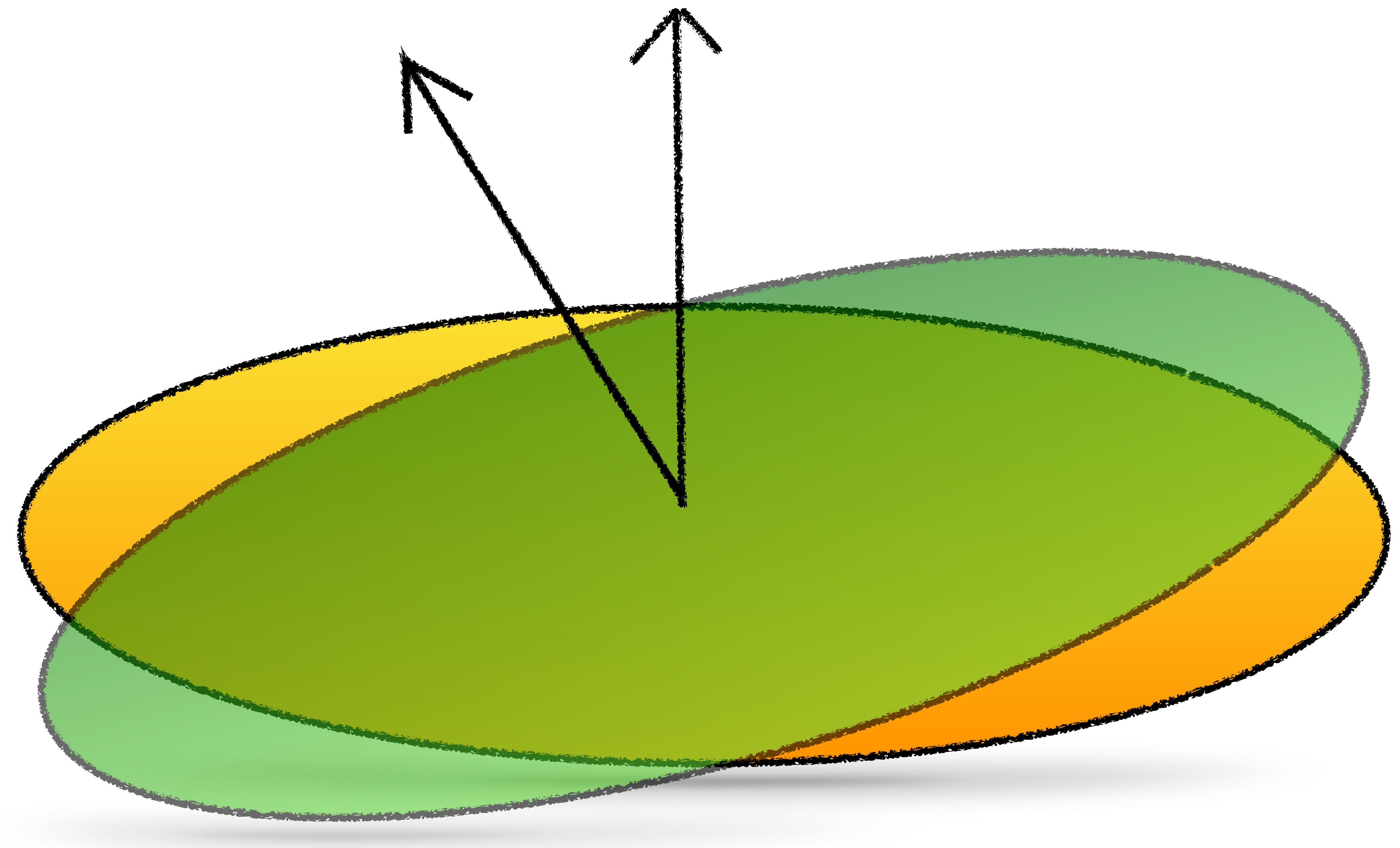

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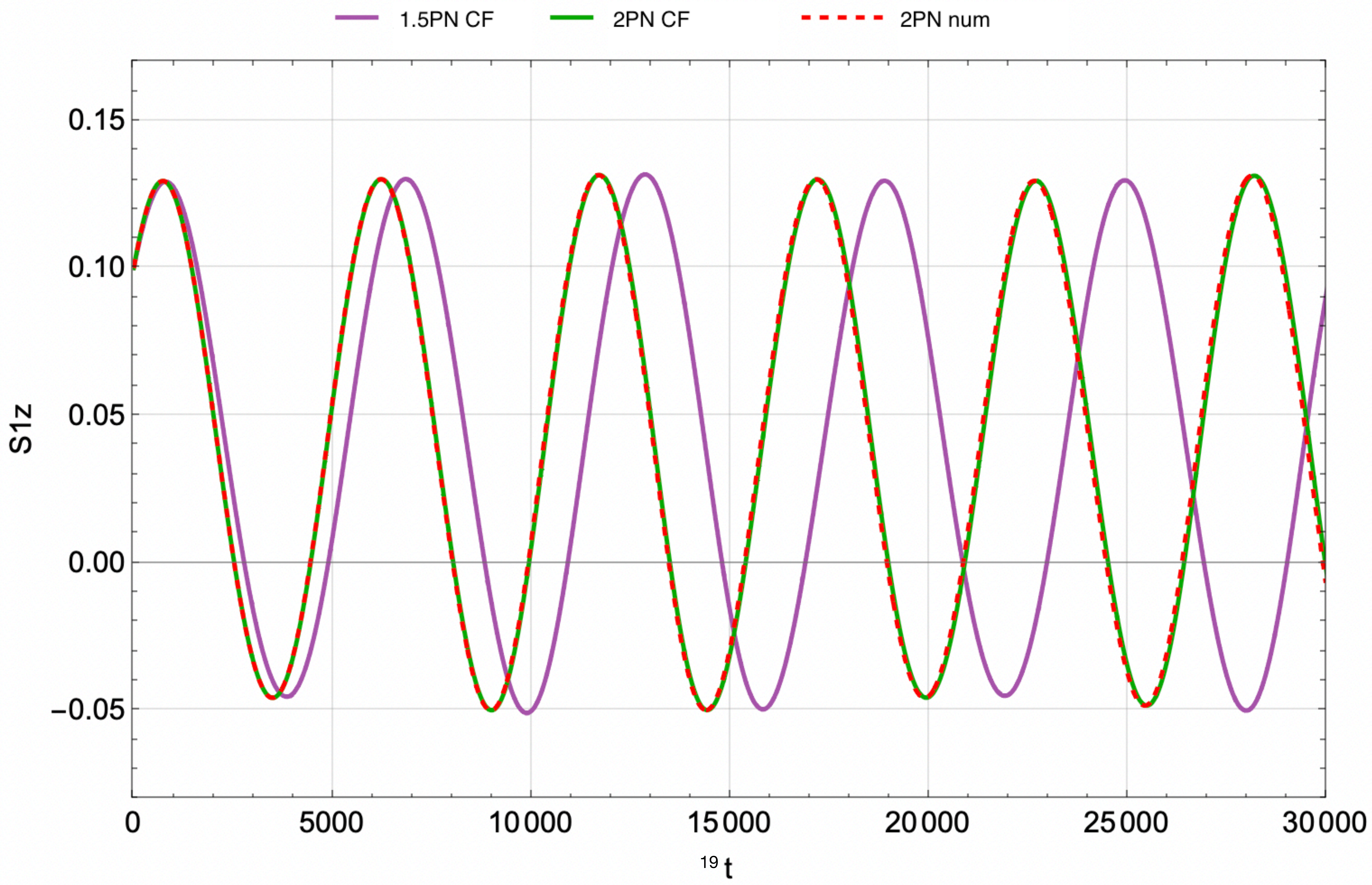
- **Result:** We (almost) have an analytical solution for $(\vec{R}, \vec{S}_1, \vec{S}_2)$ at 2PN.

