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SECOND ORDER GRAVITATIONAL WAVES: PAVING THE WAY FOR A FULL CALCULATION

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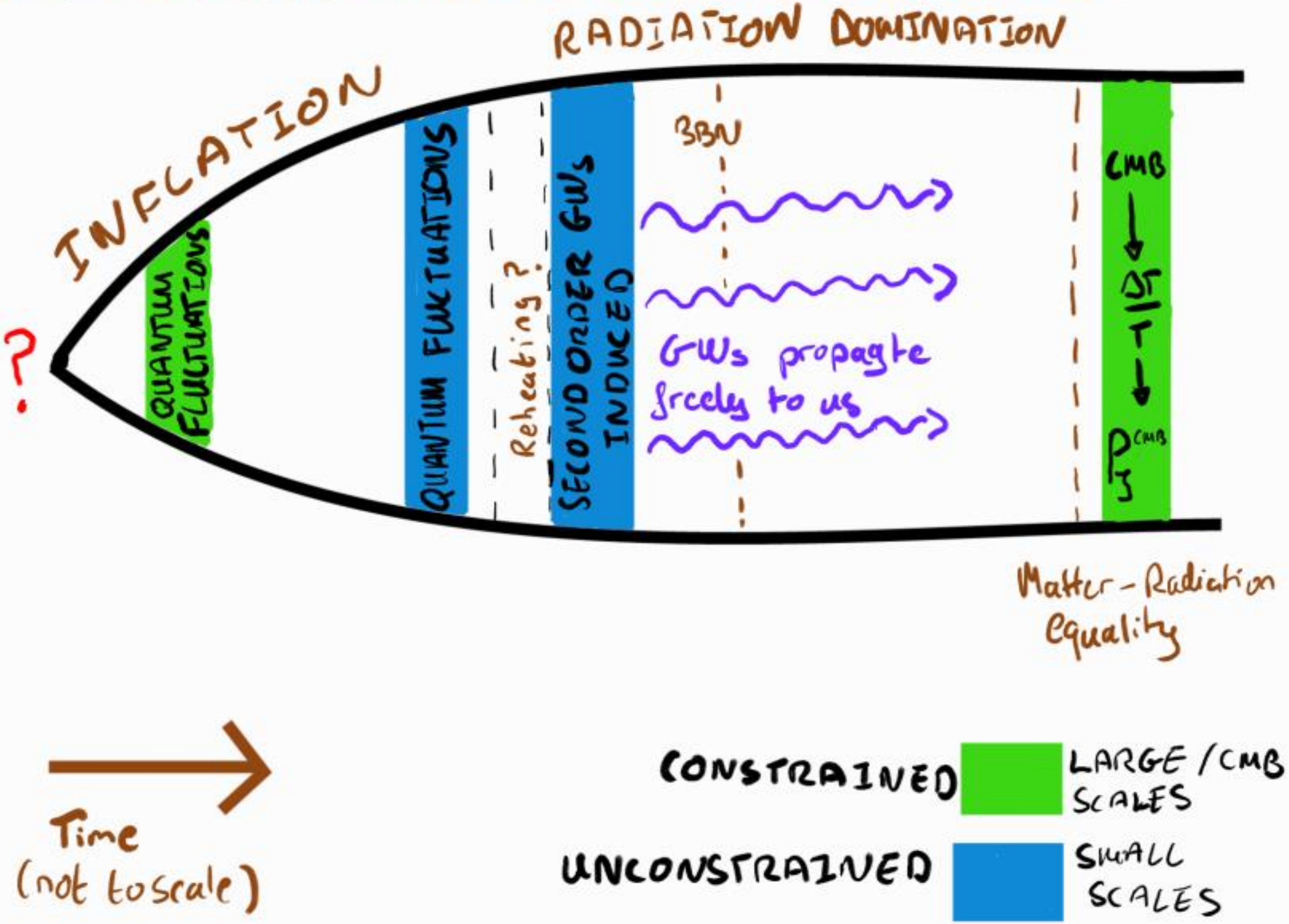
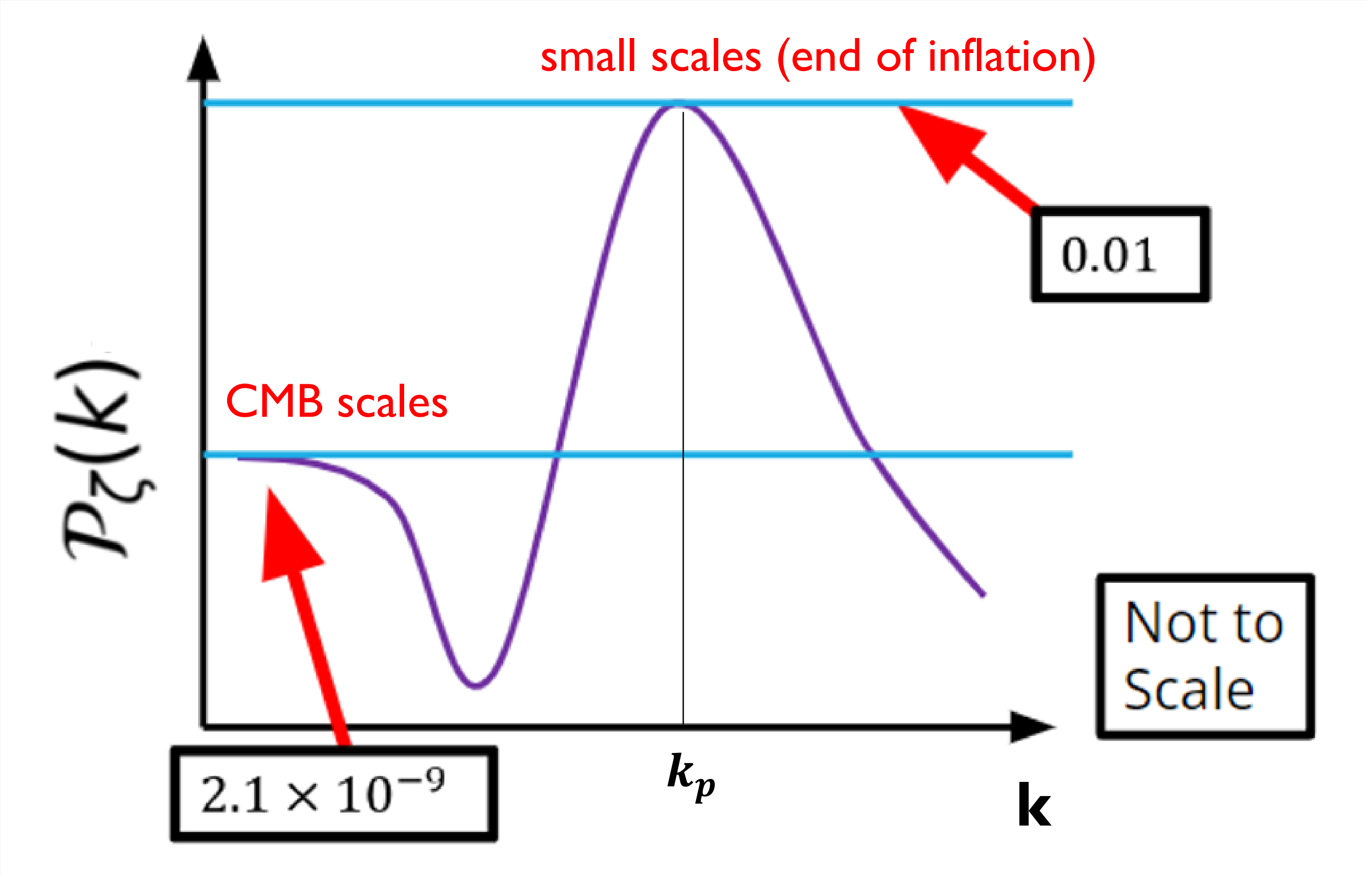
Work based on **2311.14513** with **K.A. Malik**
and 24XX.XXXXX with **L.E. PADILLA AND K.A. MALIK**

