Tidal contributions to the gravitational waveform amplitude to the 2.5 post-Newtonian order

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Introduction

Systems

Non-spinning compact binary systems (BNS or BH-NS)

Project : continuation of previous work

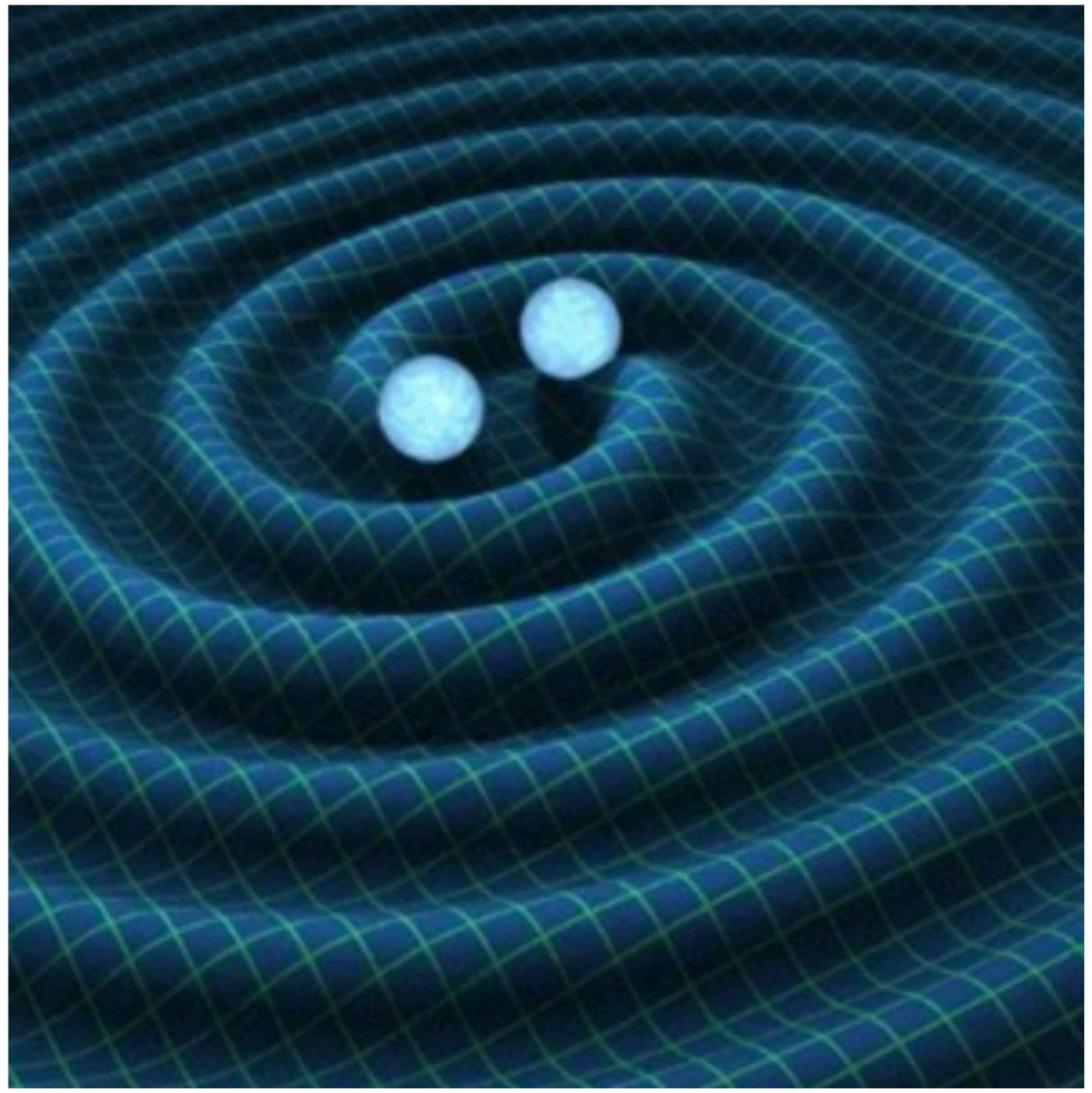
Quentin Henry, Guillaume Faye, Luc Blanchet

Tidal effects in the equations of motion of compact binary systems to next-to-next-to-leading post-Newtonian order, Phys. Rev. D.101, 064047, 2020

Tidal effects in the gravitational-wave phase evolution of compact binary systems to next-to-next-to-leading post-Newtonian order, Phys. Rev. D.102, 044033, 2020

Aims at furnishing the complete waveform amplitude including tidal effects at 2.5PN order consistently with the precision of the orbital phase

Overview



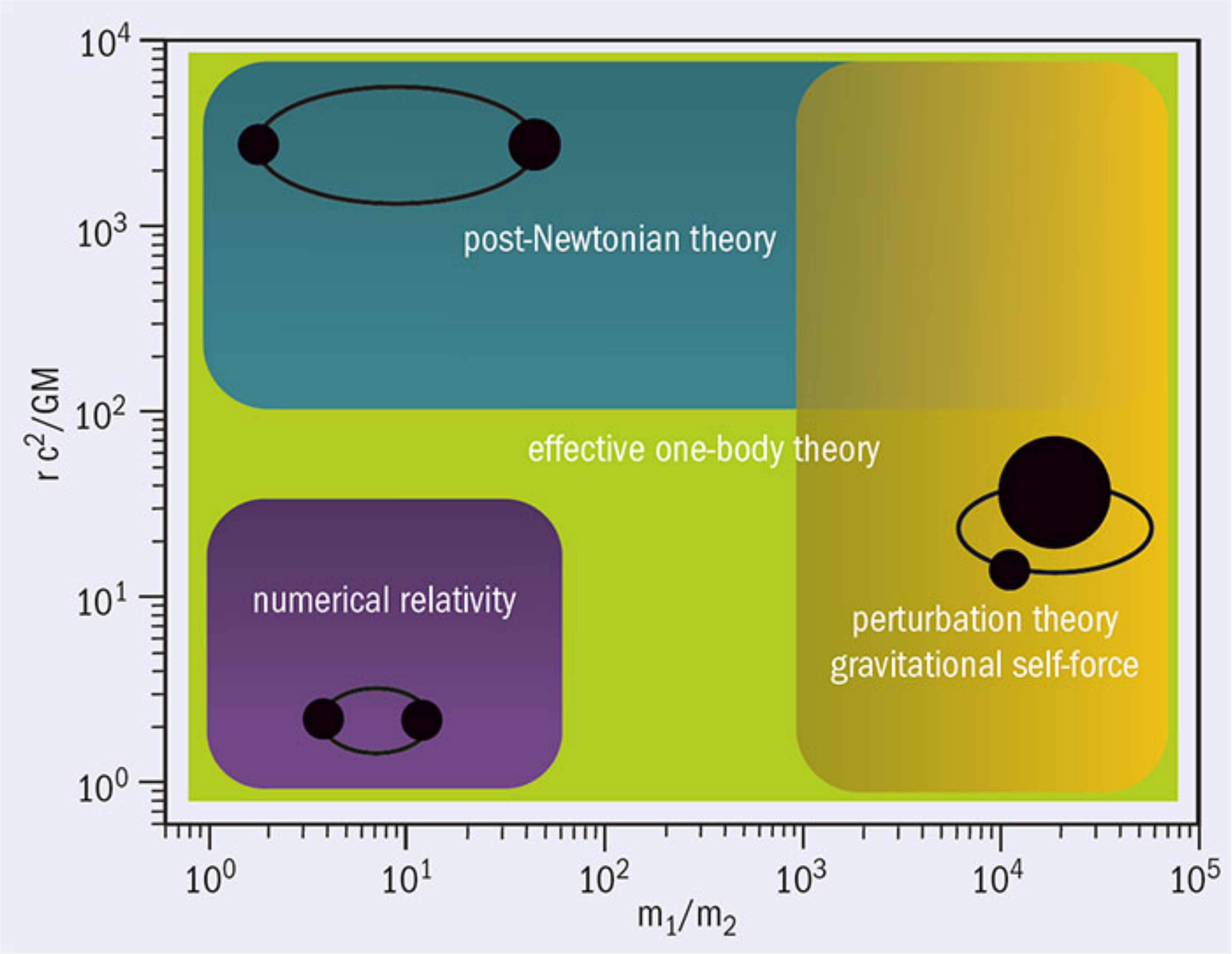
Analytical waveform modeling for inspiraling binaries