Result: spins with $(\epsilon \sim 0.01)$



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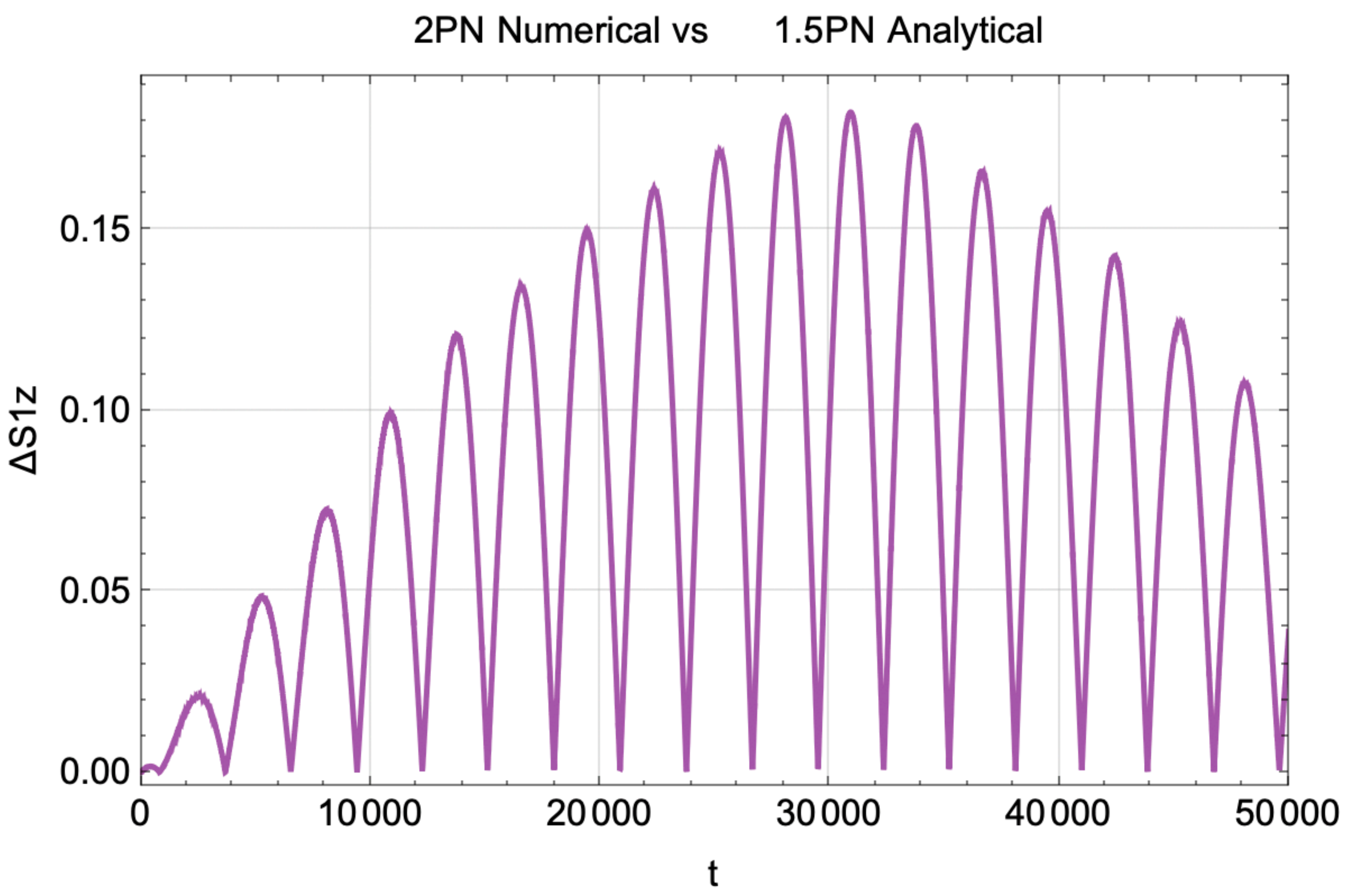
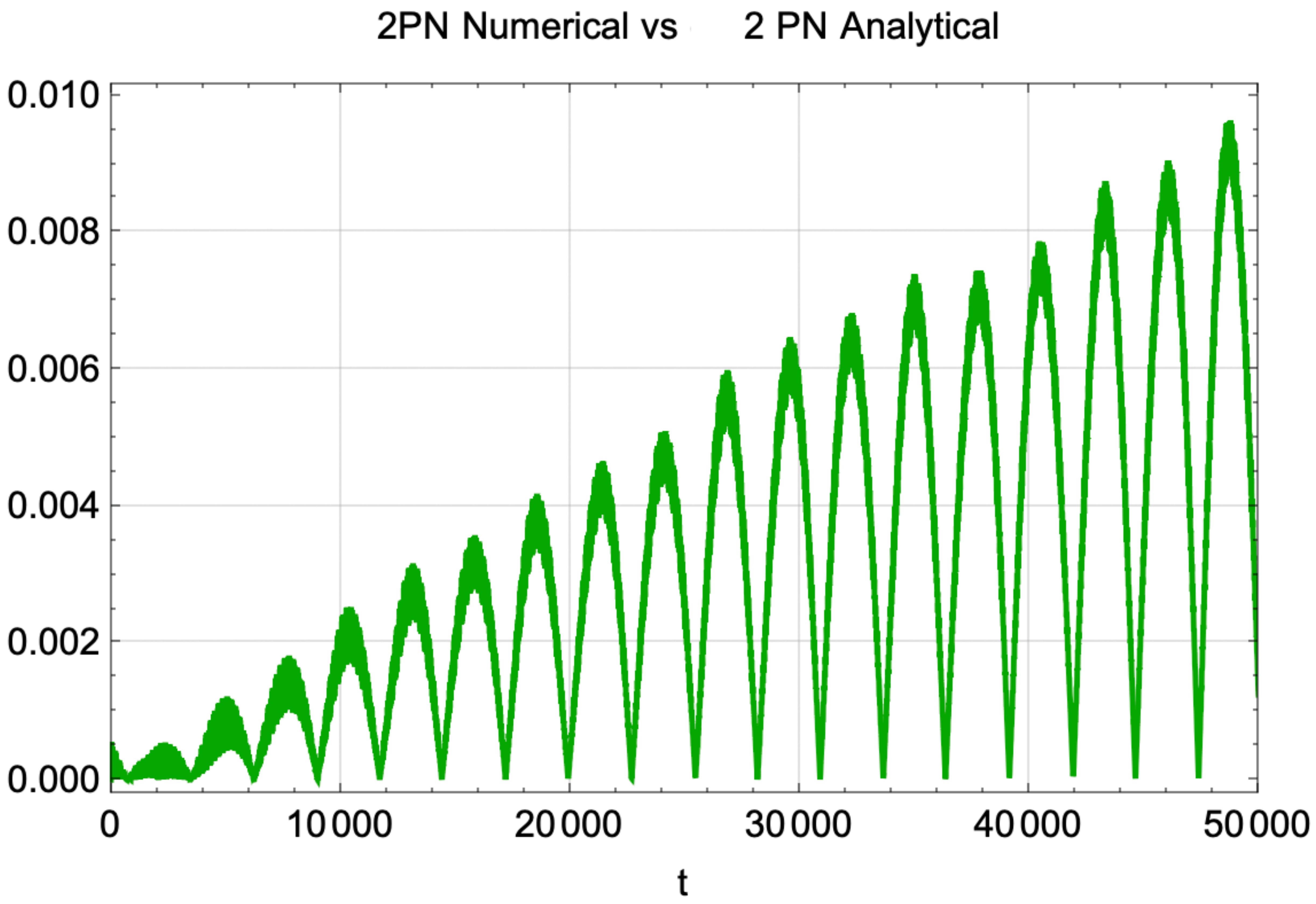
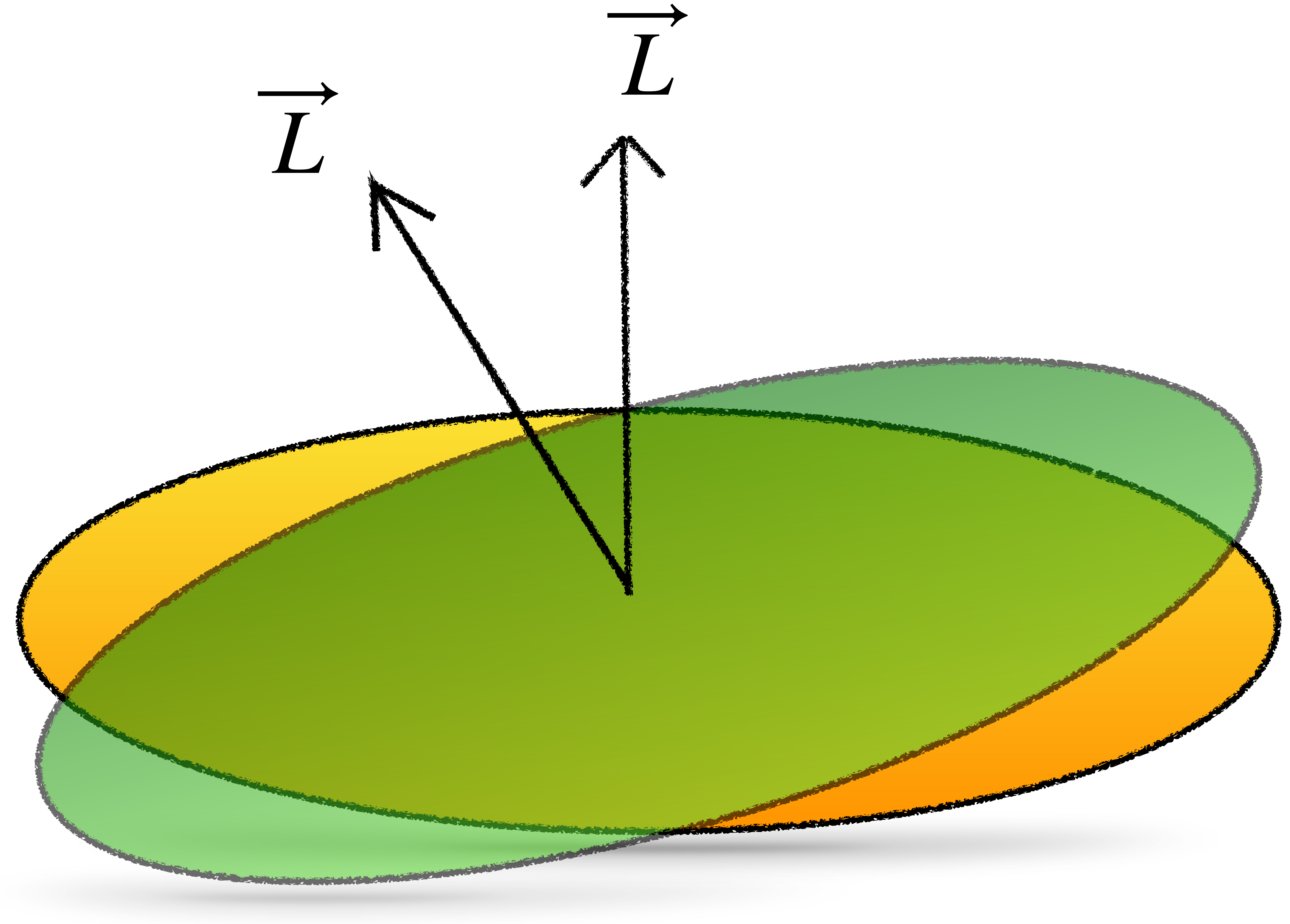