Science and Technology Facilities Council

The early universe (NOT TO SCALE)



THE POWER SPECTRUM ON SMALL SCALES



(M. Davies)

SET UP

Metric

$$g_{00} = -a^2(\eta)\left(1 + 2\Psi^{(1)}\right)$$

$$g_{0i} = 0 = g_{i0}$$

$$g_{ij} = a^2(\eta)\left(\left(1 - 2\Psi^{(1)}\right)\delta_{ij} + \bar{\boldsymbol{h}}_{\boldsymbol{ij}}^{(\boldsymbol{2})}\right)$$

$$\Lambda_{ab}^{ij} G_{ij}^{(2)} = \frac{1}{M_{Pl}^2} \Lambda_{ab}^{ij} T_{ij}^{(2)}$$

Matter

$$T_{ij} = (\rho + P)u_i u_j + P g_{ij}$$

SCALAR INDUCED GRAVITATIONAL WAVES

- First order scalar perturbations couple and source GWs at second order in perturbation theory (*Tomita*, *Ananda et al*, *Baumann et al*)

- GW equation of motion:

$$\boldsymbol{h_{ab}^{\prime\prime(2)} + 2\mathcal{H}h_{ab}^{\prime(2)} - \nabla^2 h_{ab}^{(2)} = \Lambda_{ab}^{ij} S_{ij}^{ss}}$$

$$S_{ij}^{ss} = \frac{8}{3(1+w)}\left[\left(\partial_i\Psi + \frac{\partial_i\Psi'}{\mathcal{H}}\right)\left(\partial_j\Psi + \frac{\partial_j\Psi'}{\mathcal{H}}\right)\right] + 4\partial_i\Psi\partial_j\Psi$$

SCALAR INDUCED GRAVITATIONAL WAVES

- Induced during radiation domination

- They are directly 'sourced' by inflation and therefore a probe of the smallest scales and counterpart signal of PBHs

- Solution in Fourier space

$$h_{ss}^{\lambda(2)}(\eta, \boldsymbol{k}) = 4 \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} q^{\lambda,ss}(\boldsymbol{k}, \boldsymbol{p}) I_{ss}(\eta, k\ , p) \zeta_{\boldsymbol{p}} \zeta_{\boldsymbol{k}-\boldsymbol{p}}$$

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Polarization

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↓

Kernel

SCALAR INDUCED GRAVITATIONAL WAVES

- Induced during radiation domination
- They are directly 'sourced' by inflation and therefore a probe of the smallest scales and counterpart signal of PBHs
- Solution in Fourier space

$$h_{ss}^{\lambda(2)}(\eta,\boldsymbol{k}) = 4\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} q^{\lambda,ss}(\boldsymbol{k},\boldsymbol{p}) I_{ss}(\eta,k\,,p)\zeta_{\boldsymbol{p}}\zeta_{\boldsymbol{k}-\boldsymbol{p}}$$

Primordial values

POWER SPECTRUM / SPECTRAL DENSITY

$$\left\langle \mathrm{h}_{\lambda}^{(n)}(\eta, \boldsymbol{k}) h_{\lambda'}^{(m)}(\eta, \boldsymbol{k}') \right\rangle = \delta^{\lambda\lambda'} \delta^{(3)}(\boldsymbol{k} + \boldsymbol{k}') \frac{2\pi^2}{k^3} \mathcal{P}_{\lambda}^{(nm)}(\eta, k)$$

Plug in equation of motion

$$\Omega(\eta, k) \sim \overline{\mathcal{P}_{\lambda}^{(nm)}(\eta, k)}$$

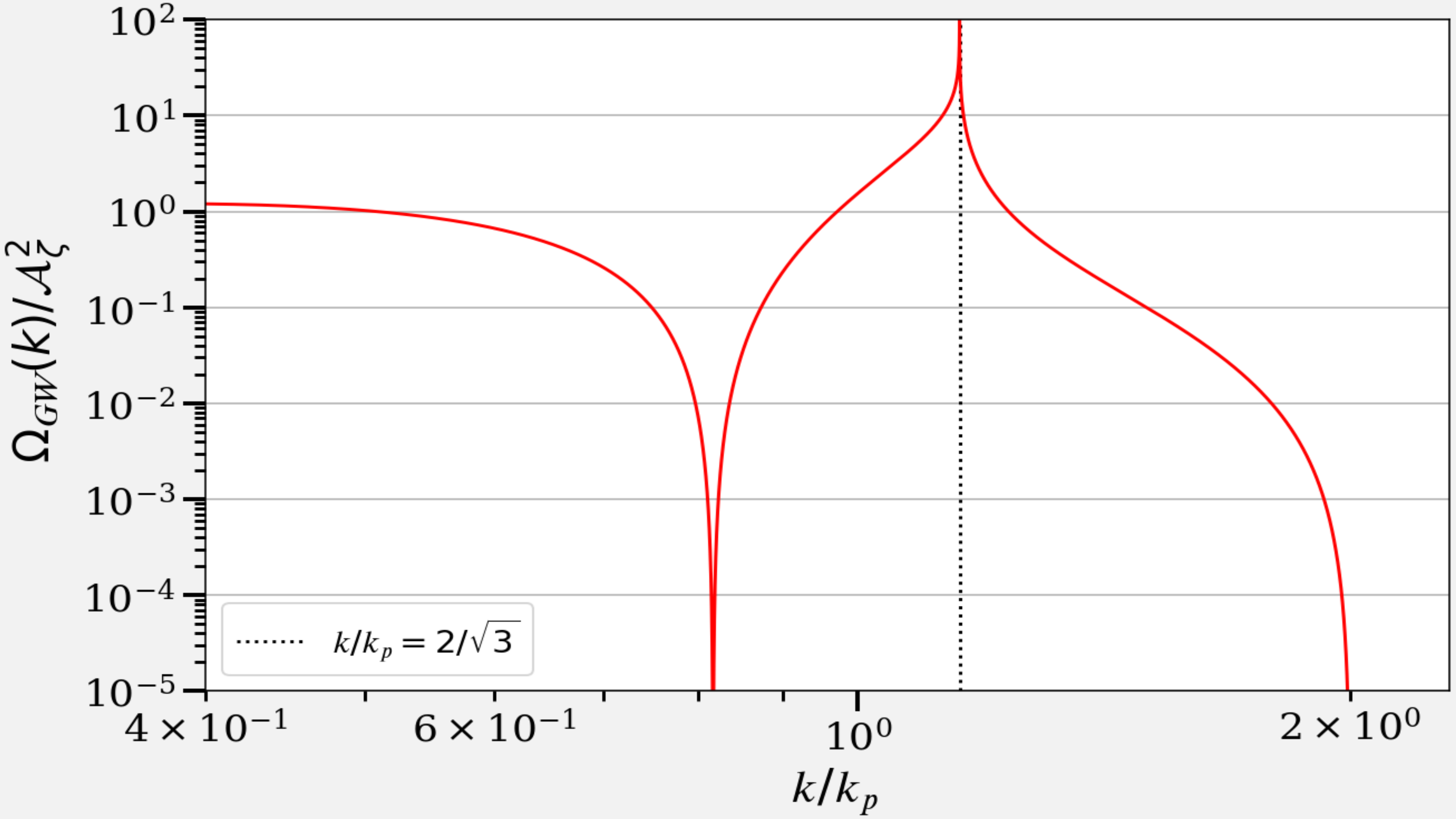
SCALAR **I**NDUCED **G**RAVITATIONAL **W**AVES