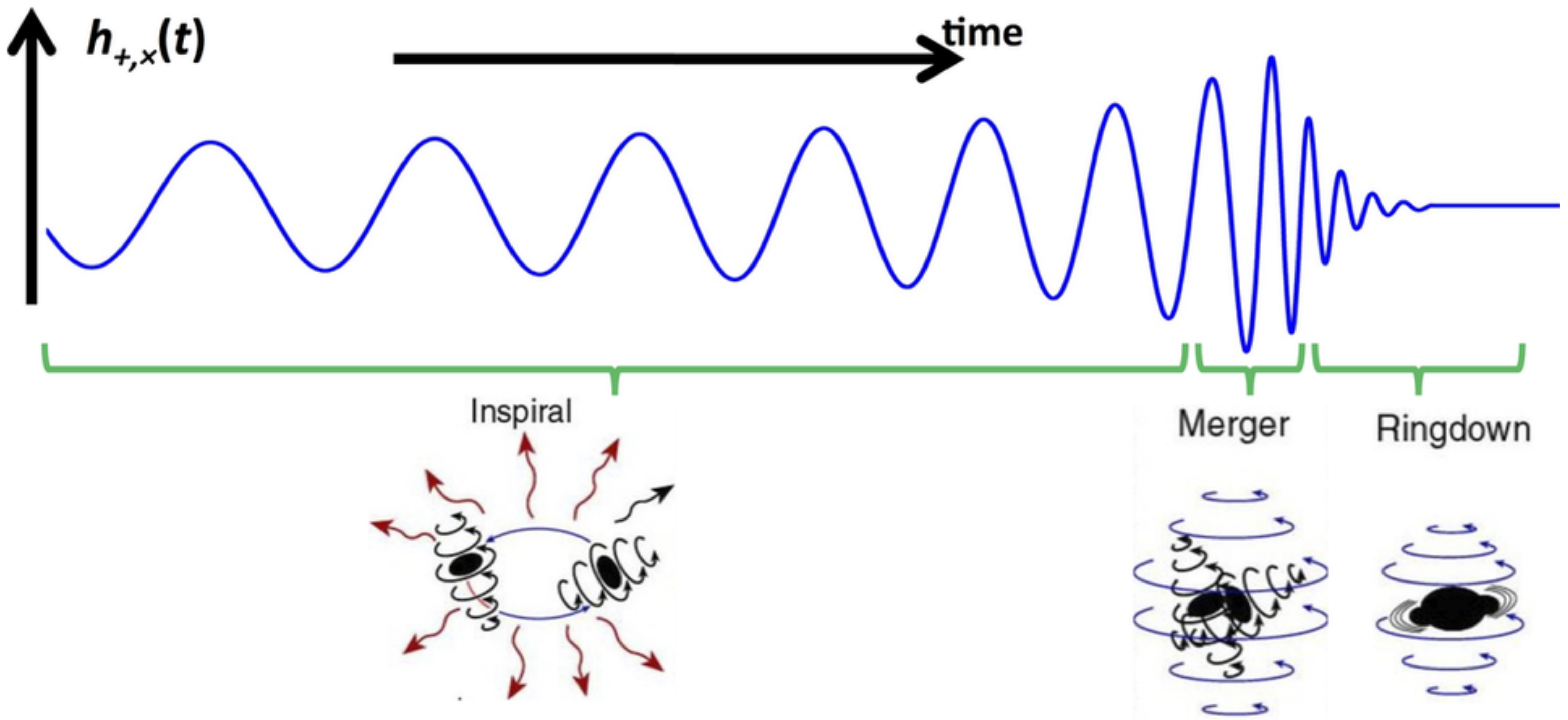
Two-body problem in GR

Tidal effects and their impact on the GW amplitude

PN-expanded and EOB-factorized modes

Approaches to computing the waveform



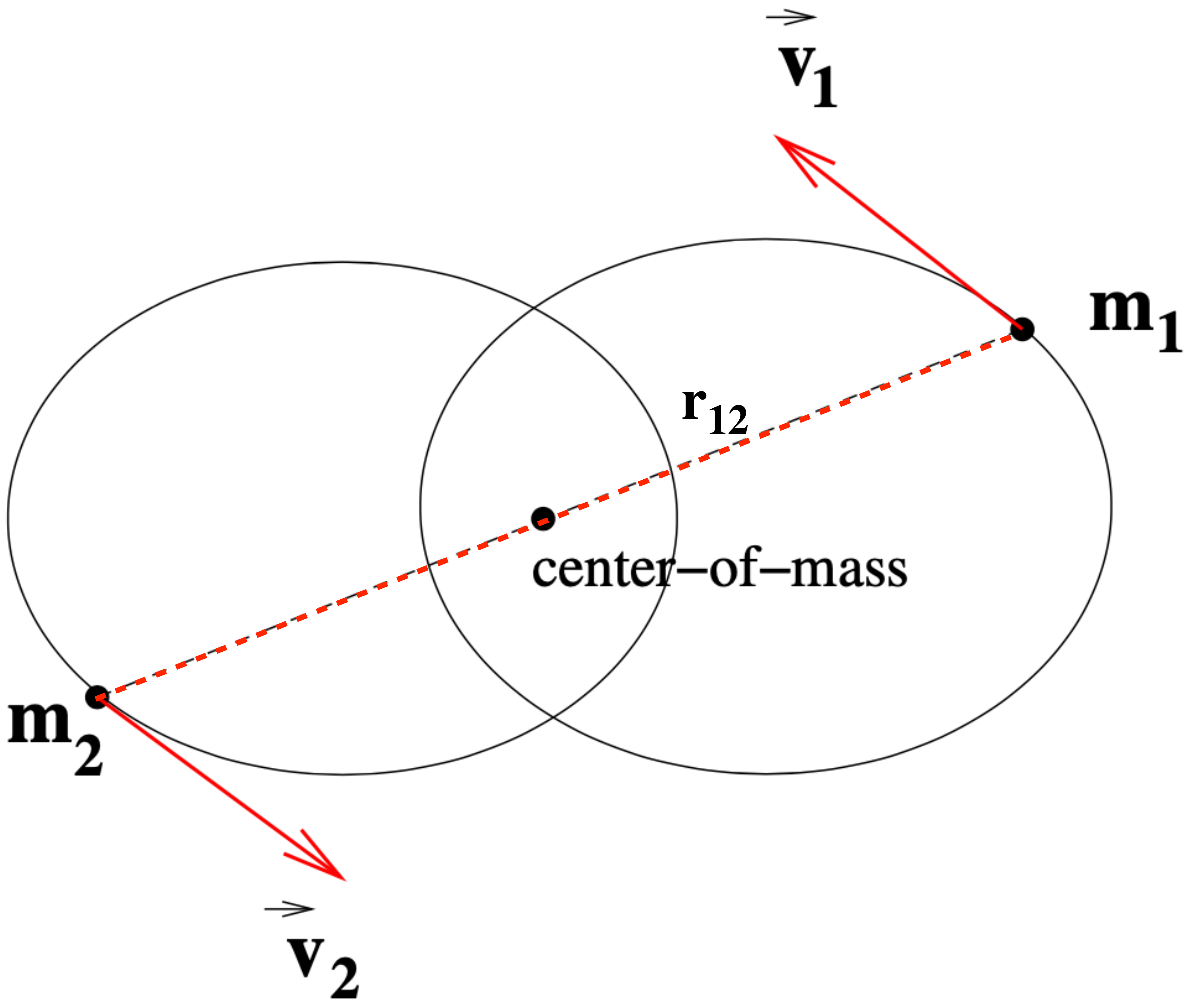


Post-Newtonian theory

Numerical Relativity

Perturbation theory

Post-Newtonian formalism



- **Slow motion** and **weak field** regimes

- PN power series in the small parameter

$$\varepsilon = \frac{{v_{12}}^2}{c^2} \sim \frac{Gm}{r_{12}\, c^2} \ll 1$$

- PN orders : nPN $= \mathcal{O}(\epsilon^n)$

Solving the Relativistic Two-Body Problem

Dynamical sector

- Effective action $S = S_{EH} + S_m$
- Solving iteratively the EFEs :

$$\Box_\eta h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu} = \frac{16\pi G}{c^4}|g|T^{\mu\nu} + \Lambda^{\mu\nu}(h, \partial h, \partial^2 h)$$