Result: plane of motion

$$\vec{L} = \vec{R} \times \vec{P} = \vec{J} - \vec{S}_1 - \vec{S}_2 \,.$$



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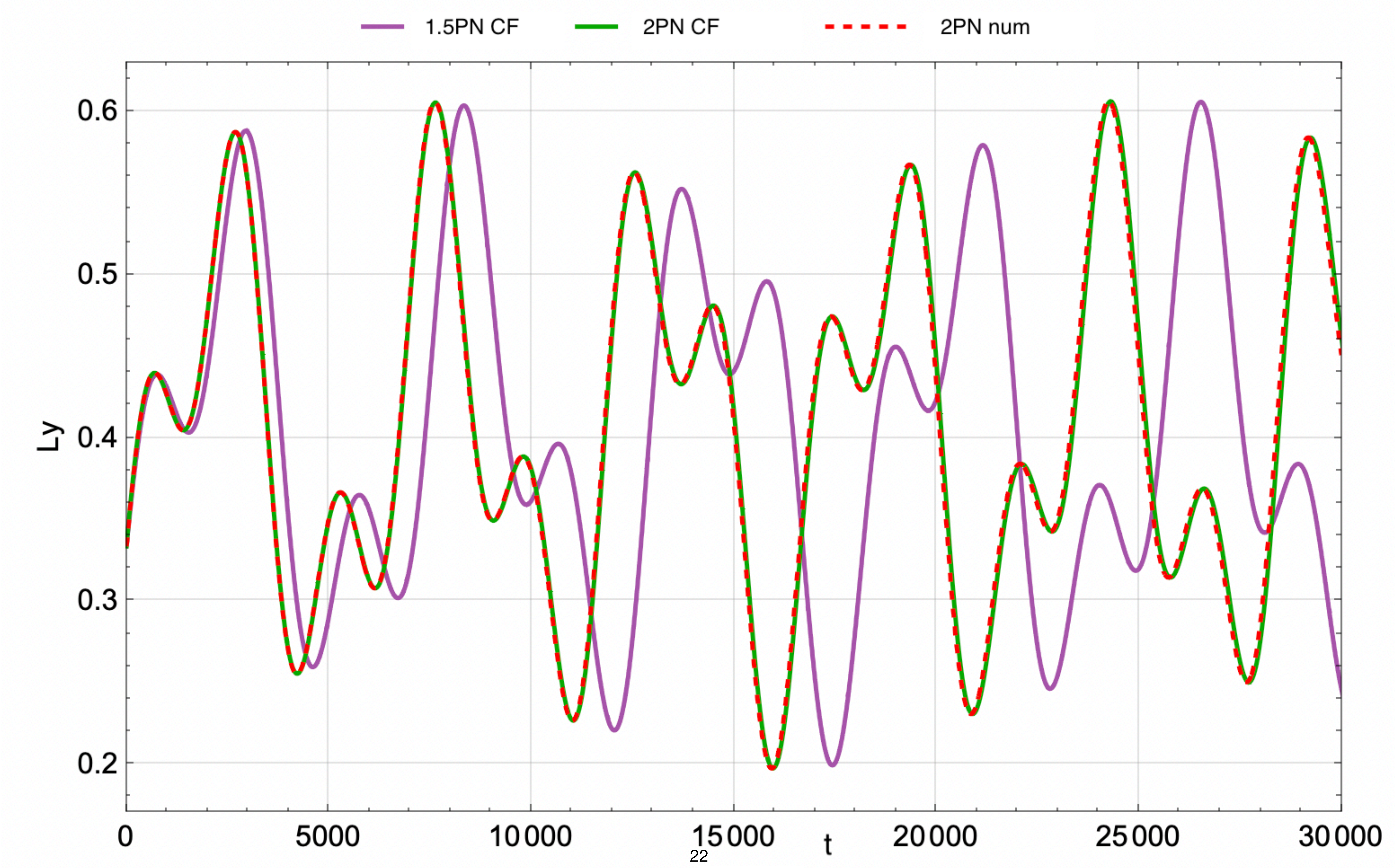

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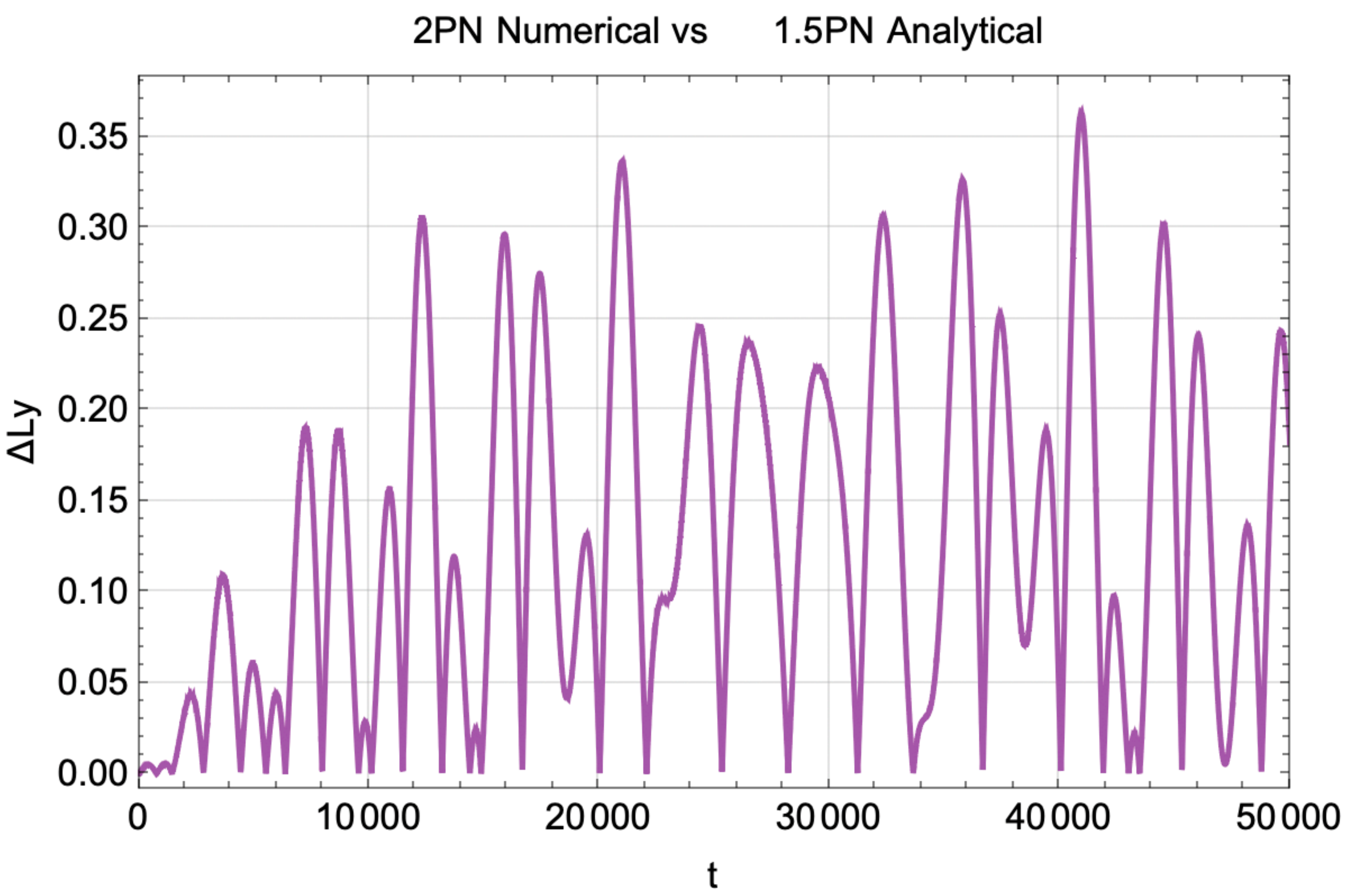