- The two-point function of second order tensors is proportional to the four-point function of scalars

$$\boldsymbol{\mathcal{P}_{\lambda}^{(22)}(\eta, k) \sim \left\langle h_{\lambda}^{(2)}(\eta, k) h_{\lambda'}^{(2)}(\eta, k') \right\rangle \propto \mathcal{P}_{\zeta}^{2}}$$

Scalar Induced Gravitational Waves

- What happens for some peaked input power spectra on small scales? (Dirac delta input peak)



INCLUSION OF FIRST ORDER TENSORS

Metric

$$g_{00} = -a^2(\eta)\left(1 + 2\Psi^{(1)}\right)$$

$$g_{0i} = 0 = g_{i0}$$

$$g_{ij} = a^2(\eta)\left(\left(1 - 2\Psi^{(1)}\right)\delta_{ij} + 2\bar{h}_{ij}^{(1)} + \bar{\boldsymbol{h}}_{\boldsymbol{ij}}^{(\mathbf{2})}\right)$$

$$\Lambda_{ab}^{ij} G_{ij}^{(2)} = \frac{1}{M_{Pl}^2} \Lambda_{ab}^{ij} T_{ij}^{(2)}$$

Matter

$$T_{ij} = (\rho + P)u_i u_j + P g_{ij}$$

SECOND ORDER GRAVITATIONAL WAVES

- First order **scalar and tensor** perturbations couple and source GWs as they re-enter the horizon during radiation domination
- GW equation (*Zhang et al*, *Bari et al*, *Yu et al*)

$$\boldsymbol{h''^{(2)}_{ab} + 2\mathcal{H}h'^{(2)}_{ab} - \nabla^2 h^{(2)}_{ab} = \Lambda^{ij}_{ab} S_{ij}}$$

$$S^{ss}_{ij} = \frac{8}{3(1+w)}\left[\left(\partial_i\Psi + \frac{\partial_i\Psi'}{\mathcal{H}}\right)\left(\partial_j\Psi + \frac{\partial_j\Psi'}{\mathcal{H}}\right)\right] + 4\partial_i\Psi\partial_j\Psi$$

$$S^{tt}_{ij} = -4h^{cd}\partial_c\partial_d h_{ij} + 4\partial_d h_{jc}\partial^c h^d_i - 4\partial_d h_{jc}\partial^d h^c_i + 8h^{dc}\partial_i\partial_c h_{jd} + 4h^{c\prime}_i h'_{jc} + 2\partial_i h^{cd}\partial_j h_{cd}$$

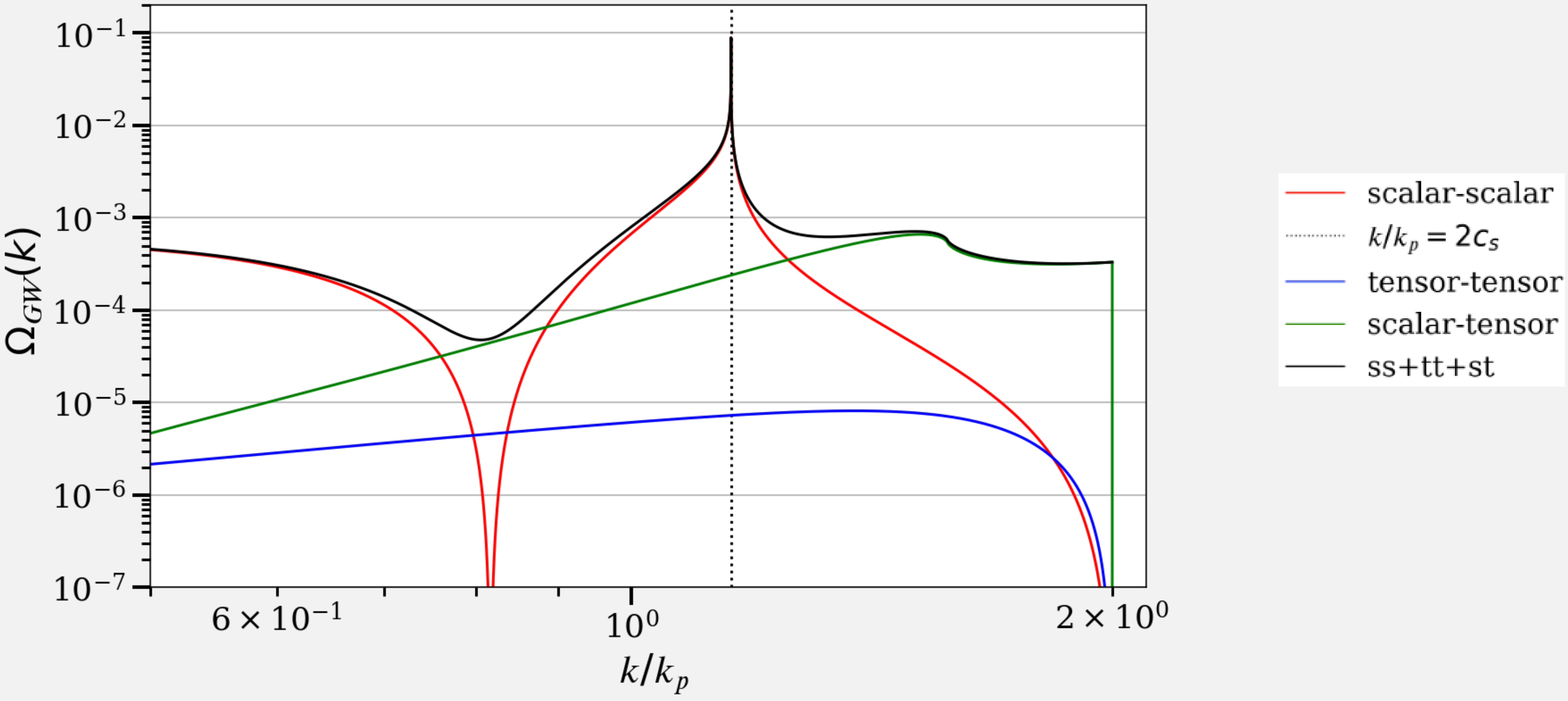
$$S^{st}_{ij} = 8\Psi\nabla^2 h_{ij} + 8\partial_c h_{ij}\partial^c\Psi + 4h_{ij}(\mathcal{H}(1+3c_s^2)\Psi' + (1-c_s^2)\nabla^2\Psi)$$