- Fokker Lagrangian $L_{fokker} = L\,[y_A, v_A, a_A^k]$

⟶ (a_1^i, a_2^i) : conservative EOM
E : conserved energy

Radiative sector

- **Gravitational wave generation formalism** [Blanchet Living Review]
  - **mPM** expansion of the field outside the source
  - **PN** expansion of the field in the near zone
  - **Matching of MPM and PN expansions** in exterior near zone where both expansions are valid

⟶ $\mathscr{F}$: radiated energy flux parametrized by a set of radiative multipole moments (U_L, V_L)

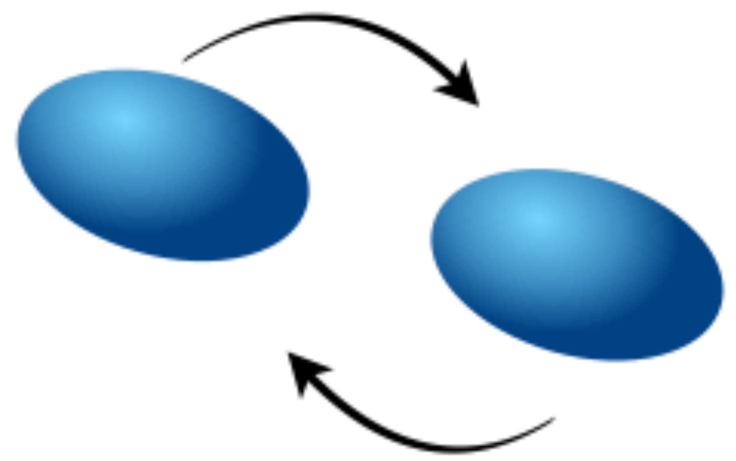
Orbital phase

Flux balance equation :

$$\frac{dE}{dt} = -\mathscr{F} \Rightarrow \phi = \int \omega dt = -\int \frac{\omega dE}{\mathscr{F}}$$

Adiabatic tidal effects

Motivations



- Main influence of NS matter on the GW signals in the inspiral due to adiabatic tidal effects

→ very promising way to **probe the internal structure of NS**

- A way to **distinguish signals** coming from BBH, BH-NS, BNS or systems involving more exotic objects such as boson stars

- Affects both the dynamics and the GW emission of compact binaries

→ results in a **change in the orbital phase and waveform amplitude, which are directly observable**

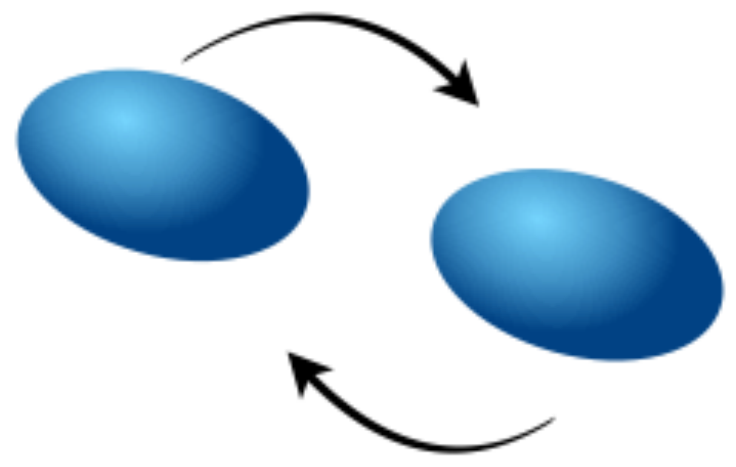
- Becomes more important in the late inspiral and for extended NS

→ **could be measurable**, in particular with 3G detectors (ET, CE …)

Effective action at 2PN

Go beyond the point-particule approximation :

$$S_m = -\sum_{A=1,2}\int d\tau_A \left\{ m_A c^2 + \frac{\mu_A^{(2)}}{4} G^A_{\mu\nu} G^{\mu\nu}_A + \frac{\sigma_A^{(2)}}{6c^2} H^A_{\mu\nu} H^{\mu\nu}_A + \frac{\mu_A^{(3)}}{12} G^A_{\mu\nu\rho} G^{\mu\nu\rho}_A \right\}$$



$G_{\mu\nu} \equiv -c^2 R_{\mu\alpha\nu\beta} u^\alpha u^\beta$: tidal mass-type quadrupole moment

$H_{\mu\nu} \equiv 2c^3 R^*_{\mu(\alpha\underline{\nu}\beta)} u^\alpha u^\beta$: tidal current-type quadrupole moment

$G_{\lambda\mu\nu} \equiv -c^2 \nabla^\perp_{(\lambda} R_{\mu\underline{\alpha}\nu)\beta} u^\alpha u^\beta$: tidal mass-type octupole moment

$$\nabla^\perp_\mu = \perp^\nu_\mu \nabla_\nu = (\delta^\nu_\mu + u_\mu u^\nu)\nabla_\nu$$

Tidal deformability of the NS characterized by a set of deformation parameters $(\mu_A^{(l)}, \sigma_A^{(l)})$

→ linked to the **Tidal Love Numbers** $(k_A^{(l)}, j_A^{(l)})$

$$G\mu_A^{(l)} = \frac{2}{(2l-1)!!} k_A^{(l)} R_A^{2l+1}$$

$$G\sigma_A^{(l)} = \frac{l-1}{4(l+2)(2l-1)!!} j_A^{(l)} R_A^{2l+1}$$

+

Compactness

$$\mathscr{C} \sim \frac{Gm}{Rc^2} \sim 1$$

for compact objects

⇒

$\mu_A^{(2)} \sim \sigma_A^{(2)} \sim \mathcal{O}\left(\frac{1}{c^{10}}\right)$: 5PN effect (LO/0PN)