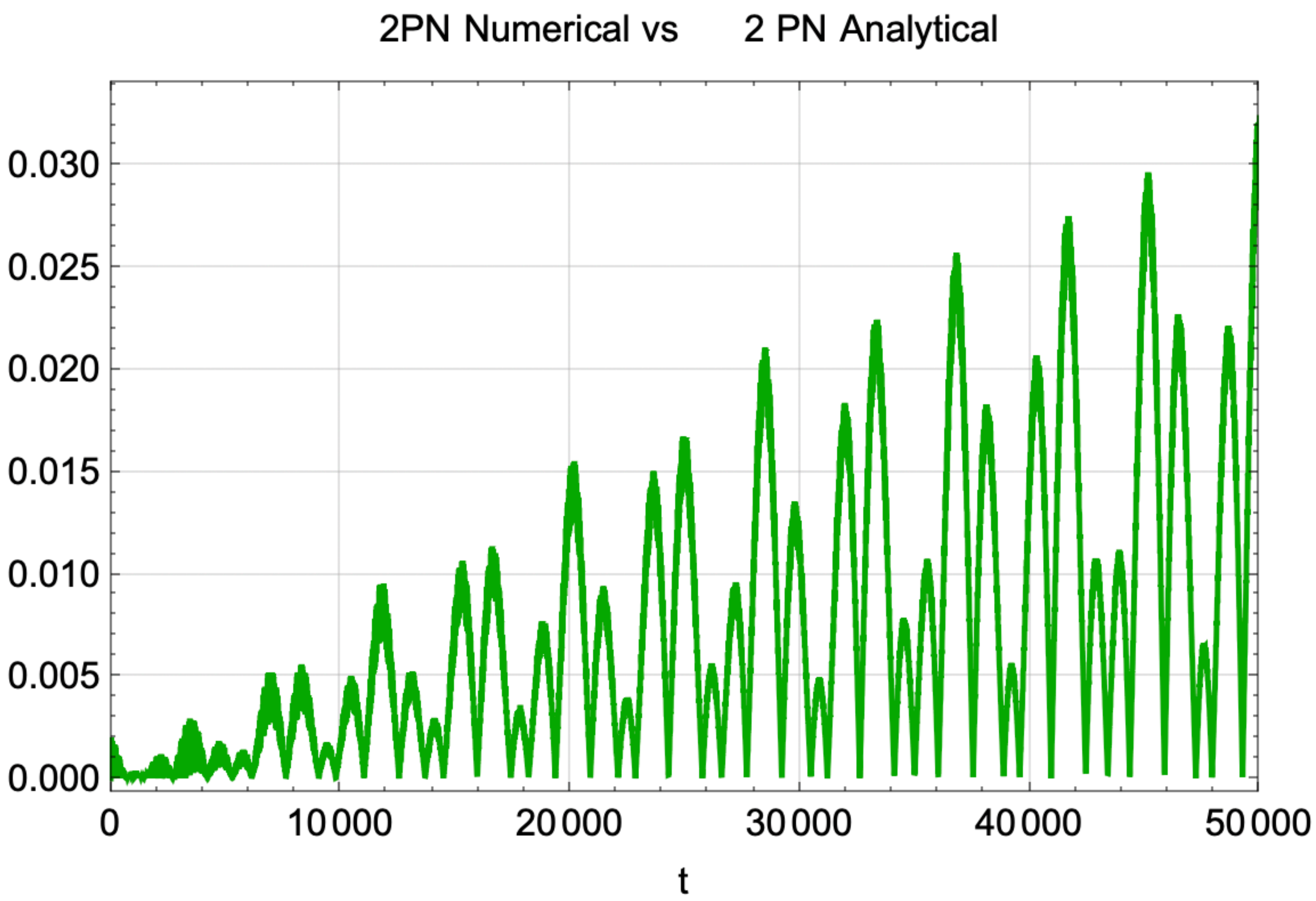

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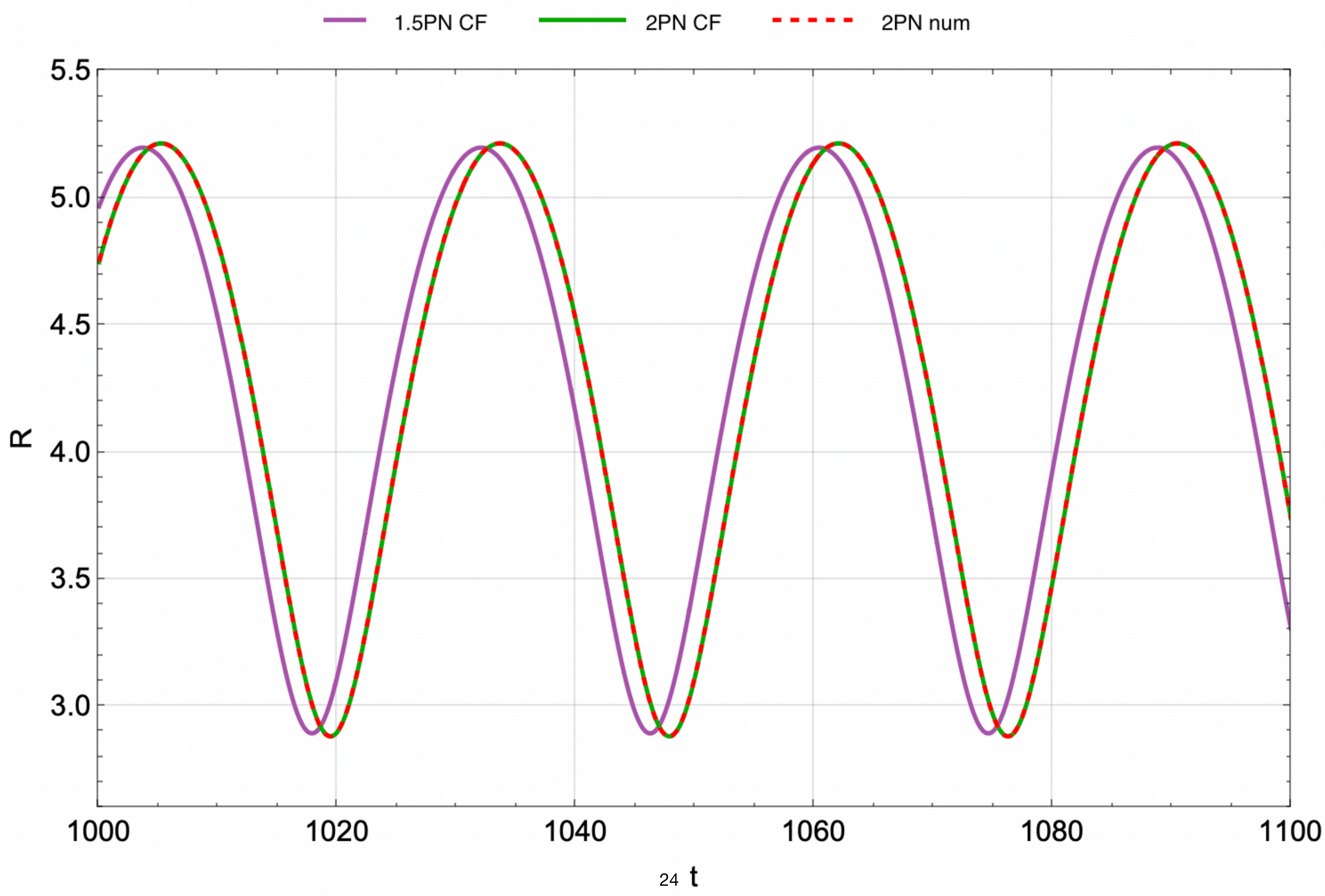