SECOND ORDER GRAVITATIONAL WAVES

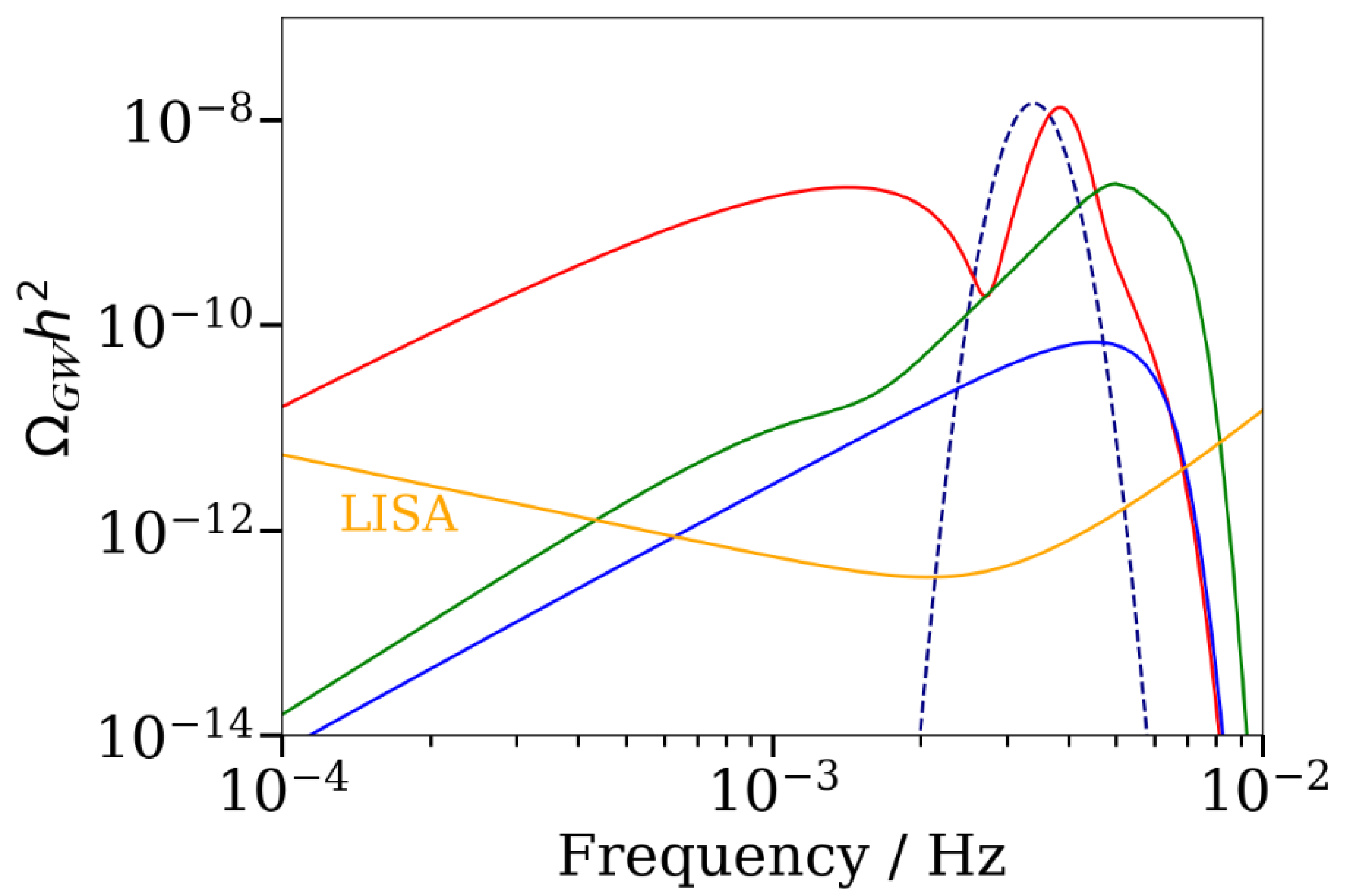
- First order **scalar and tensor** perturbations couple and source GWs as they re-enter the horizon during radiation domination
- They are directly 'sourced' by inflation and therefore give us information about the scalar **and** tensor power spectrum on smallest scales

$$\mathcal{P}_{\lambda}^{(22)}(\eta, k) \sim \left\langle h_{\lambda}^{(2)}(\eta, k) h_{\lambda'}^{(2)}(\eta, k') \right\rangle \propto \mathcal{P}_{\zeta}^{2} + \mathcal{P}_{h}^{2} + \mathcal{P}_{h}\, \mathcal{P}_{\zeta}$$

RESULTS: DIRAC DELTA INPUT

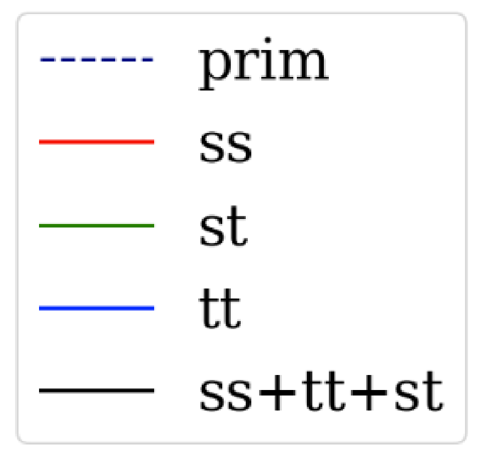


RESULTS: LOGNORMAL PEAK

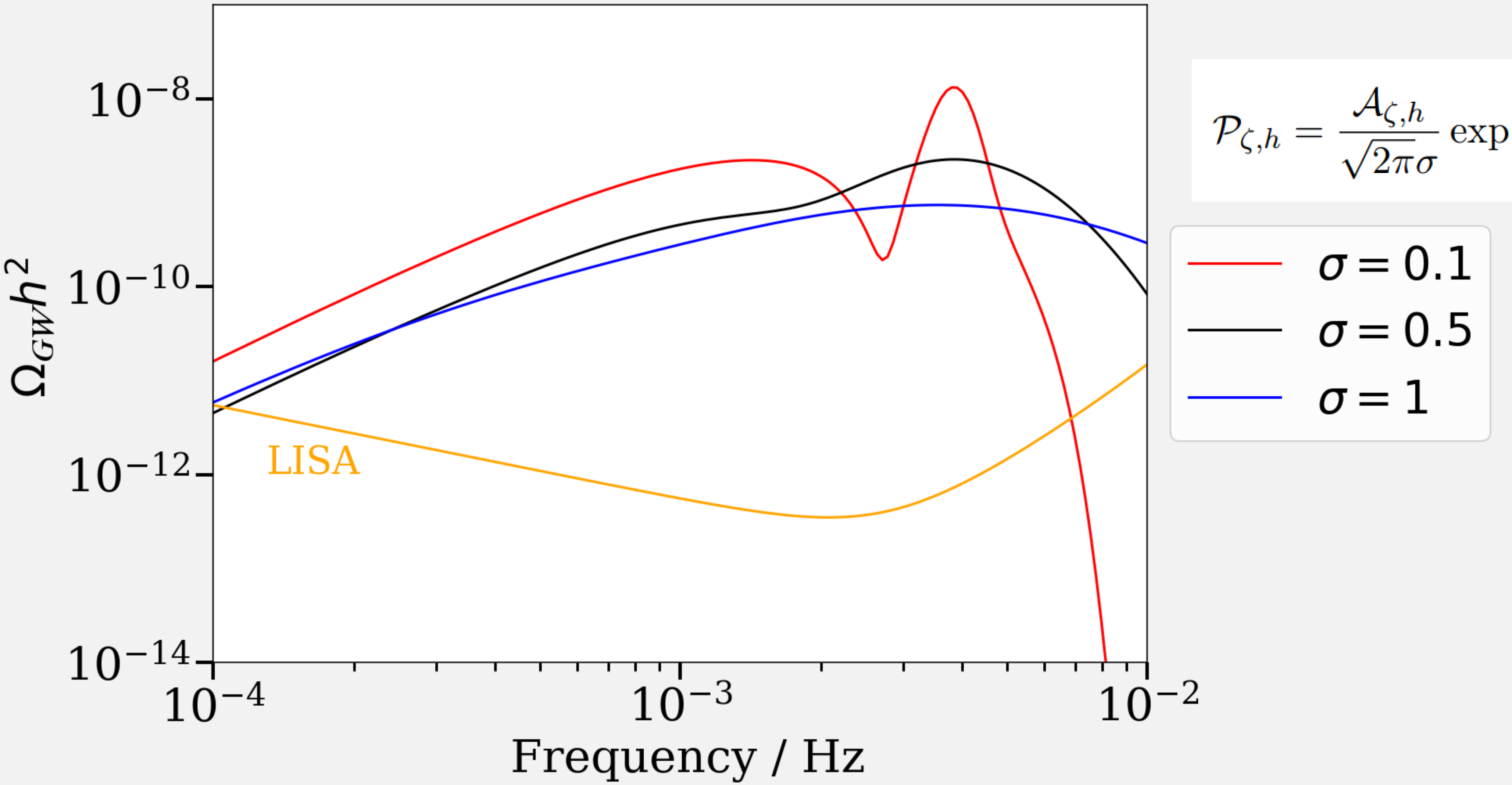


$$\mathcal{P}_{\zeta,h} = \frac{\mathcal{A}_{\zeta,h}}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\log^2(k/k_{\zeta,h})}{2\sigma^2}\right)$$