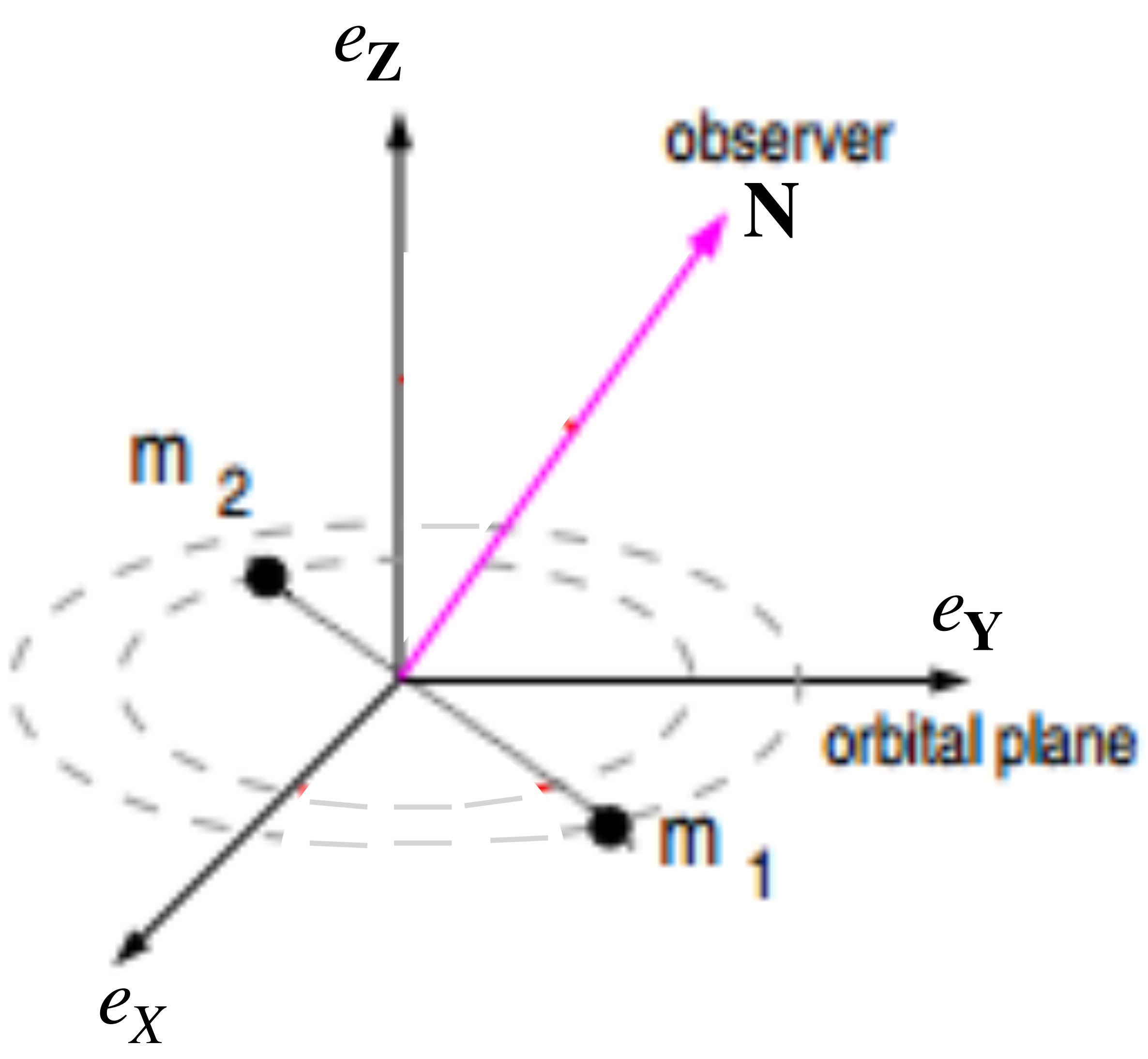
$\mu_A^{(3)} \sim \mathcal{O}\left(\frac{1}{c^{14}}\right)$: 7PN effect (NNLO/2PN relative)

Waveform amplitude

Radiative coordinate system : $X^\mu = (cT, \mathbf{X})$

The TT projection of the metric uniquely decomposed, at LO in 1/R, in terms of the STF radiative multipole moments (U_L, V_L)

$$h_{ij}^{\mathrm{TT}} = \frac{4G}{c^2 R} \mathcal{P}_{ijkl}(\boldsymbol{N}) \sum_{\ell=2}^{+\infty} \frac{1}{c^\ell \ell!} \left\{ N_{L-2}\, \mathrm{U}_{klL-2}(T_R) - \frac{2\ell}{c(\ell+1)} N_{aL-2}\, \varepsilon_{ab(k}\, \mathrm{V}_{l)bL-2}(T_R) \right\} + \mathcal{O}\left(\frac{1}{R^2}\right)$$



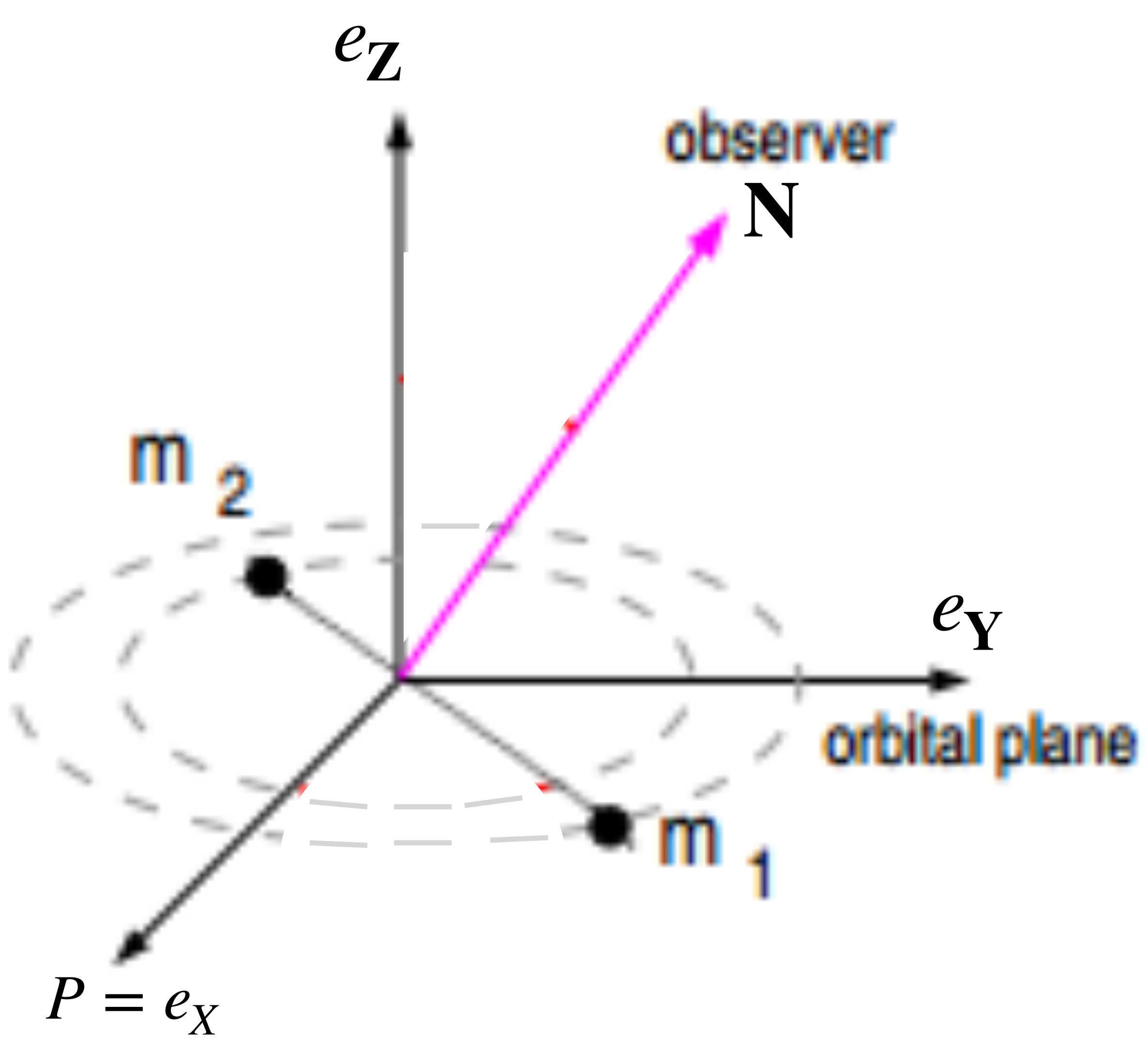
with :