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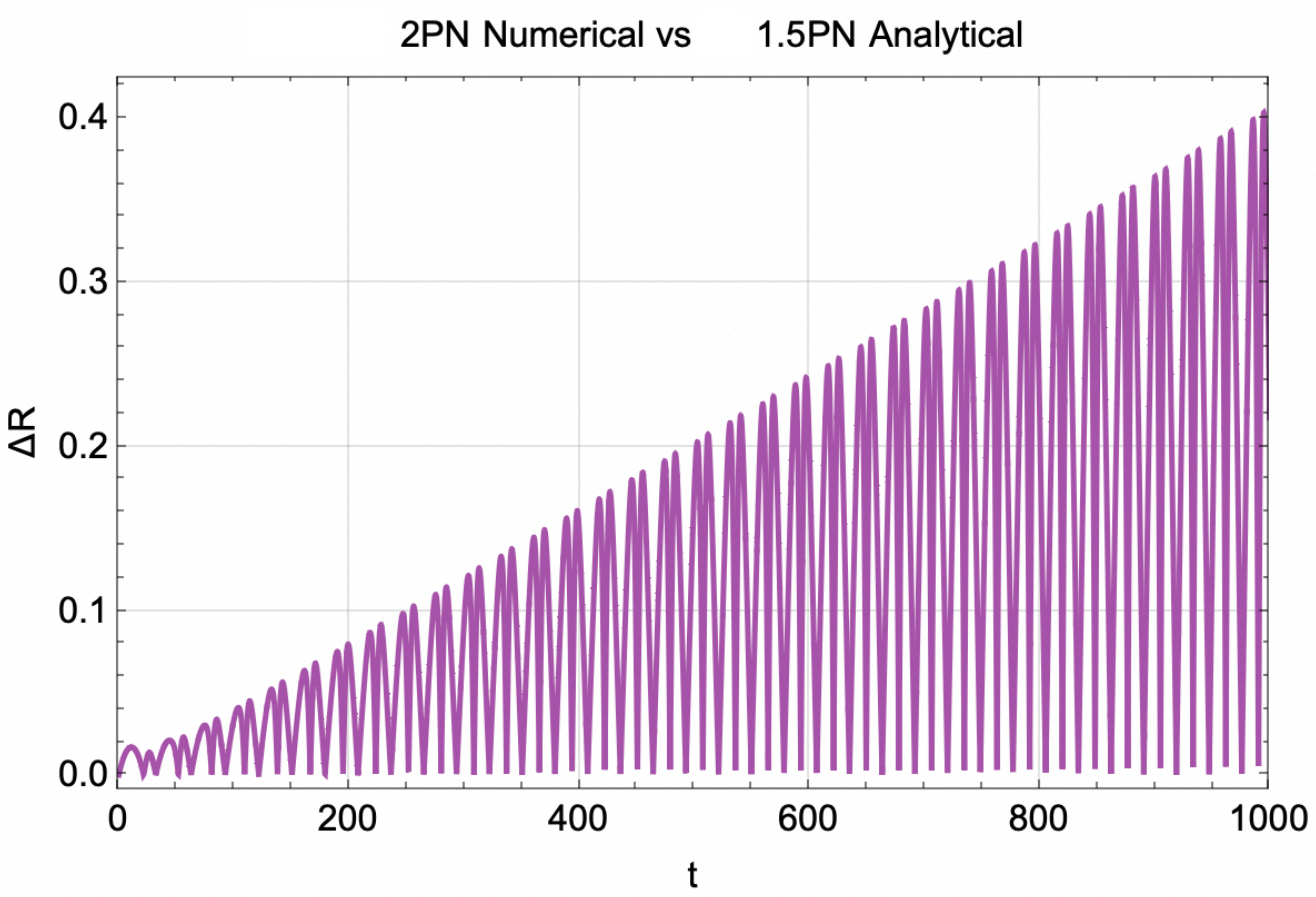

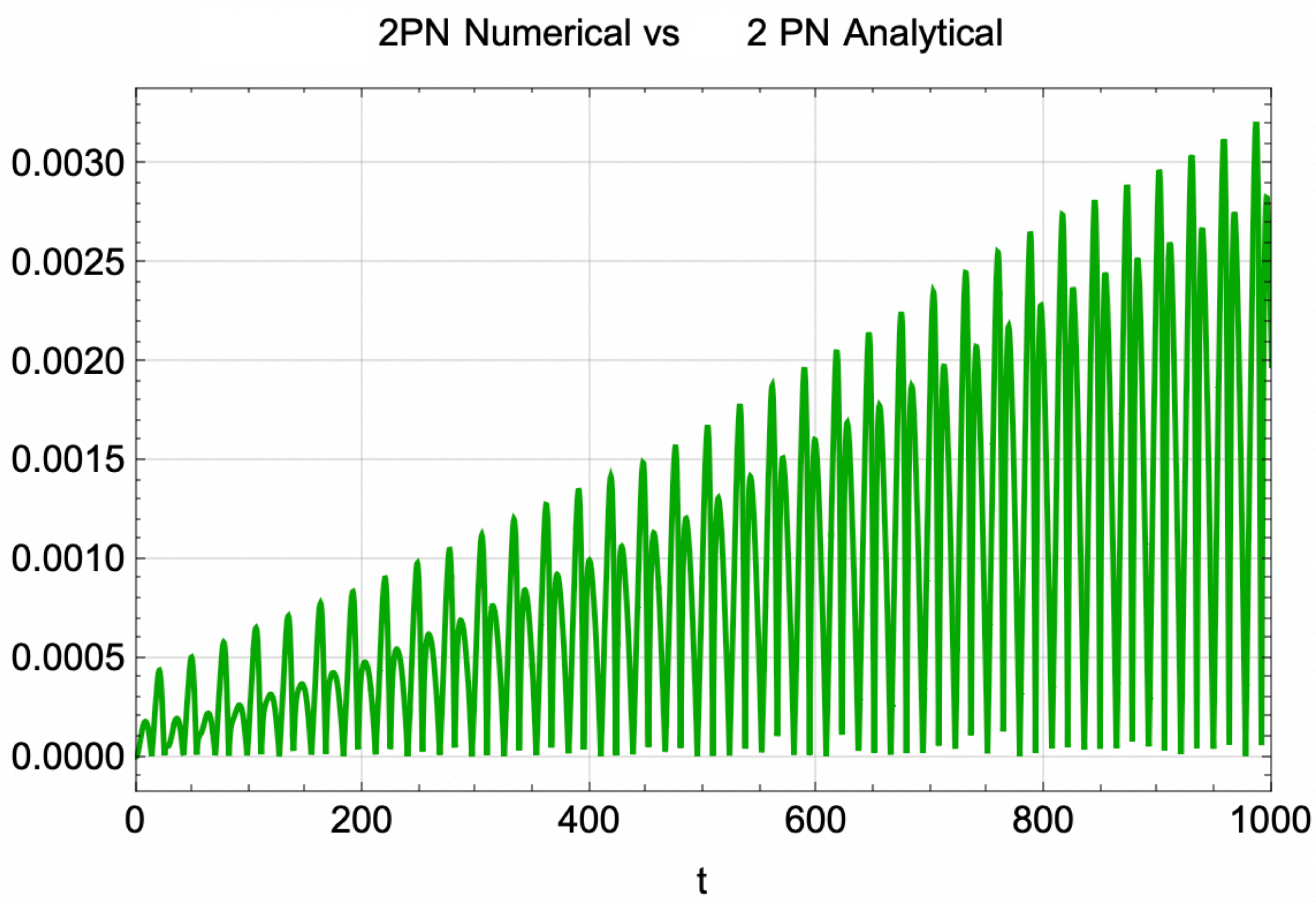

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- **Massive reduction:** 9-dim coupled ODEs → 1-dim ODE.