- $k_\zeta = k_h \sim k_{LISA}$
- $A_\zeta = 10A_h = 0.01$
- $\sigma = 0.1$

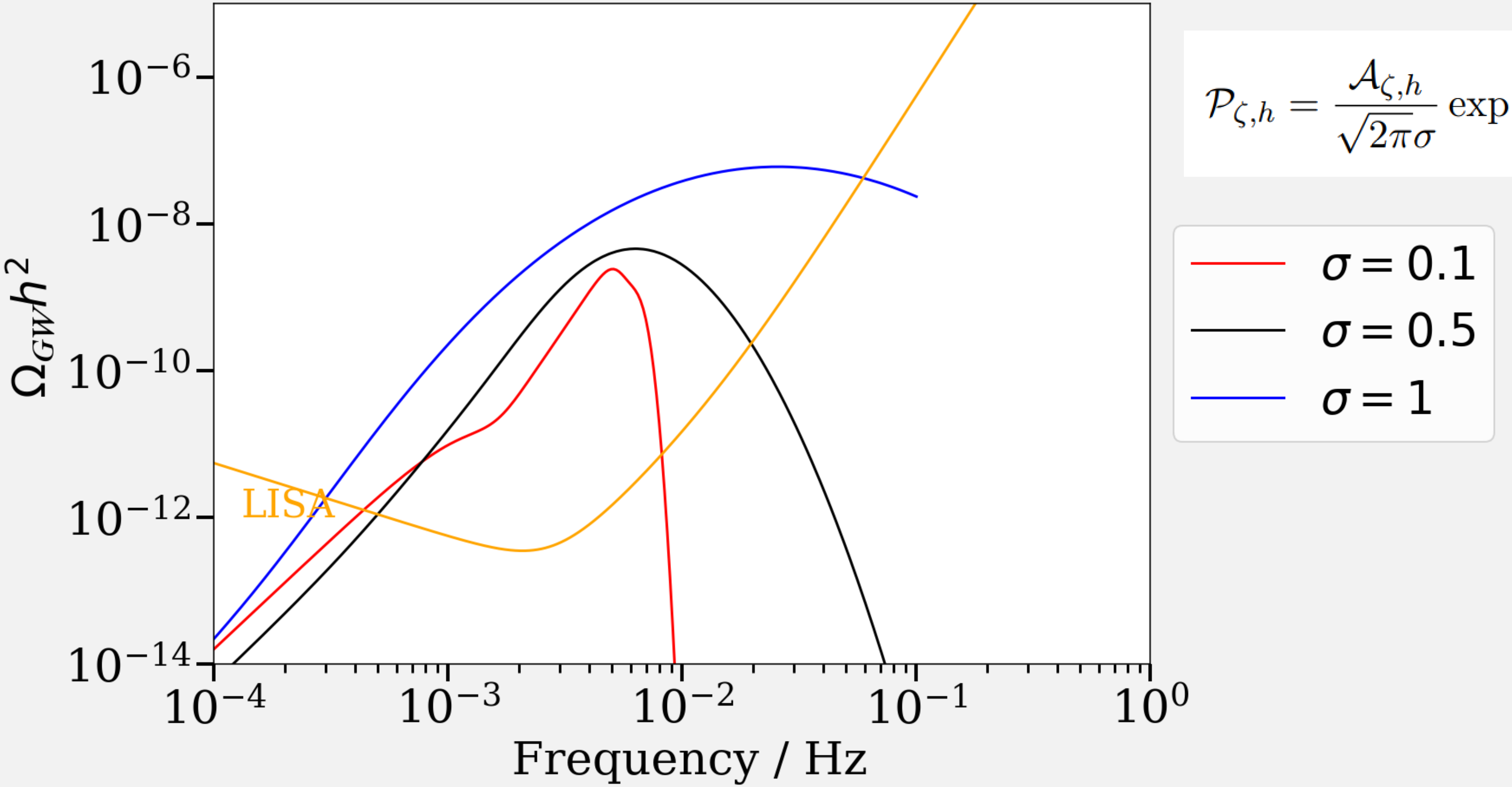


VARYING SIGMA - SIGW



$$\mathcal{P}_{\zeta,h} = \frac{\mathcal{A}_{\zeta,h}}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\log^2(k/k_{\zeta,h})}{2\sigma^2}\right)$$

ENHANCEMENT OF THE ST TERM?



$$\mathcal{P}_{\zeta,h} = \frac{\mathcal{A}_{\zeta,h}}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\log^2(k/k_{\zeta,h})}{2\sigma^2}\right)$$

ARE WE MISSING SOMETHING?

$$\Omega(\eta, k) \sim \overline{\mathcal{P}_{\lambda}^{(22)}(\eta, k)}$$

ARE WE MISSING SOMETHING? YES!

$$\Omega(\eta, k) \sim \overline{\mathcal{P}_\lambda^{(22)}(\eta, k)} + 2\overline{\mathcal{P}_\lambda^{(13)}(\eta, k)}$$

THIRD ORDER PERTURBATION THEORY

Metric

$$g_{00} = -a^2(\eta)\left(1 + 2\Psi^{(1)} + \Phi^{(2)}\right)$$

$$g_{0i} = a^2(\eta)\frac{1}{2}B_i^{(2)} = g_{i0}$$

$$g_{ij} = a^2(\eta)\left(\left(1 - 2\Psi^{(1)} - 2\Psi^{(2)}\right)\delta_{ij} + 2\overline{h}_{ij}^{(1)} + \overline{h}_{ij}^{(2)} + \frac{1}{3}\overline{\boldsymbol{h}}_{\boldsymbol{ij}}^{(\mathbf{3})}\right)$$

Matter

$$T_{ij} = (\rho + P)u_i u_j + P g_{ij}$$

$$\Lambda_{ab}^{ij} G_{ij}^{(3)} = \frac{1}{M_{Pl}^2}\Lambda_{ab}^{ij} T_{ij}^{(3)}$$