- R : distance between the source and the observer
- **N** : direction of propagation of the GW
- $T_R = T - R/c$: retarded time
- $P_{ijkl} = P_{i(k}P_{l)j} - \frac{1}{2}P_{ij}P_{kl}$: TT projection operator
- $P_{ij} = \delta_{ij} - N_i N_j$

Waveform amplitude

The 2 GW propagation modes expressed in the orthonormal triad $(\mathbf{P}, \mathbf{Q}, \mathbf{N})$:

$$h_+ = \frac{1}{2}\left(P_i P_j - Q_i Q_j\right) h_{ij}^{\mathrm{TT}}$$
$$h_\times = \frac{1}{2}\left(P_i Q_j + Q_i P_j\right) h_{ij}^{\mathrm{TT}}$$



$h_+ - ih_\times$ decomposed in a spin-weighted spherical harmonics basis of weight -2 :

$$h \equiv h_+ - \mathrm{i}h_\times = \sum_{l=0}^{\infty} \sum_{m=-\ell}^{\ell} h_{\ell m} Y_{-2}^{\ell m}(\Theta, \Phi)$$

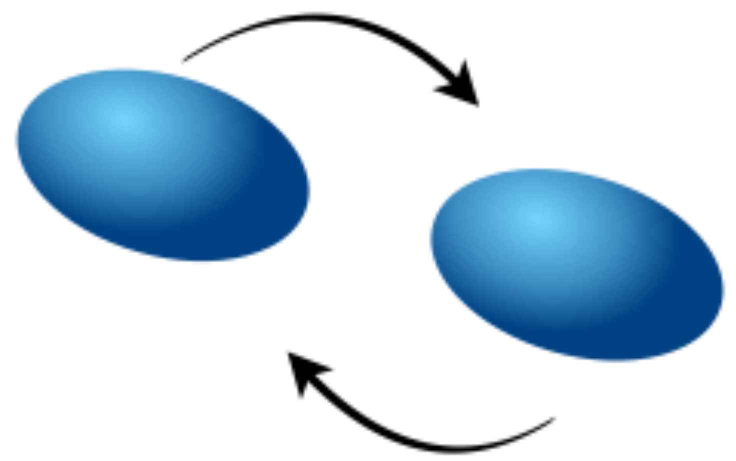
Amplitude modes h^{lm} computed directly from radiative moments :

$$h_{\ell m} = -\frac{2G}{Rc^{\ell+2}\ell!}\sqrt{\frac{(\ell+1)(\ell+2)}{\ell(\ell-1)}}\,\alpha_L^{\ell m}\left(U_L + \frac{2\ell}{\ell+1}\frac{\mathrm{i}}{c}V_L\right)$$

→ To get the full waveform amplitude at 2.5PN, we need to compute all the h^{lm} for $l \leq 7$ and $|m| \leq l$ at 2.5PN

Radiative moments

Precision of the radiative moments needed to get **the full GW amplitude to 2.5PN** :



Moments	U_{ij}	V_{ij} & U_{ijk}	V_{ijk} & U_{ijkl}	V_{ijkl} & U_{ijklm}	V_{ijklm} & U_{ijklmp}	V_{ijklmp} & $\mathrm{U}_{ijklmpq}$
Order	2.5PN	2PN	1.5PN	1PN	0.5PN	0PN

In comparison, for the computation of the flux (and orbital phase) to 2.5PN :

$$\mathcal{F} = \frac{G}{c^5}\left\{\frac{1}{5}U^{(1)}_{ij}U^{(1)}_{ij} + \frac{1}{c^2}\left[\frac{1}{189}U^{(1)}_{ijk}U^{(1)}_{ijk} + \frac{16}{45}V^{(1)}_{ij}V^{(1)}_{ij}\right] + \frac{1}{c^4}\left[\frac{1}{9072}U^{(1)}_{ijkm}U^{(1)}_{ijkm} + \frac{1}{84}V^{(1)}_{ijk}V^{(1)}_{ijk}\right] + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}$$

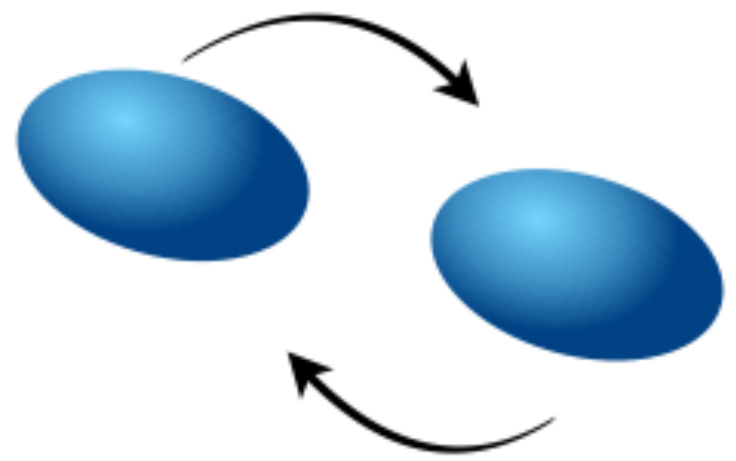
Moments	U_{ij}	V_{ij} & U_{ijk}	V_{ijk} & U_{ijkl}
Order	2.5PN	1.5PN	0.5PN

→ **More PN information is needed** to derive the modes at a given PN order than to derive the energy flux at that same order

Stress-energy tensor and potentials

Start from the matter action :

$$S_m = -\sum_{A=1,2} \int d\tau_A \left\{ m_A c^2 + \frac{\mu_A^{(2)}}{4} G^A_{\mu\nu} G^{\mu\nu}_A + \frac{\sigma_A^{(2)}}{6c^2} H^A_{\mu\nu} H^{\mu\nu}_A + \frac{\mu_A^{(3)}}{12} G^A_{\mu\nu\rho} G^{\mu\nu\rho}_A \right\}$$



In [Henry+20], they derived the stress-energy tensor :

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}}$$

We define the matter source densities : $\sigma \equiv \frac{T^{00} + T^{ii}}{c^2}$, $\sigma_i \equiv \frac{T^{0i}}{c}$ and $\sigma_{ij} \equiv T^{ij}$

The metric parametrized by PN potentiels $g_{\mu\nu} = g_{\mu\nu}\,[V, V_i, W_{ij}, R_i, X]$ satisfying wave equations sourced by $(\sigma\,, \sigma_i\,, \sigma_{ij})$:

$$\begin{aligned}
\Box V &= -4\pi G \sigma\,, \\
\Box V_i &= -4\pi G \sigma_i\,, \\
\Box \hat{W}_{ij} &= -4\pi G \big(\sigma_{ij} - \delta_{ij}\sigma_{kk}\big) - \partial_i V \partial_j V\,, \\
\Box \hat{R}_i &= -4\pi G \big(V \sigma_i - V_i \sigma\big) - 2\partial_k V \partial_i V_k - \frac{3}{2}\partial_t V \partial_i V\,, \\
\Box \hat{X} &= -4\pi G \sigma_{kk} + 2 V_i \partial_t \partial_i V + V \partial_t^2 V + \frac{3}{2}(\partial_t V)^2 - 2\partial_i V_j \partial_j V_i + \hat{W}_{ij}\partial^2_{ij} V
\end{aligned}$$

Matter source densities

[Henry+20]

(σ at 2PN , σ_i at 1PN , σ_{ij} at 0PN)

In this work, we need :

(σ at 2PN , σ_i at 2PN , σ_{ij} at 1PN)