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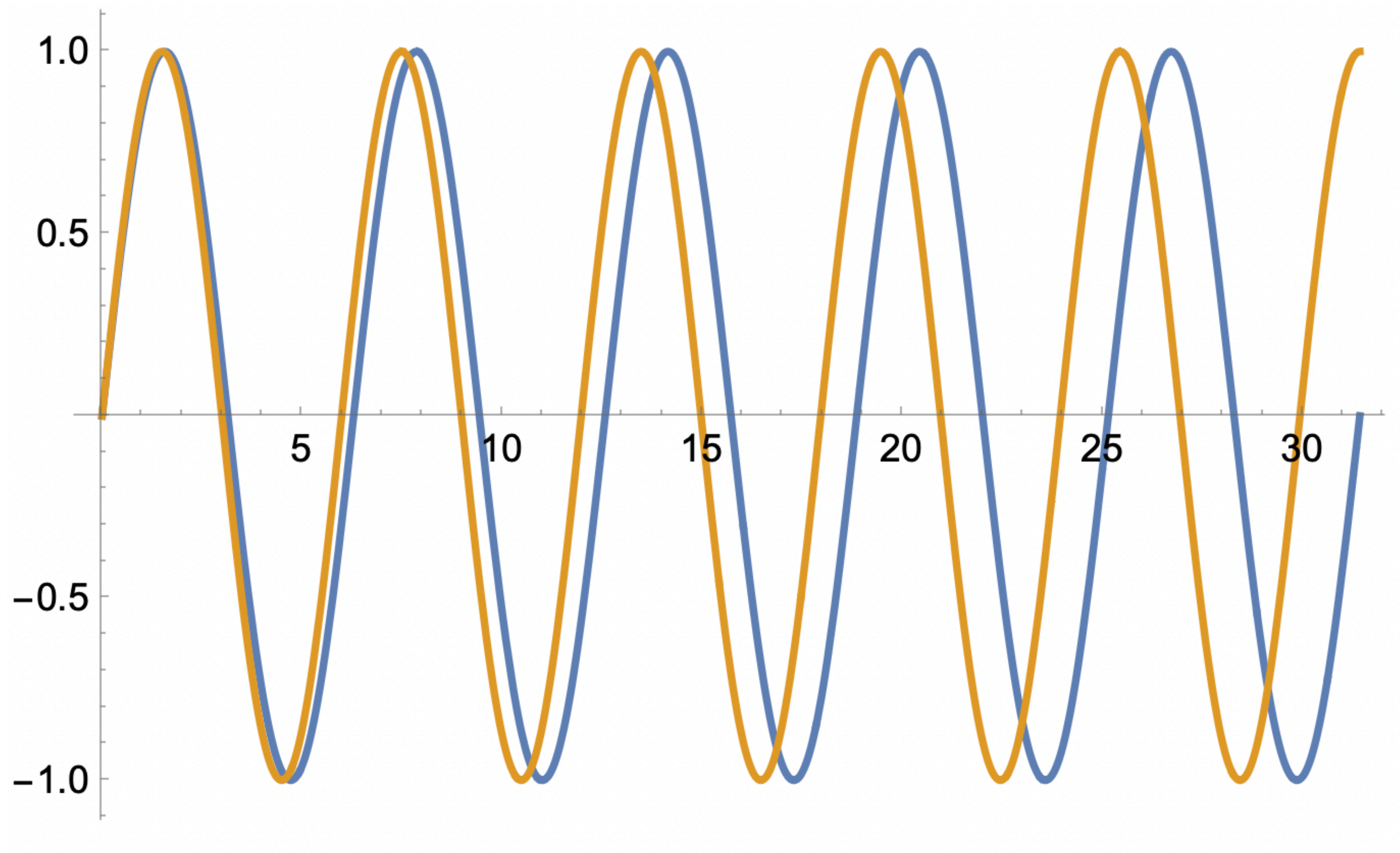

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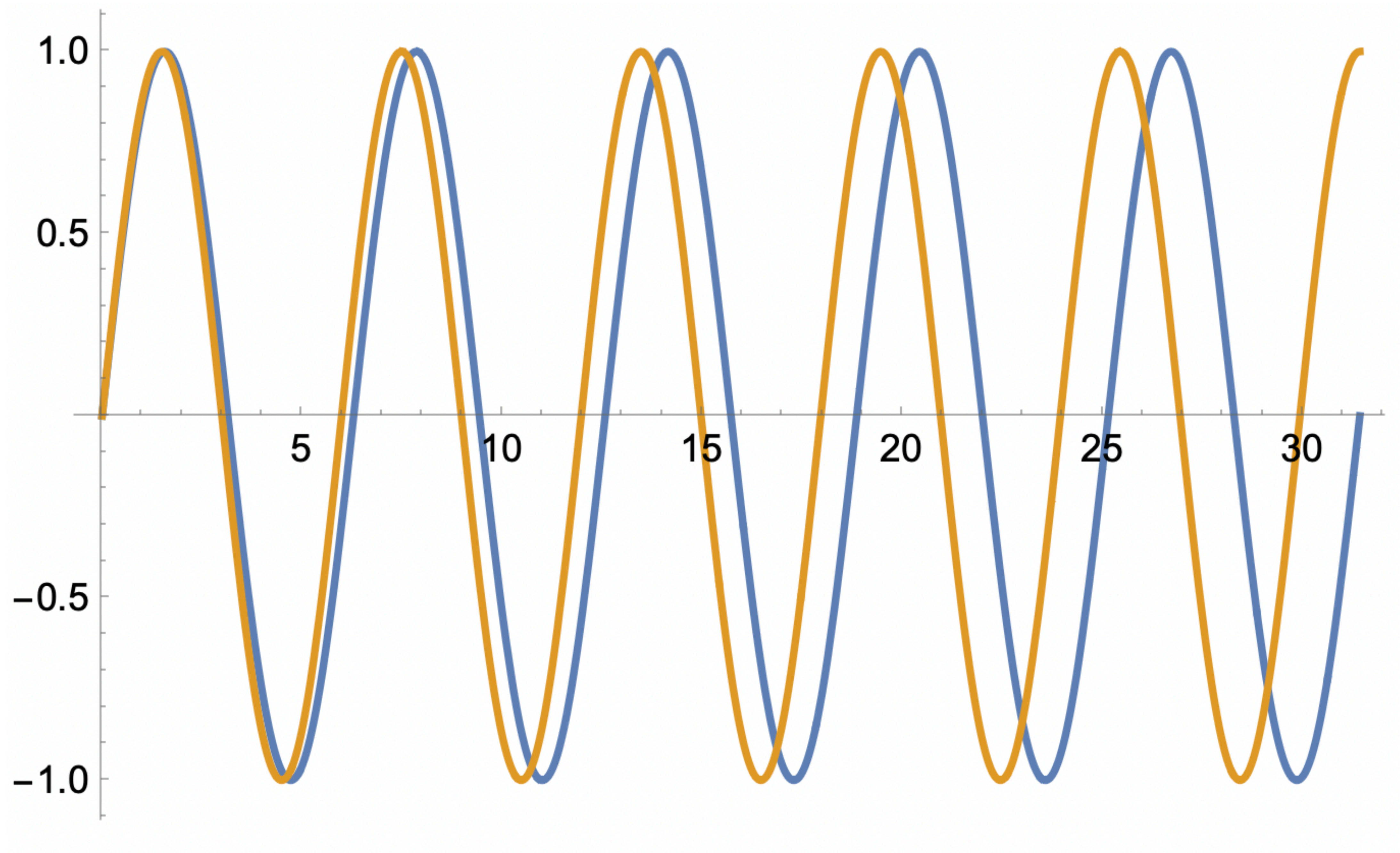

Difference due to frequency perturbation

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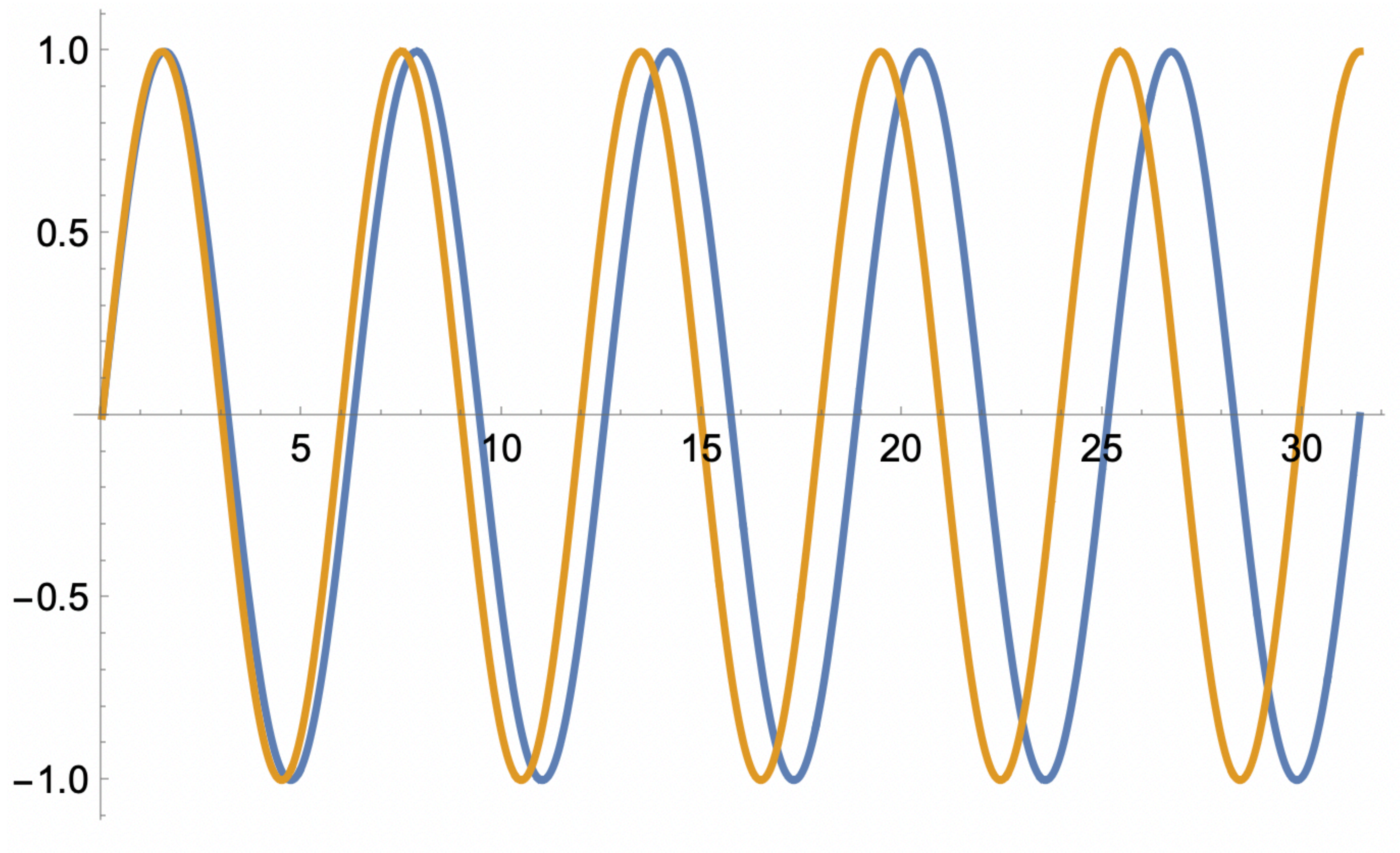