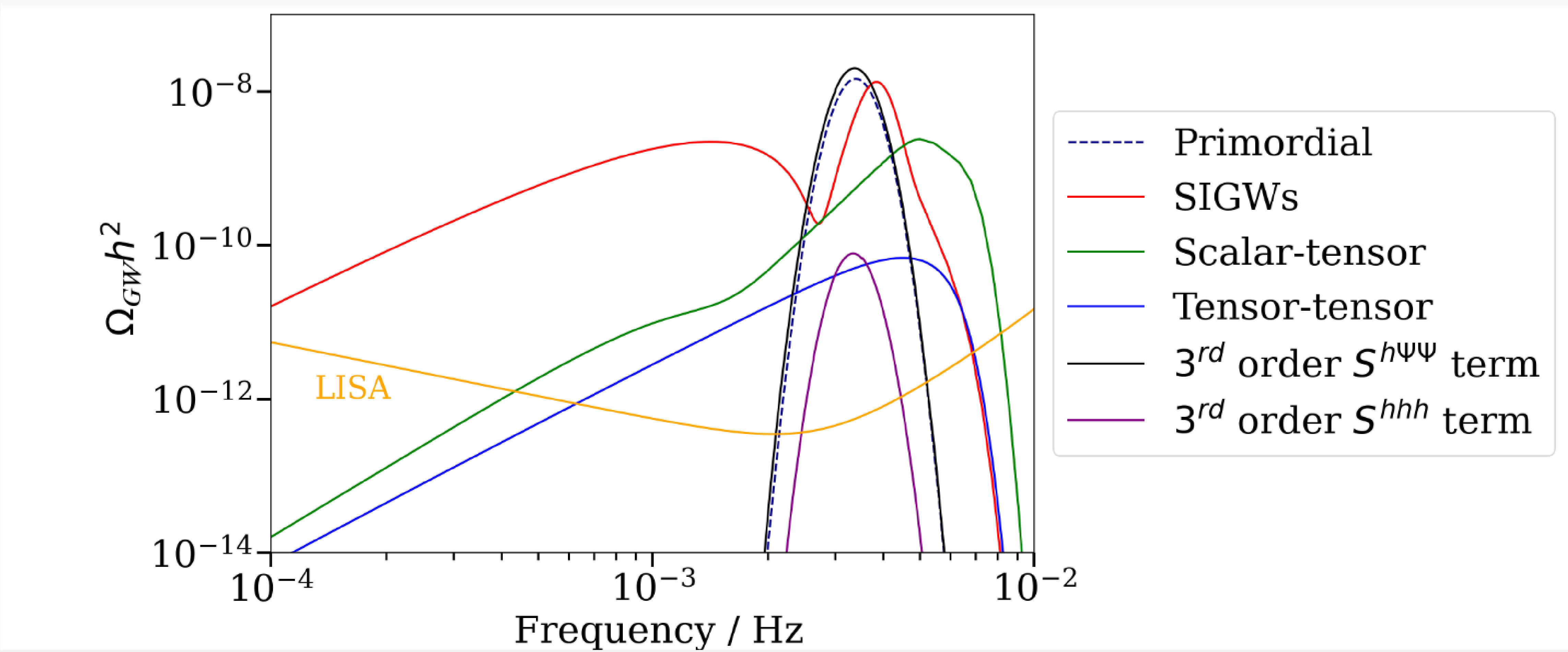
NOT EVERYTHING AT THIRD ORDER WILL CONTRIBUTE

- These were studies in the IR regime previously (*Chen et al*)
- Ultimately, we correlate the third order solution with a first order tensor.
- Since first order scalars and tensors do not correlate, we can drop a few terms

$$S_{ij}^{(3)} = \boldsymbol{S_{ij}^{hhh}} + \boldsymbol{S_{ij}^{h\Psi\Psi}} + S_{ij}^{h^{(2)}h} + S_{ij}^{h^{(2)}\Psi} + S_{ij}^{B^{(2)}h} + S_{ij}^{B^{(2)}\Psi} + S_{ij}^{\Psi^{(2)}h} + S_{ij}^{\Psi^{(2)}\Psi}$$

- The difficulty: the kernels of source terms containing second order perturbations have to be computed numerically.

SO FAR…

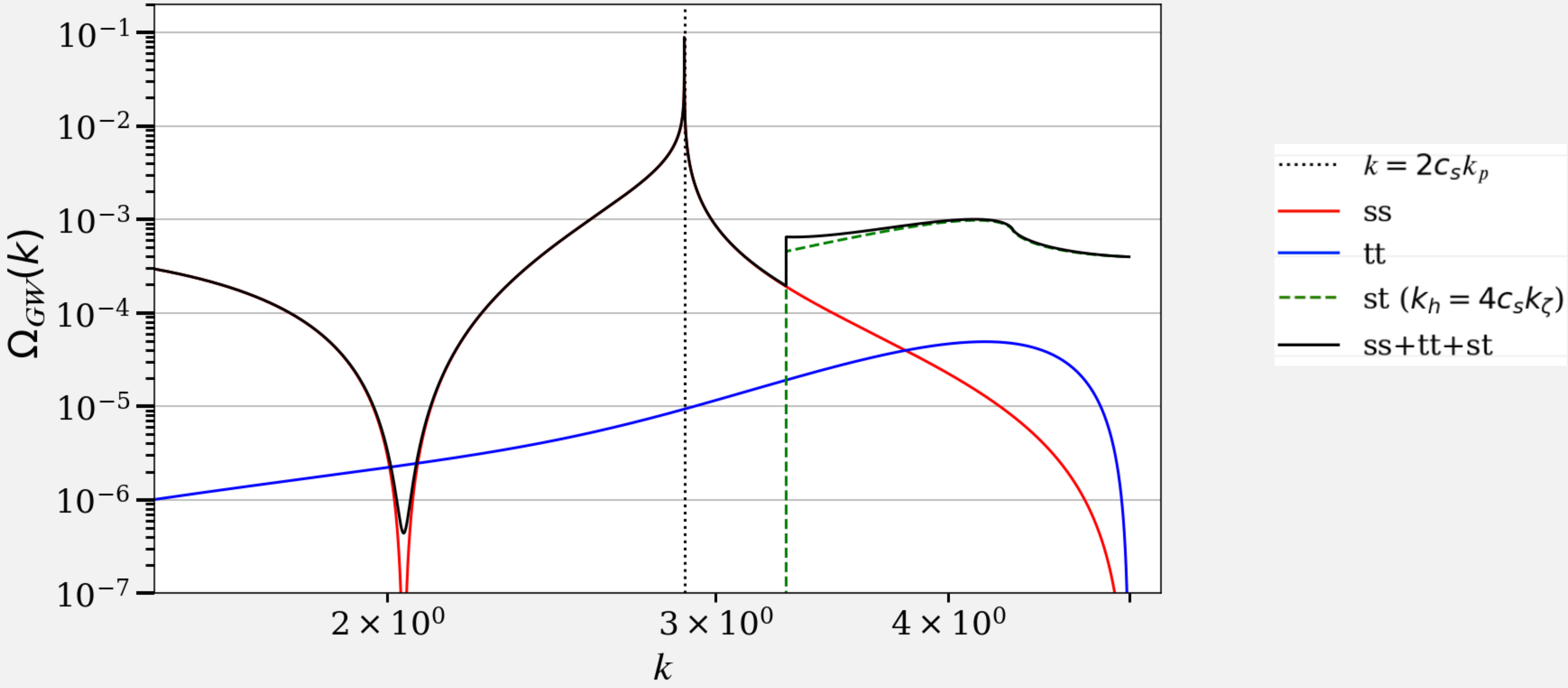


10^{-8}
10^{-10}
10^{-12}
10^{-14}
$\Omega_{GW}h^2$
LISA
10^{-4}
10^{-3}
10^{-2}
Frequency / Hz
Primordial
SIGWs
Scalar-tensor
Tensor-tensor
3^{rd} order $S^{h\Psi\Psi}$ term
3^{rd} order S^{hhh} term