0PN tidal effect

1PN tidal effect

2PN tidal effect

$$
\begin{aligned}
\sigma_{\text{tidal}} = &-\frac{1}{\sqrt{-g}}\partial_{ab}\bigg\{\delta_1\bigg(\mu_1^{(2)}\bigg[-\frac{1}{2}\hat{G}_{1ab}+\frac{1}{c^2}(-\frac{3}{4}\hat{G}_{1ab}v_1^2+\frac{3}{2}\hat{G}_{1ai}v_1^b v_1^i+\frac{1}{2}\hat{G}_{1ab}V) \\
&+\frac{1}{c^4}\bigg(-\frac{7}{16}\hat{G}_{1ab}v_1^4-\frac{1}{8}(\hat{G}_{1ij}v_1^i v_1^j)v_1^a v_1^b+\frac{7}{8}\hat{G}_{1ai}v_1^2 v_1^b v_1^i-\frac{1}{4}\hat{G}_{1ab}v_1^2 V+\frac{1}{2}\hat{G}_{1ai}V v_1^b v_1^i-\frac{1}{4}\hat{G}_{1ab}V^2 \\
&+2\hat{G}_{1ab}(v_1^i V_i)-2\hat{G}_{1ai}v_1^i V_b-2\hat{G}_{1ai}v_1^b V_i+\hat{G}_{1bi}\hat{W}_{ai}+\hat{G}_{1ai}\hat{W}_{bi}\bigg)\bigg] \\
&+\sigma_1^{(2)}\bigg(-\frac{4\varepsilon_{aij}\hat{H}_{1bj}v_1^i}{3c^2}+\frac{1}{c^4}\bigg(-\frac{2}{3}\varepsilon_{aij}\hat{H}_{1bj}v_1^2 v_1^i+\frac{2}{3}\varepsilon_{ajk}\hat{H}_{1ik}v_1^b v_1^i v_1^j+\frac{4}{3}\varepsilon_{aij}\hat{H}_{1bj}V v_1^i+\frac{8}{3}\varepsilon_{aij}\hat{H}_{1bj}V_i\bigg)\bigg)\bigg\} \\
&-\frac{1}{\sqrt{-g}}\bigg(\partial_t\partial_a\bigg\{\mu_1^{(2)}\delta_1\bigg[\frac{\hat{G}_{1ab}v_1^b}{c^2}+\frac{1}{c^4}\bigg(\frac{1}{2}(\hat{G}_{1ij}v_1^i v_1^j)v_1^a-\hat{G}_{1ab}V v_1^b\bigg)\bigg]\bigg\} \\
&+\partial_t\bigg\{\frac{\mu_1^{(2)}\delta_1}{c^4}\bigg((\hat{G}_{1ab}v_1^a\partial_b V)+2(\hat{G}_{1ab}\partial_b V_a)\bigg)\bigg\}\bigg) \\
&-\frac{1}{\sqrt{-g}}\partial_a\bigg\{\delta_1\bigg(\mu_1^{(2)}\bigg[-\frac{\hat{G}_{1ab}\partial_b V}{c^2}+\frac{1}{c^4}\bigg(\hat{G}_{1ab}v_1^b\partial_t V+\frac{7}{2}(\hat{G}_{1ij}v_1^i\partial_j V)v_1^a+\frac{7}{2}\hat{G}_{1ab}(v_1^i\partial_i V)v_1^b \\
&-2(\hat{G}_{1ij}v_1^i v_1^j)\partial_a V-\frac{7}{2}\hat{G}_{1ab}v_1^2\partial_b V-4\hat{G}_{1ab}\partial_t V_b+5\hat{G}_{1ab}V\partial_b V+4(\hat{G}_{1ij}\partial_j V_i)v_1^a+2\hat{G}_{1bi}v_1^b\partial_a V_i \\
&-8\hat{G}_{1ai}v_1^b\partial_b V_i-2\hat{G}_{1bi}v_1^b\partial_i V_a+4\hat{G}_{1ai}v_1^b\partial_i V_b+\hat{G}_{1bi}\partial_a\hat{W}_{bi}-2\hat{G}_{1bi}\partial_i\hat{W}_{ab}\bigg)\bigg] \\
&+\frac{\sigma_1^{(2)}}{c^4}\bigg(\frac{8}{3}\varepsilon_{bij}\hat{H}_{1aj}v_1^b\partial_i V-\frac{8}{3}\varepsilon_{abj}\hat{H}_{1ij}v_1^b\partial_i V-\frac{8}{3}\varepsilon_{bij}\hat{H}_{1aj}\partial_i V_b-\frac{8}{3}\varepsilon_{aij}\hat{H}_{1bj}\partial_i V_b\bigg)\bigg)\bigg\} \\
&+\delta_1\bigg(\mu_1^{(2)}\bigg[\frac{1}{c^2}\bigg(-(\hat{G}_{1ab}\partial_{ab}V)+\frac{3}{4}(\hat{G}_{1ab}\hat{G}_{1ab})\bigg)+\frac{1}{c^4}\bigg(2(\hat{G}_{1ab}v_1^a\partial_t\partial_b V)+(\hat{G}_{1ai}v_1^a v_1^b\partial_{ib}V) \\
&-\frac{3}{2}(\hat{G}_{1ab}\partial_{ab}V)v_1^2-4(\hat{G}_{1ab}\partial_t\partial_b V_a)+6(\hat{G}_{1ab}\partial_a V\partial_b V)+7(\hat{G}_{1ab}\partial_{ab}V)V-4(\hat{G}_{1bi}v_1^a\partial_{ia}V_b) \\
&+4(\hat{G}_{1bi}v_1^a\partial_{ib}V_a)+\frac{9}{8}(\hat{G}_{1ab}\hat{G}_{1ab})v_1^2-\frac{3}{4}(\hat{G}_{1ab}\hat{G}_{1ab})V\bigg)\bigg] \\
&+\frac{\sigma_1^{(2)}}{c^4}\bigg(-\frac{8}{3}(\varepsilon_{aij}\hat{H}_{1bi}v_1^a\partial_{jb}V)-\frac{16}{3}(\varepsilon_{bij}\hat{H}_{1ab}\partial_{ja}V_i)+\frac{1}{2}(\hat{H}_{1ab}\hat{H}_{1ab})\bigg)\bigg)-\frac{1}{\sqrt{-g}}\partial_t^2\bigg\{\frac{\mu_1^{(2)}\delta_1(\hat{G}_{1ab}v_1^a v_1^b)}{2c^4}\bigg\} \\
&-\frac{1}{\sqrt{-g}}\partial_{abi}\bigg\{\frac{1}{6}\mu_1^{(3)}\delta_1\hat{G}_{1abi}\bigg\}+1\leftrightarrow 2\,, \qquad \text{(B1a)}
\end{aligned}
$$

Source moments I_L and J_L

From the **PN-MPM formalism :**

→ The outer field is **PM-expanded** as $h^{\mu\nu} = Gh_1^{\mu\nu} + G^2 h_2^{\mu\nu} + \dots$

→ Assuming the **harmonic coordinate condition**, the linear field satisfies :

$$\Box h_1^{\mu\nu} = 0$$
$$\partial_\mu h_1^{\alpha\mu} = 0$$

→ The solution of this system can be written as a multipolar expansion of **2 STF sources moments** (I_L, J_L) and **some gauge moments** (W_L, X_L, Y_L, Z_L)

Canonical moments

$$h_1^{\mu\nu} \sim \sum_{l=0}^{+\infty} \partial_L \left(\frac{K^{\mu\nu}\left[I_L, J_L; W_L, X_L, Y_L, Z_L\right]}{r} \right) \sim \sum_{l=0}^{+\infty} \partial_L \left(\frac{K^{\mu\nu}\left[M_L, S_L\right]}{r} \right) \sim \sum_{l=0}^{+\infty} \partial_L \left(\frac{K^{\mu\nu}\left[I_L, J_L\right]}{r} \right)$$

Only in this work

→ The explicit formula for I_L and J_L is obtained by matching to the inner field that is PN-expanded