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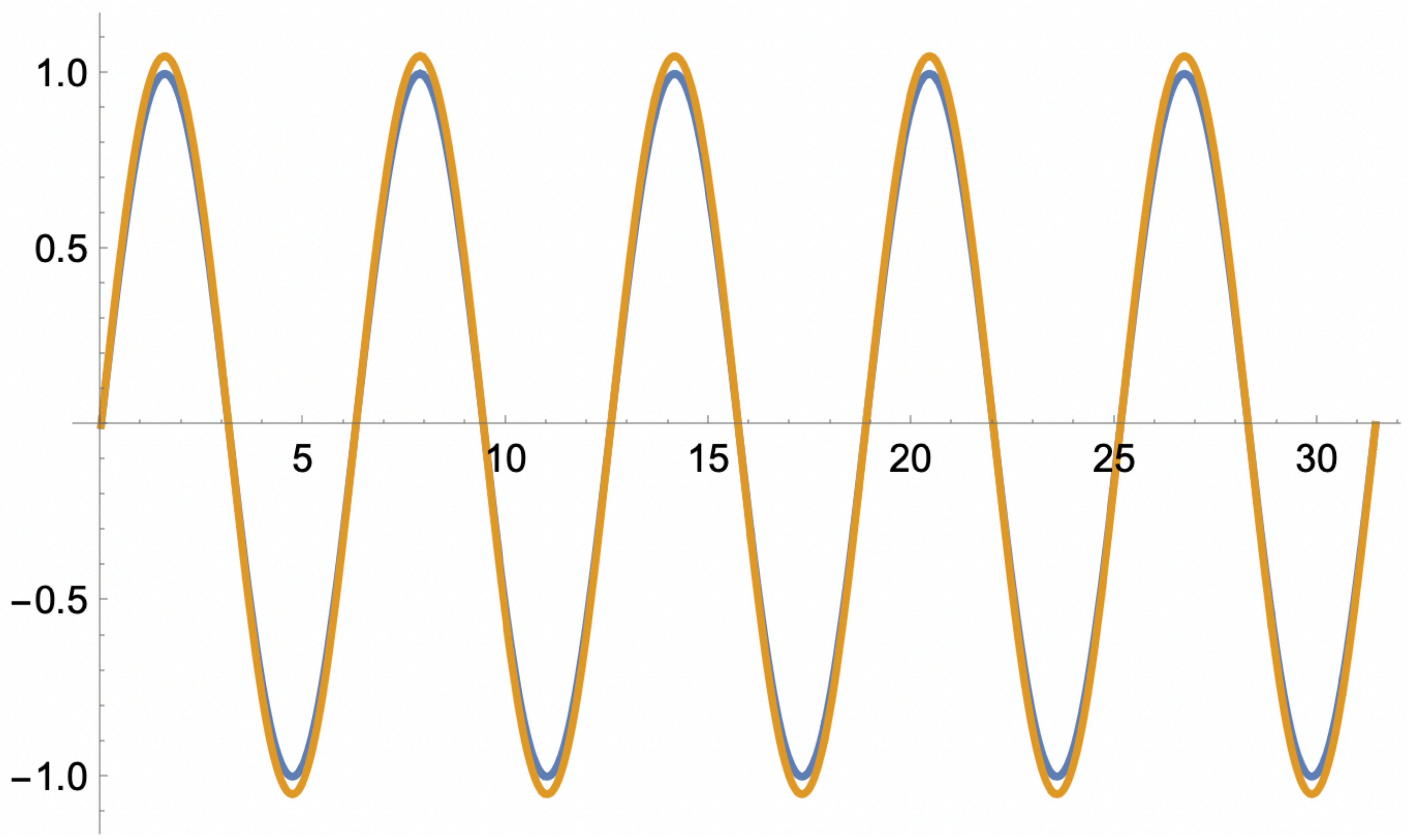

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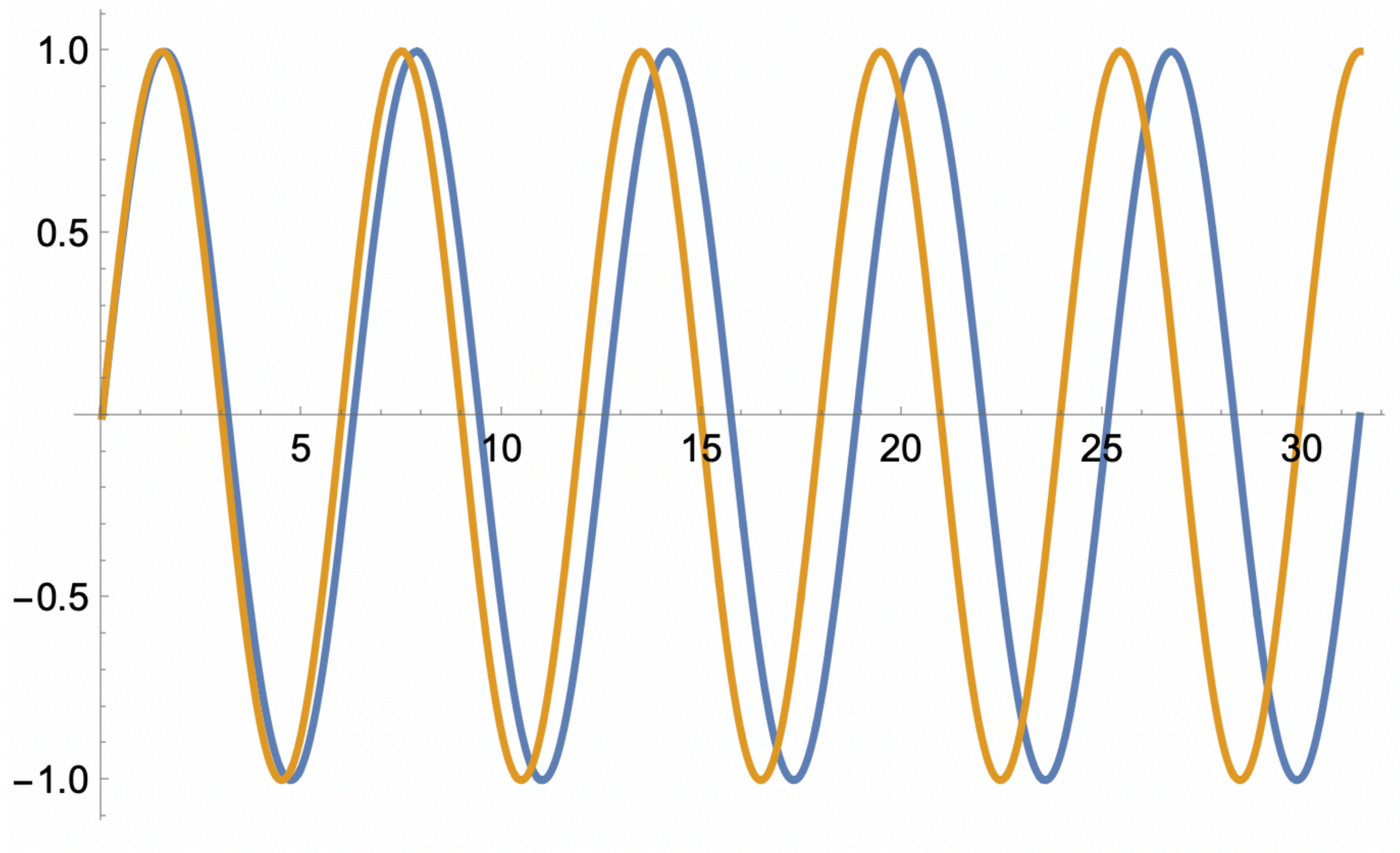