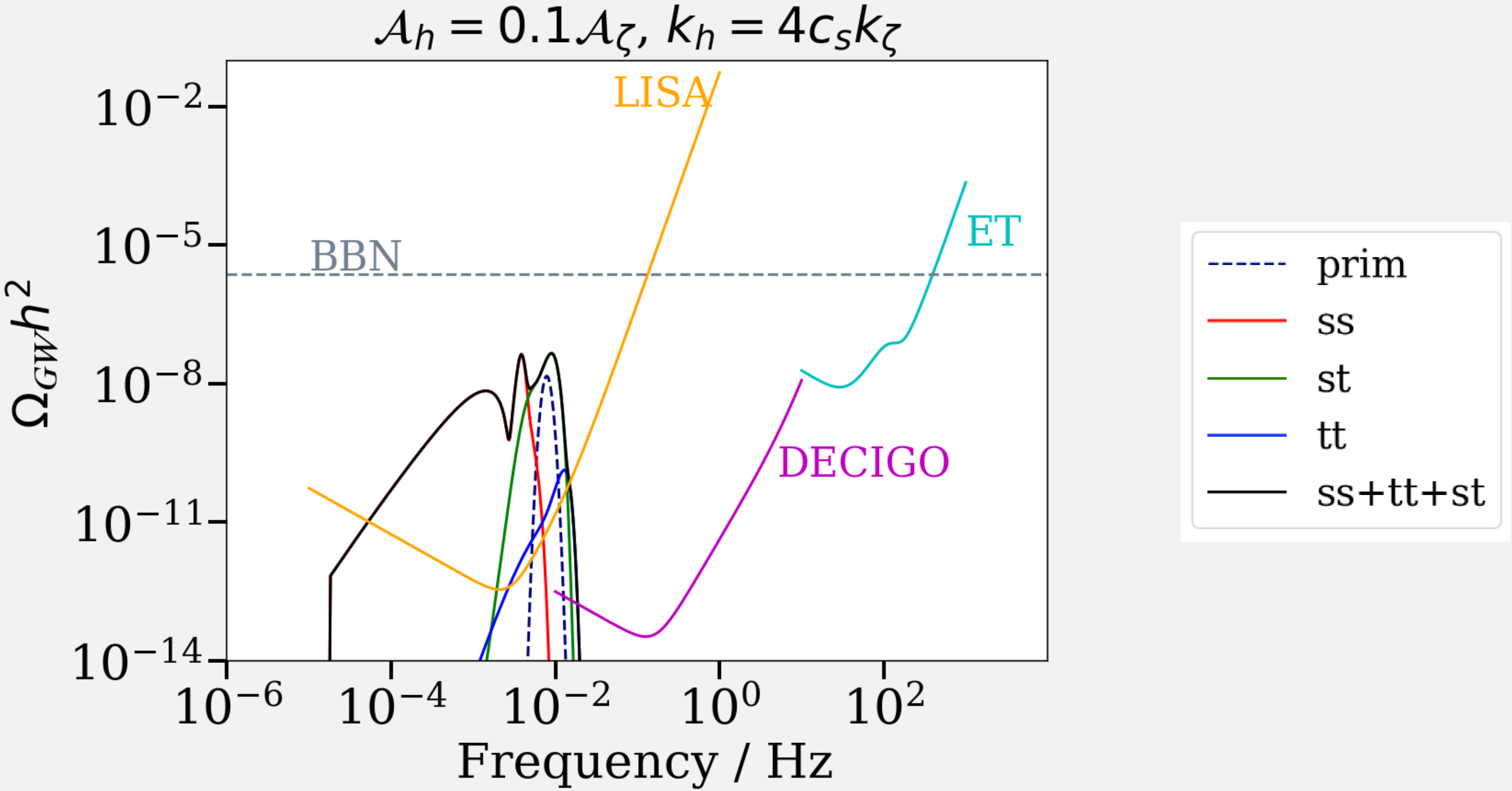
CONCLUSION

- Our work can be used to constrain models of inflation that contain a peak in the power spectrum on small scales for both scalars and tensors.
- Scalar-tensor induced waves (and tensor-tensor) suffer on small scales. We get an unphysical enhancement of the observable when the primordial input spectrum is not peaked enough.
- Hopefully, we can fix this by also considering the correlations of first and third order tensors…