Source moments I_L and J_L

From the **PN-MPM formalism**, the STF source multipole moments I_L (**mass-type**) and J_L (**current-type**) given at any PN order by ($l \geq 2$) :

$$I_L(u) = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x}\, \tilde{r}^B \int_{-1}^{1} \mathrm{d}z \Big[\delta_\ell(z) \hat{x}_L \Sigma - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)} \delta_{\ell+1}(z) \hat{x}_{iL} \Sigma_i^{(1)} + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)} \delta_{\ell+2}(z) \hat{x}_{ijL} \Sigma_{ij}^{(2)} \Big](\mathbf{x}, u + zr/c)\,,$$

$$J_L(u) = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x}\, \tilde{r}^B \int_{-1}^{1} \mathrm{d}z\, \varepsilon_{ab\langle i_\ell} \Big[\delta_\ell(z) \hat{x}_{L-1\rangle a} \Sigma_b - \frac{2\ell+1}{c^2(\ell+2)(2\ell+3)} \delta_{\ell+1}(z) \hat{x}_{L-1\rangle ac} \Sigma_{bc}^{(1)} \Big](\mathbf{x}, u + zr/c)$$

→ The source terms Σ, Σ_i and Σ_{ij} contain the **matter source densities** (σ, σ_i, σ_{ij}) as well the **PN potentials** (V, V_i, W_{ij}, R_i, X)

→ The integrations over z are transformed into infinite PN series:

$$\int_{-1}^{1} \mathrm{d}z\; \delta_\ell(z)\, \Sigma(\mathbf{x}, t + zr/c) = \sum_{k=0}^{+\infty} \frac{(2\ell+1)!!}{(2k)!!(2\ell+2k+1)!!} \left(\frac{r}{c}\right)^{2k} \Sigma^{(2k)}(\mathbf{x}, t)$$

→ The **finite part (FP) regularization** is there to cure the IR divergences at spatial infinity

Source moments I_L and J_L

Source mass-type quadrupole at 2.5PN

→ Function of $(y_1^i, y_2^i, v_1^i, v_2^i)$

→ Reduce to the COM frame and to quasi circular orbits with $\gamma = \dfrac{GM}{rc^2}$

0PN p.p + tidal effect

1PN p.p + tidal effect

2PN p.p + tidal effect

$$
\begin{aligned}
\mathrm{I}_{ij} = M r^2 \Bigg[& n^{\langle i}n^{j\rangle}\Bigg\{\nu\bigg[1+\left(-\frac{1}{42}-\frac{13}{14}\nu\right)\gamma+\left(-\frac{461}{1512}-\frac{18395}{1512}\nu-\frac{241}{1512}\nu^2\right)\gamma^2\bigg] \\
&+\left(3\widetilde{\mu}_+^{(2)}+3\delta\,\widetilde{\mu}_-^{(2)}\right)\gamma^5+\bigg[\widetilde{\mu}_+^{(2)}\left(-\frac{3}{2}+\frac{1}{7}\nu-\frac{222}{7}\nu^2\right)+\delta\,\widetilde{\mu}_-^{(2)}\left(-\frac{3}{2}-\frac{67}{7}\nu\right)+\frac{160}{3}\nu\widetilde{\sigma}_+^{(2)}\bigg]\gamma^6 \\
&+\bigg[\widetilde{\mu}_+^{(2)}\left(\frac{871}{56}-\frac{1613}{168}\nu-\frac{17237}{168}\nu^2+\frac{929}{42}\nu^3\right)+\delta\,\widetilde{\mu}_-^{(2)}\left(\frac{871}{56}+\frac{1493}{24}\nu-\frac{7201}{168}\nu^2\right)+\widetilde{\sigma}_+^{(2)}\left(\frac{388}{9}\nu-\frac{2504}{7}\nu^2\right) \\
&+\frac{1732}{63}\delta\nu\widetilde{\sigma}_-^{(2)}\bigg]\gamma^7\Bigg\}+\lambda^{\langle i}\lambda^{j\rangle}\Bigg\{\nu\bigg[\left(\frac{11}{21}-\frac{11}{7}\nu\right)\gamma+\left(\frac{1013}{378}+\frac{299}{378}\nu-\frac{365}{378}\nu^2\right)\gamma^2\bigg]+\bigg[\widetilde{\mu}_+^{(2)}\left(3+\frac{104}{7}\nu-\frac{198}{7}\nu^2\right) \\
&+\delta\,\widetilde{\mu}_-^{(2)}\left(3-\frac{38}{7}\nu\right)+\frac{128}{3}\nu\widetilde{\sigma}_+^{(2)}\bigg]\gamma^6+\bigg[\widetilde{\mu}_+^{(2)}\left(-\frac{19}{2}+\frac{617}{42}\nu+\frac{5039}{42}\nu^2+\frac{260}{21}\nu^3\right) \\
&+\delta\,\widetilde{\mu}_-^{(2)}\left(-\frac{19}{2}+\frac{1291}{42}\nu-\frac{1649}{42}\nu^2\right)+\widetilde{\sigma}_+^{(2)}\left(-\frac{64}{9}\nu-\frac{1696}{7}\nu^2\right)+\frac{2048}{63}\delta\nu\widetilde{\sigma}_-^{(2)}\bigg]\gamma^7\Bigg\} \\
&+n^{\langle i}\lambda^{j\rangle}\Bigg\{\frac{48}{7}\nu^2\gamma^{5/2}+\bigg[\widetilde{\mu}_+^{(2)}\left(-\frac{64}{5}+\frac{2336}{35}\nu+\frac{1296}{7}\nu^2\right)+\delta\,\widetilde{\mu}_-^{(2)}\left(-\frac{64}{5}+\frac{288}{7}\nu\right)\bigg]\gamma^{15/2}\Bigg\}\Bigg], \qquad (4.18\text{a})
\end{aligned}
$$

2.5PN p.p + tidal effect

Radiative moments U_L and V_L

The MPM algorithm relates the radiative moments (U_L, V_L) to the canonical moments (M_L, S_L)

In this work, $(M_L, S_L) \rightarrow (I_L, J_L)$

Taking the exemple of the mass quadrupole at 2.5PN:

$$U_{ij} = \overset{(2)}{M}_{ij} + \frac{2G\mathcal{M}}{c^3}\int_0^{\infty} \mathrm{d}\tau \left[\ln\left(\frac{\tau}{2b_0}\right) + \frac{11}{12}\right] \mathrm{M}_L^{(4)}(t-\tau) \quad \textbf{Tails effects}$$

$$- \frac{2G}{7c^5}\int_0^{\infty} \mathrm{d}\tau\, \mathrm{M}^{(3)}_{a\langle i}(t-\tau)\mathrm{M}^{(3)}_{j\rangle a}(t-\tau) \quad \textbf{Non-linear memory effects}$$

$$+ \frac{G}{7c^5}\left[\mathrm{M}^{(5)}_{a\langle i}\mathrm{M}_{j\rangle a} - 5\mathrm{M}^{(4)}_{a\langle i}\mathrm{M}^{(1)}_{j\rangle a} - 2\mathrm{M}^{(3)}_{a\langle i}\mathrm{M}^{(2)}_{j\rangle a} + \frac{7}{3}\epsilon_{ab\langle i}\mathrm{M}^{(4)}_{j\rangle a}\mathrm{S}_b\right] + \mathcal{O}\left(\frac{1}{c^6}, \frac{\epsilon_{tidal}}{c^6}\right) \quad \textbf{Instantaneous effects}$$

- Tail effects: GW are backscattered on the spacetime curvature generated by the mass monopole I
- Memory effects: GW radiated by the GW themselves

→ The non-linear propagation effects are only **quadratic**: $M \times M_{ij}$ (tails) and $M_{ij} \times M_{ij}$ (memory effects)