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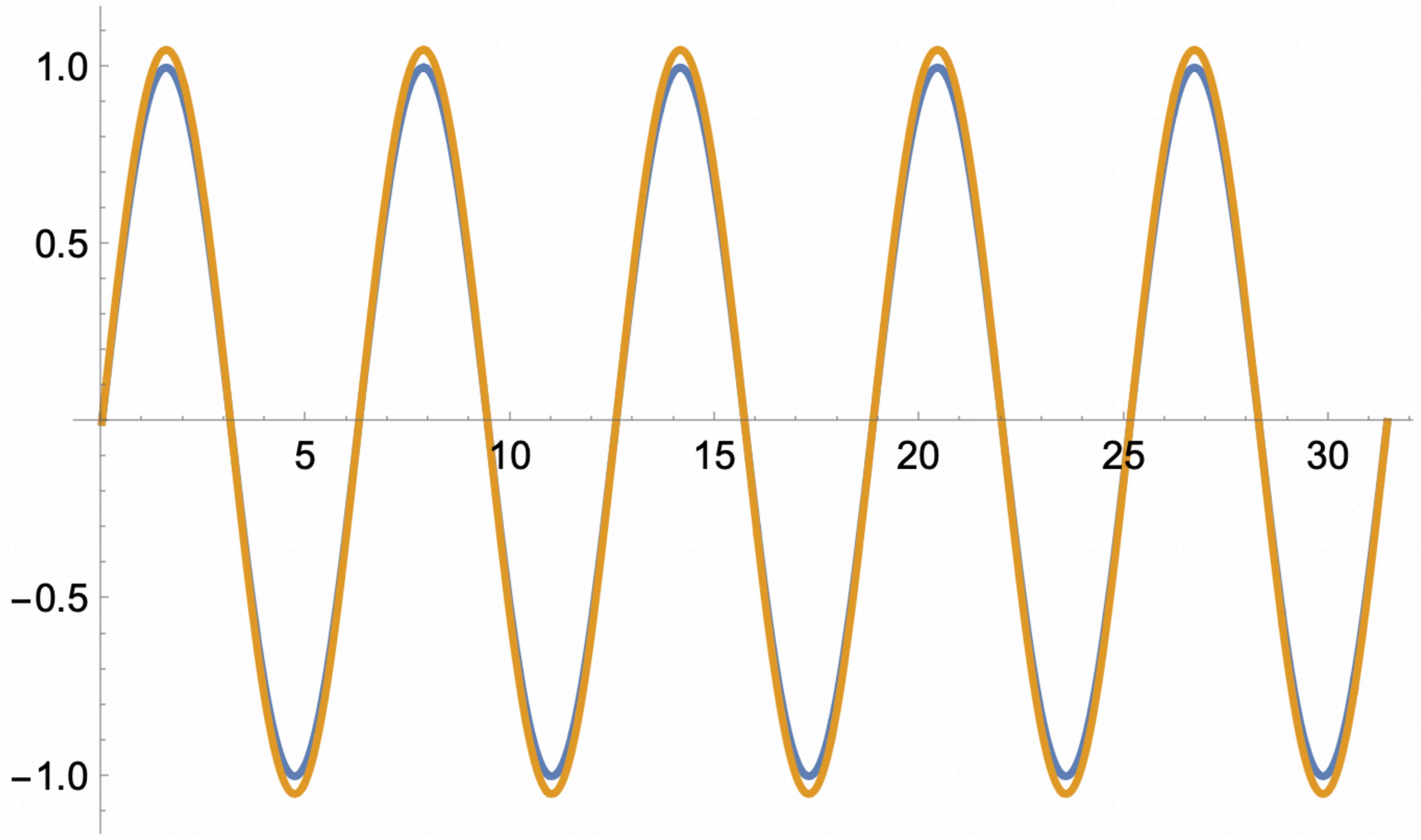

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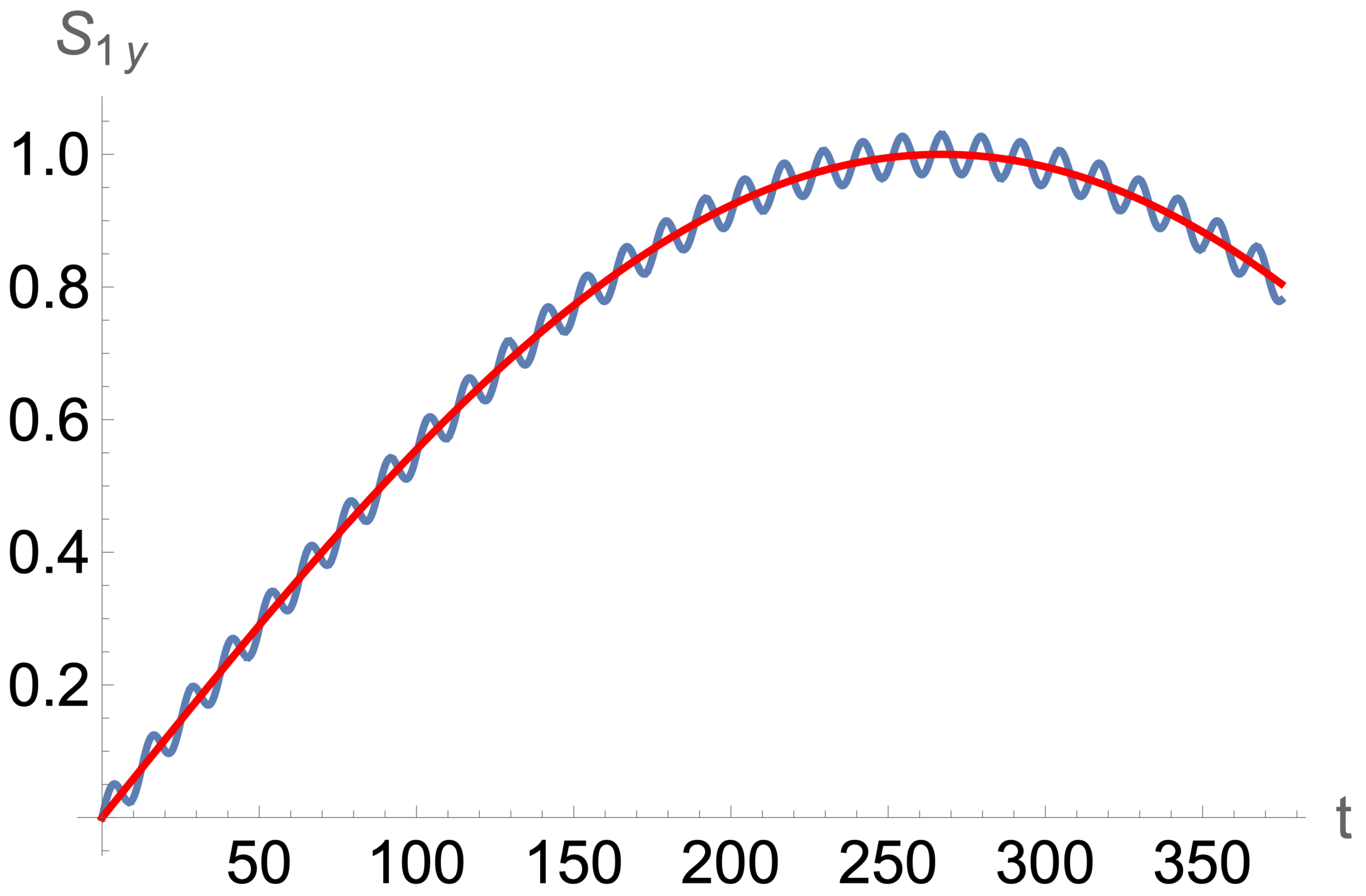
    - orbit-averaging for spins.

Orbit averaging



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- Solution not fully 2PN, but still much better than the 1.5PN solution (by ~ 50 times).

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In[3]:=
```
m1 = 5 / 2; m2 = 1; G = 1;
R0 = 2 {1, 1, 1};
P0 = {1 / 2, -1 / 2, 1 / 3};
S10 = {0, 1, 1} Sqrt[ϵ];
```
racine carrée
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S20 = {1, -3 / 10, 0} Sqrt[ϵ];
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ϵ = 0.01;
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Out[9]= 0.0101036

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