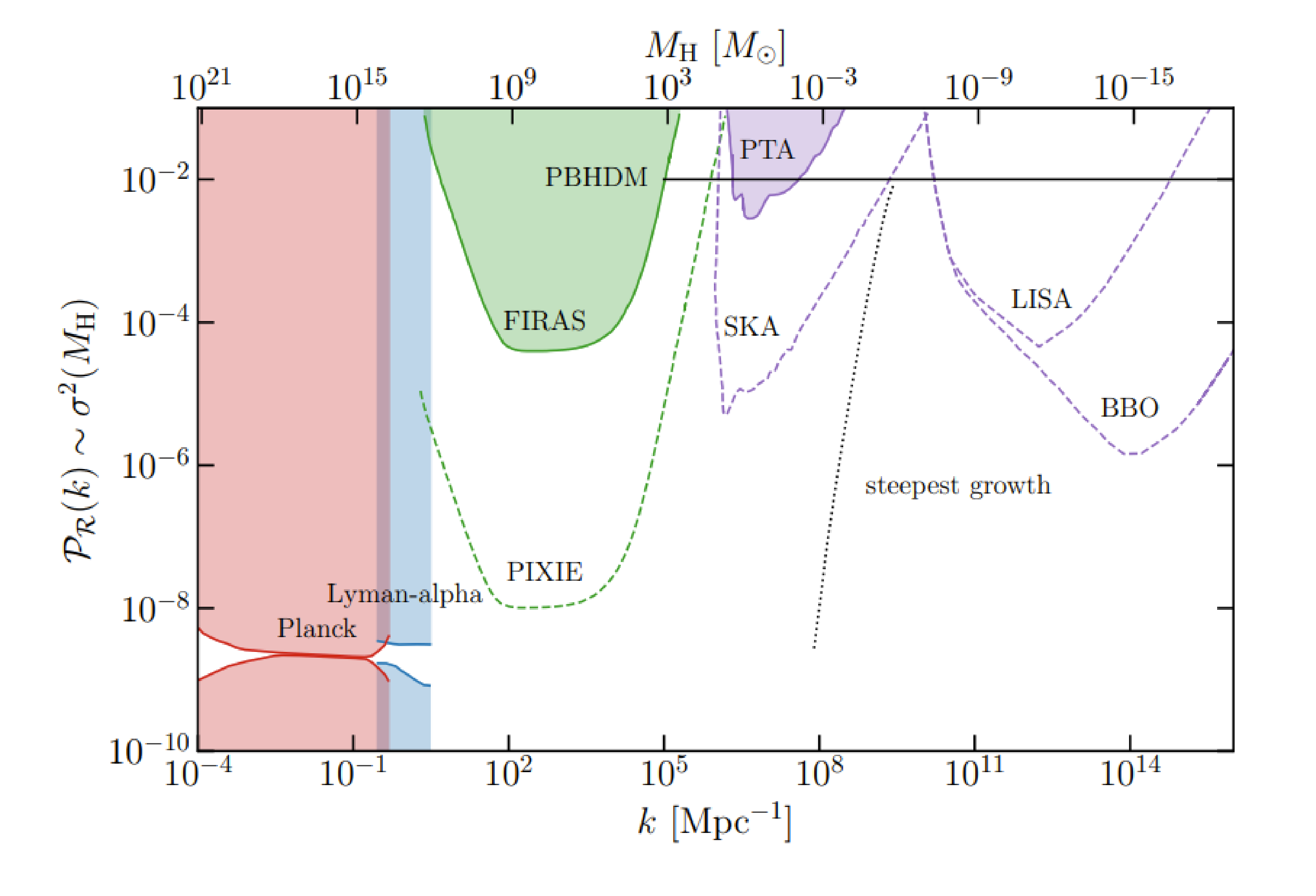
RESULTS: DIRAC DELTA INPUT



RESULTS: LOGNORMAL PEAK

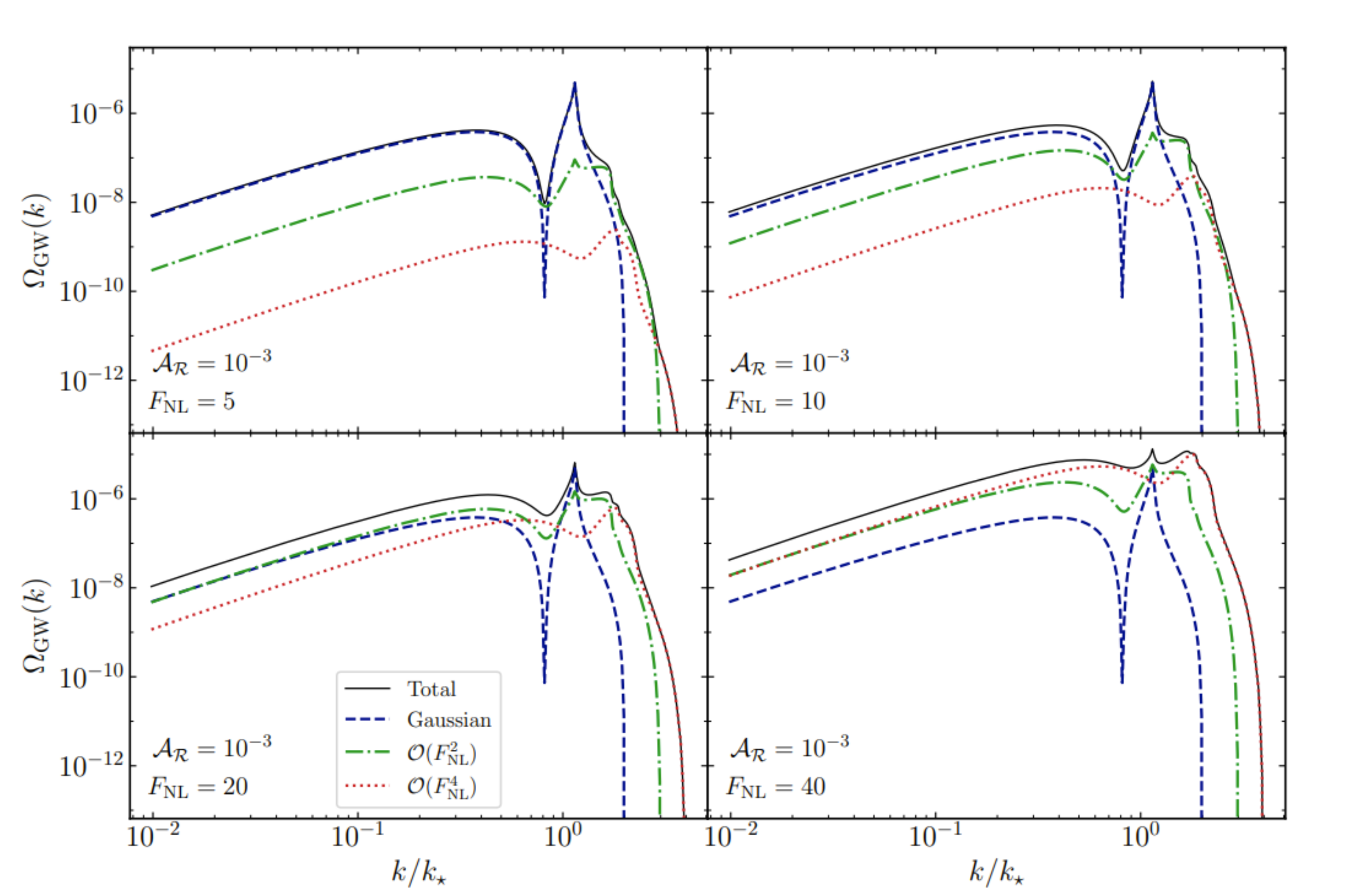


EXTRA SLIDES



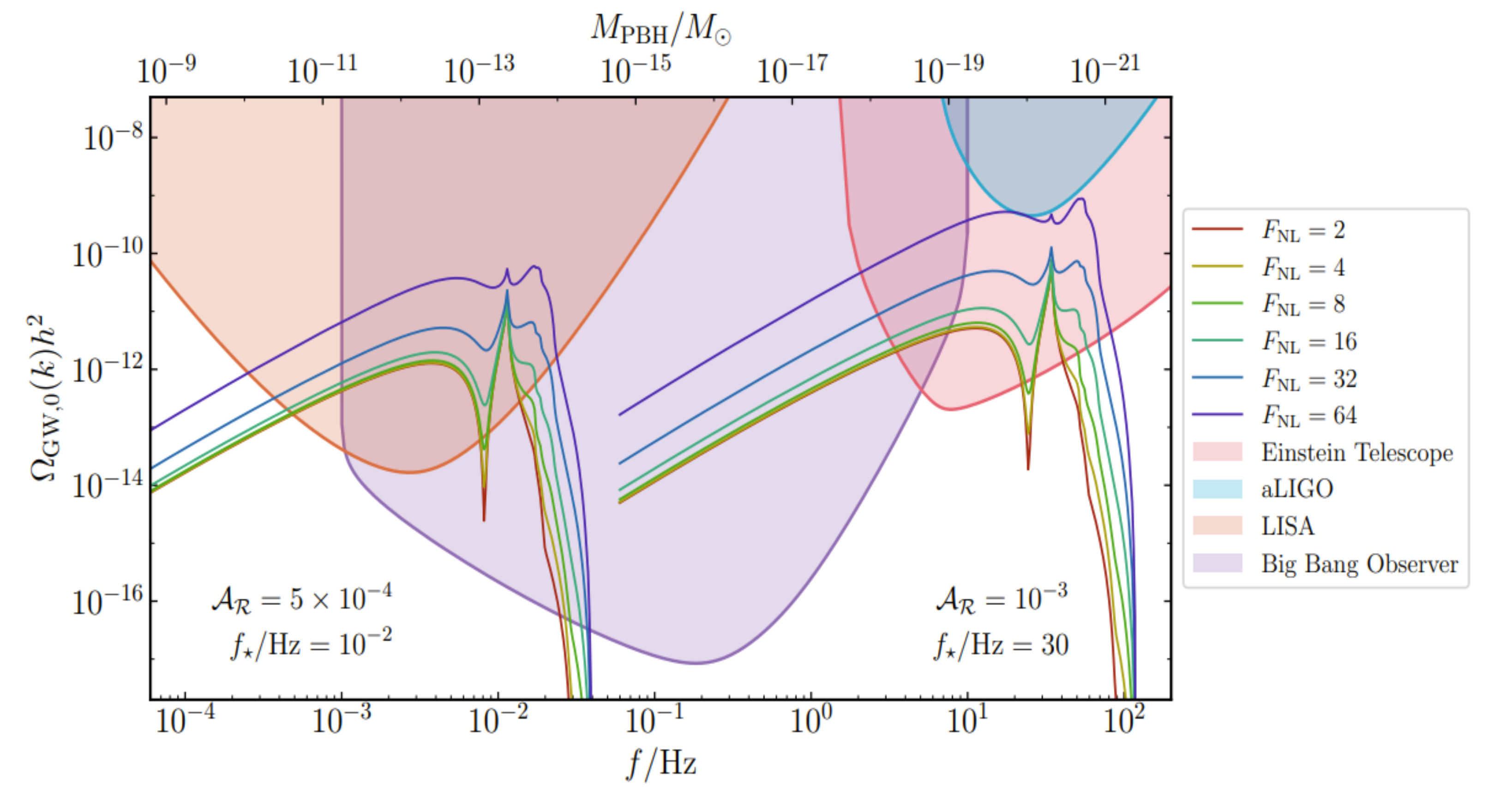
(A. Green)

EXTRA SLIDES



(Adshead et al)

EXTRA SLIDES



(Adshead et al)