Radiative moments U_L and V_L

The relations required to derive the full waveform amplitude to 2.5PN are:

$$U_{ij} = \overset{(2)}{I_{ij}} + U_{ij}^{tail} + U_{ij}^{inst} + U_{ij}^{mem}$$
$$U_{ijk} = \overset{(3)}{I_{ijk}} + U_{ijk}^{tail}$$
$$U_{ijkl} = \overset{(4)}{I_{ijkl}} + U_{ijkl}^{tail} + U_{ijkl}^{inst} + U_{ijkl}^{mem}$$
$$V_{ij} = \overset{(2)}{J_{ij}} + V_{ij}^{tail}$$
$$V_{ijk} = \overset{(3)}{J_{ijk}} + V_{ijk}^{tail} + V_{ijk}^{inst}$$

For the rest of radiative moments, we just have :

$$U_L = \overset{(l)}{I_L} \ , \ V_L = \overset{(l)}{J_L}$$

→ These relations already well-know

→ We included the tidal contributions consistently with the precision required for each radiative moment

Amplitude modes: PN expanded form

orbital phase — orbital frequency

$$h_{\ell m} = \frac{8GM\nu x}{Rc^2}\sqrt{\frac{\pi}{5}}\left(\hat{H}^{\rm pp}_{\ell m} + x^5\,\hat{H}^{\rm tidal}_{\ell m}\right) e^{-im\psi} \qquad \text{with} \qquad \psi \equiv \phi - \frac{2G\mathcal{M}\omega}{c^3}\ln\left(\frac{\omega}{\omega_0}\right)$$

We computed the $\hat{H}^{lm}$ for $l \leq 7$ and $|m| \leq l$ up to the **relative 2.5PN order**.

The **dominant mode** is the **(2,2) mode**:

0PN tidal effect; 1PN tidal effect; 1.5PN tidal effect; 2PN tidal effect; 2.5PN tidal effect

$$\begin{aligned}\hat{H}^{\rm tidal}_{22} = \frac{1}{\nu}\Bigg\{ & \widetilde{\mu}^{(2)}_+(3+12\nu) + 3\delta\,\widetilde{\mu}^{(2)}_- + \left[\widetilde{\mu}^{(2)}_+\left(\frac{9}{2} - 20\nu + \frac{45}{7}\nu^2\right) + \delta\,\widetilde{\mu}^{(2)}_-\left(\frac{9}{2} + \frac{125}{7}\nu\right) + \frac{224}{3}\nu\,\widetilde{\sigma}^{(2)}_+\right]x \\ & + 6\pi\left[\widetilde{\mu}^{(2)}_+(1+4\nu) + \delta\,\widetilde{\mu}^{(2)}_-\right]x^{3/2} + \left[\widetilde{\mu}^{(2)}_+\left(\frac{1403}{56} - \frac{9227}{168}\nu - \frac{19367}{168}\nu^2 - \frac{274}{21}\nu^3\right)\right. \\ & \left. + \delta\,\widetilde{\mu}^{(2)}_-\left(\frac{1403}{56} + \frac{887}{56}\nu + \frac{103}{24}\nu^2\right) + \widetilde{\sigma}^{(2)}_+\left(\frac{11132}{63}\nu - \frac{6536}{63}\nu^2\right) + \frac{8084}{63}\delta\,\nu\,\widetilde{\sigma}^{(2)}_- + 80\nu\,\widetilde{\mu}^{(3)}_+\right]x^2 \\ & + \left[\widetilde{\mu}^{(2)}_+\left(\frac{\mathrm{i}}{5}(64 - 108\nu - 8640\nu^2) + \frac{\pi}{7}(63 - 301\nu + 132\nu^2)\right)\right. \\ & \left. + \delta\,\widetilde{\mu}^{(2)}_-\left(\frac{\mathrm{i}}{5}(64+20\nu) + \frac{\pi}{7}(63+229\nu)\right) + \frac{448}{3}\pi\,\nu\,\widetilde{\sigma}^{(2)}_+\right]x^{5/2}\Bigg\},\end{aligned}$$

Amplitude modes: EOB-factorized form

In EOB waveform models, there is a freedom on the choice of resumming the waveform modes

→ historical choice to **lower the mismatch** with Numerical Relativity

Modes factorized in **5 blocks**:

$$h^{\rm F}_{\ell m} = h^{\rm N}_{\ell m}\, \hat{S}_{\rm eff}\, T_{\ell m}\, f_{\ell m}\, e^{{\rm i}\delta_{\ell m}}$$

- h^N_{lm} : the leading order PN contribution
- $\hat{S}_{eff}$: the effective source term
- T_{lm}
- f_{lm} : the remaining amplitude
- δ_{lm} : the residual phase

Related to the ADM mass

$$\hat{S}_{\rm eff} = \begin{cases} \frac{H_{\rm eff}(x)}{M\nu c^2} & \text{for } \ell + m \text{ even} \\ J(x) & \text{for } \ell + m \text{ odd} \end{cases}$$

$$T_{\ell m} = \frac{\Gamma\left(\ell + 1 - 2{\rm i}\hat{k}\right)}{\Gamma(\ell+1)} e^{\pi \hat{k}} e^{2{\rm i}\hat{k}\ln(2m\omega b_0)}$$

$$\hat{k} \equiv m\frac{G\mathcal{M}\omega}{c^3}$$

→ h^F_{lm} coincides with the PN-expanded modes

Amplitude modes: EOB-factorized form

The dominant (2,2) mode has a **remaining amplitude** :

$$f_{\ell m} = f_{\ell m}^{\rm pp} + x^5 f_{\ell m}^{\rm tidal}$$

0PN tidal effect **1PN tidal effect**

$$\begin{aligned} f_{22}^{\rm tidal} = \frac{1}{\nu}\Bigg\{ & \widetilde{\mu}_+^{(2)}(3+12\nu) + 3\delta\,\widetilde{\mu}_-^{(2)} + \left[\widetilde{\mu}_+^{(2)}\left(6 - 23\nu + \frac{45}{7}\nu^2\right) + \delta\,\widetilde{\mu}_-^{(2)}\left(6 + \frac{125}{7}\nu\right) + \frac{224}{3}\nu\widetilde{\sigma}_+^{(2)}\right]x \\ & + \left[\widetilde{\mu}_+^{(2)}\left(\frac{377}{14} - \frac{13985}{168}\nu - \frac{17615}{168}\nu^2 - \frac{274}{21}\nu^3\right) + \delta\,\widetilde{\mu}_-^{(2)}\left(\frac{377}{14} + \frac{589}{56}\nu + \frac{103}{24}\nu^2\right) + \widetilde{\sigma}_+^{(2)}\left(\frac{7940}{63}\nu - \frac{6536}{63}\nu^2\right) \right. \\ & \left. + \frac{8084}{63}\delta\,\nu\,\widetilde{\sigma}_-^{(2)} + 80\nu\,\widetilde{\mu}_+^{(3)}\right]x^2 \Bigg\} + \mathcal{O}\left(\frac{\epsilon_{\rm tidal}}{c^6}\right), \qquad (5.8) \end{aligned}$$

2PN tidal effect

And **residual phase** :

$$\delta_{22} = \frac{7}{3}x^{3/2} - \frac{151}{6}\nu x^{5/2} + \frac{64}{5\nu}\left[\widetilde{\mu}_+^{(2)}\left(1 + \frac{63}{16}\nu - \frac{7095}{64}\nu^2\right) + \delta\,\widetilde{\mu}_-^{(2)}\left(1 + \frac{95}{16}\nu\right)\right]x^{15/2} + \mathcal{O}\left(\frac{1}{c^6}, \frac{\epsilon_{\rm tidal}}{c^6}\right)$$

1.5PN p.p **2.5PN p.p + tidal effect**

Conclusion

- We computed the full waveform amplitude including tidal effects up to 2.5PN consistently with the precision of the orbital phase

→ **Results will soon be available on arXiv !**

- Outlook

  - Improve the modeling of **physical effects** : mixed tidal-EM effects in GR …

  - Study the effects of **dynamic tides** on the dynamics and the waveform